Trigonometry

Trigonometry is the branch of mathematics that studies the relations between the sides and angles of triangles. The word "**trigonometry**" comes from the Greek **trigōnon** (triangle) and **metron** (measure.) It was first studied by the Babylonians, Greeks, and Egyptians, and used in surveying, navigation, and astronomy. Trigonometry is a powerful tool that allows us to find the measures of angles and sides of triangles, without physically measuring them, and areas of plots of land. We begin our study of trigonometry by studying angles and their degree measures.

T1 Angles and Degree Measure

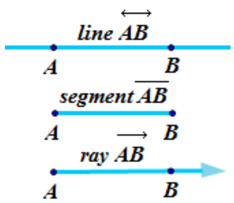


Figure 1a

Two distinct points $\boldsymbol{A}$ and $\boldsymbol{B}$ determine a line denoted $\overleftrightarrow{AB}$. The portion of the line between $\boldsymbol{A}$ and $\boldsymbol{B}$, including the points $\boldsymbol{A}$ and $\boldsymbol{B}$, is called a **line segment** (or simply, a **segment**) $\overline{AB}$. The portion of the line $\overleftrightarrow{AB}$ that starts at $\boldsymbol{A}$ and continues past $\boldsymbol{B}$ is called the **ray** $\overrightarrow{AB}$ (see *Figure 1a*.) Point $\boldsymbol{A}$ is the **endpoint** of this ray.

Two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ sharing the same endpoint $\boldsymbol{A}$, cut the plane into two separate regions. The union of the two rays and one of those regions is called an **angle**, the common endpoint $\boldsymbol{A}$ is called a **vertex**, and the two **rays** are called **sides** or **arms** of this angle. Customarily, we draw a small arc connecting the two rays to indicate which of the two regions we have in mind.

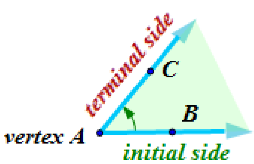


Figure 1b

In trigonometry, an **angle** is often identified with its **measure**, which is the **amount of rotation** that a ray in its initial position (called the **initial side**) needs to turn about the vertex to come to its final position (called the **terminal side**), as in *Figure 1b*. If the rotation from the initial side to the terminal side is *counterclockwise*, the angle is considered to be *positive*. If the rotation is *clockwise*, the angle is *negative* (see *Figure 1c*).

An angle is named either after its vertex, its rays, or the amount of rotation between the two rays. For example, an angle can be denoted $\angle \boldsymbol{A}$, $\angle \boldsymbol{BAC}$, or $\angle \boldsymbol{\theta}$, where the sign $\angle$ (or $\measuredangle$) simply means *an angle*. Notice that in the case of naming an angle with the use of more than one letter, like $\angle \boldsymbol{BAC}$, the middle letter ($\boldsymbol{A}$) is associated with the vertex and the angle is oriented from the ray containing the first point ($\boldsymbol{B}$) to the ray containing the third point ($\boldsymbol{C}$). Customarily, angles (often identified with their measures) are denoted by Greek letters such as $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\theta}$, etc.

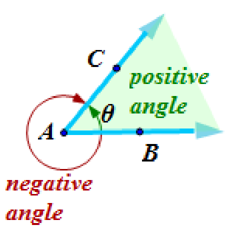


Figure 1c

An angle formed by rotating a ray counterclockwise (in short, **ccw**) exactly one **complete revolution** around its vertex is defined to have a measure of 360 degrees, which is abbreviated as **360°**.

Definition 1.1 ►

One **degree (1°)** is the measure of an angle that is $\frac{1}{360}$ part of a complete revolution.

One **minute** $(\mathbf{1}')$, is the measure of an angle that is $\frac{1}{60}$ part of a degree.

One **second** $(\mathbf{1}'')$ is the measure of an angle that is $\frac{1}{60}$ part of a minute.

Therefore $\mathbf{1}^\circ = \mathbf{60}'$ and $\mathbf{1}' = \mathbf{60}''$.

A fractional part of a degree can be expressed in decimals (e.g. 29.68°) or in minutes and seconds (e.g. 29°40'48"). We say that the first angle is given in **decimal form,** while the second angle is given in **DMS** (**D**egree, **M**inute, **S**econd) **form**.

Example 1 ► Converting Between Decimal and DMS Form

Convert as indicated.

a. $29.68°$ to DMS form

b. $46°18'21''$ to decimal degree form

Solution ►

a. $29.68°$ can be converted to DMS form, using any calculator with **DMS** or **° ′ ″** key. To do it by hand, separate the fractional part of a degree and use the conversion factor $1° = 60'$.

$$\begin{aligned}\mathbf{29.68°} &= 29° + 0.68° \\ &= 29° + 0.68 \cdot 60' = 29° + 40.8'\end{aligned}$$

Similarly, to convert the fractional part of a minute to seconds, separate it and use the conversion factor $1' = 60''$. So we have

$$29.68° = 29° + 40' + 0.8 \cdot 60'' = \mathbf{29°40'48''}$$

b. Similarly, $46°18'21''$ can be converted to the decimal form, using the **DMS** or **° ′ ″** key. To do it by hand, we use the conversions $1' = \left(\frac{1}{60}\right)^{\circ}$ and $1'' = \left(\frac{1}{3600}\right)^{\circ}$.

$$\mathbf{46°18'21''} = \left[46 + 18 \cdot \frac{1}{60} + 21 \cdot \frac{1}{3600}\right]^{\circ} \cong \mathbf{46.3058°}$$

Example 2 ► Adding and Subtracting Angles in DMS Form

Perform the indicated operations.

a. $36°58'21'' + 5°06'45''$ **b.** $36°17' - 15°46'15''$

Solution ►

a. First, we add degrees, minutes, and seconds separately. Then, we convert each $60''$ into $1'$ and each $60'$ into $1°$. Finally, we add the degrees, minutes, and seconds again.

$$\begin{aligned}36°58'21'' + 5°06'45'' &= 41° + 64' + 66'' \\ &= 41° + 1°04' + 1'06'' = \mathbf{42°05'06''}\end{aligned}$$

b. We can subtract within each denomination, degrees, minutes, and seconds, even if the answer is negative. Then, if we need more minutes or seconds to perform the remaining subtraction, we convert $1°$ into $60'$ or $1'$ into $60''$ to finish the calculation.

$$\begin{aligned}36°17' - 15°46'15'' &= 21° - 29' - 15'' \\ &= 20° + 60' - 29' - 15'' = 20° + 31' - 15'' \\ &= 20° + 30' + 60'' - 15'' = \mathbf{20°30'45''}\end{aligned}$$

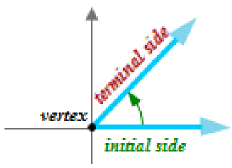


Figure 2

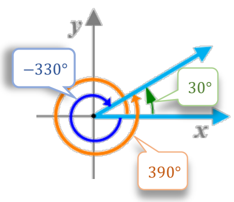


Figure 3

Angles in Standard Position

In trigonometry, we often work with angles in **standard position**, which means angles located in a rectangular system of coordinates with the vertex at the origin and the initial side on the positive x-axis, as in *Figure 2*. With the notion of angle as an amount of rotation of a ray to move from the initial side to the terminal side of an angle, the standard position allows us to represent infinitely many angles with the same terminal side. Those are the angles produced by rotating a ray from the initial side by full revolutions beyond the terminal side, either in a positive or negative direction. Such angles share the same initial and terminal sides and are referred to as **coterminal** angles.

For example, angles $-330°, 30°, 390°, 750°$, and so on, are coterminal.

Definition 1.2 ► Angles α and β are **coterminal**, if and only if there is an integer k, such that

$$\boldsymbol{\alpha = \beta + k \cdot 360°}$$

Example 3 ► **Finding Coterminal Angles**

Find one positive and one negative angle that is closest to $0°$ and coterminal with

a. $80°$

b. $-530°$

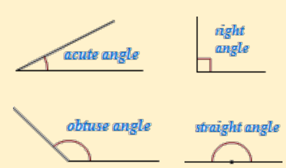
Solution ►

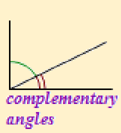
a. To find the closest to $0°$ positive angle coterminal with $80°$ we add one complete revolution, so we have $80° + 360° = \mathbf{440}°$.
Similarly, to find the closest to $0°$ negative angle coterminal with $80°$ we subtract one complete revolution, so we have $80° - 360° = \mathbf{-280}°$.

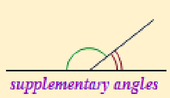
b. This time, to find the closest to $0°$ positive angle coterminal with $-530°$ we need to add two complete revolutions: $-530° + 2 \cdot 360° = \mathbf{190}°$.
To find the closest to $0°$ negative angle coterminal with $-530°$, it is enough to add one revolution: $-530° + 360° = \mathbf{-170}°$.

Definition 1.3 ► Let α be the measure of an angle. Such an angle is called

acute, if $\alpha \in (0°, 90°)$;
right, if $\alpha = 90°$; (right angle is marked by the symbol ∟)
obtuse, if $\alpha \in (90°, 180°)$; and
straight, if $\alpha = 180°$.







Angles that sum to $90°$ are called **complementary**.
Angles that sum to $180°$ are called **supplementary**.

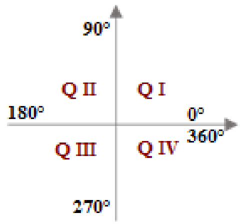


Figure 4

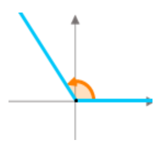
The two axes divide the plane into 4 regions, called **quadrants**. They are numbered counterclockwise, starting with the top right one, as in *Figure 4*.
An angle in standard position is said to lie in the quadrant in which its terminal side lies. For example, an **acute** angle is in *quadrant I* and an **obtuse** angle is in *quadrant II*.
Angles in standard position with their terminal sides along the x-axis or y-axis, such as $\mathbf{0°, 90°, 180°, 270°}$, and so on, are called **quadrantal angles**.

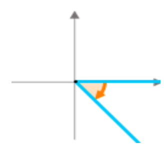
Example 4 ► **Classifying Angles by Quadrants**

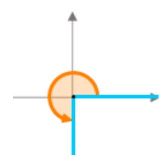
Draw each angle in standard position. Determine the quadrant in which each angle lies or classify the angle as quadrantal.

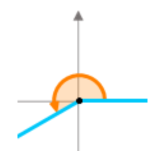
a. $125°$ **b.** $-50°$ **c.** $270°$ **d.** $210°$

Solution ►

a. 
$125°$ is in **QII**

b. 
$-50°$ is in **QIV**

c. 
quadrantal angle

d. 
$210°$ is in **QIII**

Example 5 ► **Finding Complementary and Supplementary Angles**

Find the complement and the supplement of $57°$.

Solution ► Since complementary angles add to $90°$, the complement of $57°$ is $90° - 57° = 33°$.
Since supplementary angles add to $180°$, the supplement of $57°$ is $180° - 57° = 123°$.

T.1 Exercises

*Convert each angle measure to **decimal degrees**. Round the answer to the nearest thousandth of a degree.*

1. $20°04'30''$ **2.** $71°45'$ **3.** $274°18'15''$

4. $34°41'07''$ **5.** $15°10'05''$ **6.** $64°51'35''$

*Convert each angle measure to **degrees, minutes, and seconds**. Round the answer to the nearest second.*

7. $18.0125°$ **8.** $89.905°$ **9.** $65.0015°$

10. $184.3608°$ **11.** $175.3994°$ **12.** $102.3771°$

Perform each calculation.

13. $62°18' + 21°41'$ **14.** $71°58' + 47°29'$ **15.** $65°15' - 31°25'$

16. $90° - 51°28'$ **17.** $15°57'45'' + 12°05'18''$ **18.** $90° - 36°18'47''$

Give the complement and the supplement of each angle.

19. $30°$ **20.** $60°$ **21.** $45°$ **22.** $86.5°$ **23.** $15°30'$

24. Give an expression representing the complement of a $\boldsymbol{\theta}°$ angle.

25. Give an expression representing the supplement of a $\boldsymbol{\theta}°$ angle.

Sketch each angle in standard position. Draw an arrow representing the correct amount of rotation. Give the quadrant of each angle or identify it as a quadrantal angle.

26. $75°$ **27.** $135°$ **28.** $-60°$ **29.** $270°$ **30.** $390°$

31. $315°$ **32.** $510°$ **33.** $-120°$ **34.** $240°$ **35.** $-180°$

Find the angle of least positive measure coterminal with each angle.

36. $-30°$ **37.** $375°$ **38.** $-203°$ **39.** $855°$ **40.** $1020°$

Give an expression that generates all angles coterminal with the given angle. Use $\boldsymbol{k}$ *to represent any integer.*

41. $30°$ **42.** $45°$ **43.** $0°$ **44.** $90°$ **45.** $\alpha°$

Find the degree measure of the smaller angle formed by the hands of a clock at the following times.

46.

47. $3:15$ **48.** $1:45$

T2 Trigonometric Ratios of an Acute Angle and of Any Angle

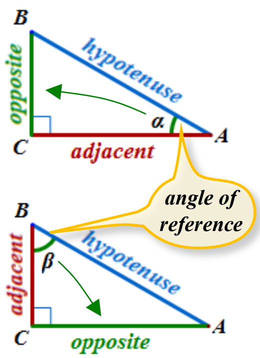


Figure 2.1

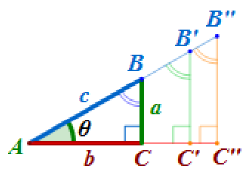


Figure 2.2

Generally, trigonometry studies ratios between sides in right angle triangles. When working with right triangles, it is convenient to refer to the side **opposite** to an angle, the side **adjacent** to (next to) an angle, and the **hypotenuse**, which is the longest side, opposite to the right angle. Notice that the opposite and adjacent sides depend on the **angle of reference** (one of the two acute angles.) However, the hypotenuse stays the same, regardless of the choice of the angle of reference. See *Figure 2.1.*

Notice that any two right triangles with the same acute angle $\boldsymbol{\theta}$ are **similar**. See *Figure 2.2.* **Similar** means that their corresponding angles are **congruent** and their corresponding sides are **proportional**. For instance, assuming notation as on *Figure 2.2*, we have

$$\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'},$$

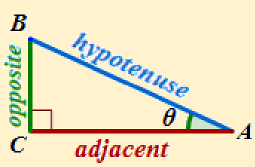
or equivalently

$$\frac{BC}{AB} = \frac{B'C'}{AB'}, \qquad \frac{AC}{AB} = \frac{AC'}{AB'}, \qquad \frac{BC}{AC} = \frac{B'C'}{AC'}.$$

Therefore, the ratios of any two sides of a right triangle does not depend on the size of the triangle but only on the size of the angle of reference. See the following demonstration. This means that we can study those **ratios** of sides as **functions** of an acute angle.

Trigonometric Functions of Acute Angles

Definition 2.1 ►



Given a **right angle triangle** with an **acute** angle $\boldsymbol{\theta}$, the three **primary trigonometric ratios** of the angle $\boldsymbol{\theta}$, called ***sine, cosine,*** and ***tangent*** (abbreviation: *sin, cos, tan*), are defined as follows:

$$\sin\boldsymbol{\theta} = \frac{\boldsymbol{Opposite}}{\boldsymbol{Hypotenuse}}, \qquad \cos\boldsymbol{\theta} = \frac{\boldsymbol{Adjacent}}{\boldsymbol{Hypotenuse}}, \qquad \tan\boldsymbol{\theta} = \frac{\boldsymbol{Opposite}}{\boldsymbol{Adjacent}}$$

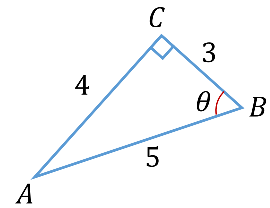
For easier memorization, we can use the acronym **SOH – CAH – TOA** (read: *so – ka – toe – ah*), formed from the first letter of the function and the corresponding ratio.

The three **reciprocal trigonometry ratios** of the angle $\boldsymbol{\theta}$, called ***cosecant***, ***secant***, and ***cotangent*** (abbreviation: *csc, sec, cot*), are reciprocals of the sine, cosine, and tangent ratios, respectively, and are defined as follows:

$$\csc\boldsymbol{\theta} = \frac{\boldsymbol{Hypotenuse}}{\boldsymbol{Opposite}}, \qquad \sec\boldsymbol{\theta} = \frac{\boldsymbol{Hypotenuse}}{\boldsymbol{Adjacent}}, \qquad \cot\boldsymbol{\theta} = \frac{\boldsymbol{Adjacent}}{\boldsymbol{Opposite}}$$

Example 1 ► **Identifying Sides of a Right Triangle to Form Trigonometric Ratios**

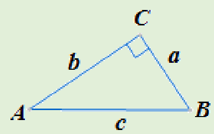
Identify the hypotenuse, opposite, and adjacent side of angle θ and state values of the six trigonometric ratios.



Solution ► Side AB is the hypotenuse, as it lies across from the right angle.
Side BC is the adjacent, as it is part of the angle θ other than hypotenuse.
Side AC is the opposite, as it lies across from angle θ.
Therefore, $\sin\theta = \frac{opp.}{hyp.} = \frac{\mathbf{4}}{\mathbf{5}}$, $\cos\theta = \frac{adj.}{hyp.} = \frac{\mathbf{3}}{\mathbf{5}}$, $\tan\theta = \frac{opp.}{adj.} = \frac{\mathbf{4}}{\mathbf{3}}$, $\csc\theta = \frac{hyp.}{opp.} = \frac{\mathbf{5}}{\mathbf{4}}$,
$\sec\theta = \frac{hyp.}{adj.} = \frac{\mathbf{5}}{\mathbf{3}}$, and $\cot\theta = \frac{adj.}{opp.} = \frac{\mathbf{3}}{\mathbf{4}}$.

The three **primary trigonometric ratios** together with the **Pythagorean Theorem** allow us to **solve** any right-angle triangle. That means that given the measurements of two sides, or one side and one angle, with a little help of algebra, we can find the measurements of all remaining sides and angles of any right triangle. See *Section T4*.

Pythagorean Theorem ► A **triangle $\boldsymbol{ABC}$ is right** with $\angle C = 90°$ if and only if $\boldsymbol{a^2 + b^2 = c^2}$.

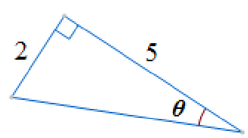


Convention: *The **side opposite** the given **vertex** (or angle) is named after the vertex, except that by a lower case rather than a capital letter. For example, the side opposite vertex **A** is called **a**.*

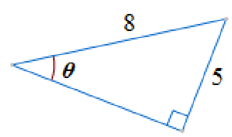
Example 2 ► **Finding Values of Trigonometric Ratios With the Aid of Pythagorean Theorem**

Given the triangle, find the exact values of the sine, cosine, and tangent ratios for angle θ.

a.



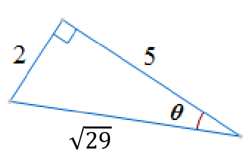
b.



Solution ► **a.** Let $\boldsymbol{h}$ denote the hypotenuse. By the **Pythagorean Theorem**, we have

$$h^2 = 2^2 + 5^2$$
$$h = \sqrt{4+25} = \sqrt{29}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{\mathbf{2\sqrt{29}}}{\mathbf{29}}$$

Note: It is customary to rationalize the denominator.

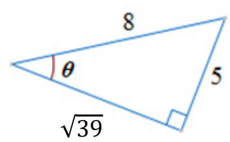
$$\cos\theta = \frac{5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{\mathbf{5\sqrt{29}}}{\mathbf{29}}$$

$$\tan\theta = \frac{\mathbf{2}}{\mathbf{5}}$$

b. Let $\boldsymbol{a}$ denote the adjacent side. By the **Pythagorean Theorem**, we have

$$a^2 + 5^2 = 8^2$$
$$a = \sqrt{8^2 - 5^2} = \sqrt{64 - 25} = \sqrt{39}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \frac{\mathbf{5}}{\mathbf{8}}$$

$$\cos\theta = \frac{\sqrt{\mathbf{39}}}{\mathbf{8}}$$

$$\tan\theta = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \frac{\mathbf{5\sqrt{39}}}{\mathbf{39}}$$

Trigonometric Functions of Any Angle

Notice that any angle of a right triangle, other than the right angle, is acute. Thus, the "SOH – CAH – TOA" definition of the trigonometric ratios refers to acute angles only. However, we can extend this definition to include all angles. This can be done by observing our right triangle within the Cartesian Coordinate System.

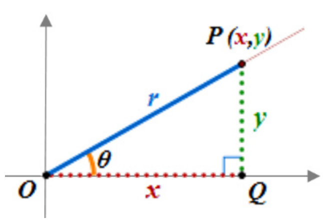


Figure 2.3

Let triangle OPQ with $\angle Q = 90°$ be placed in the coordinate system so that O coincides with the origin, Q lies on the positive part of the x-axis, and P lies in the first quadrant. See *Figure 2.3*. Let (x, y) be the coordinates of the point P, and let θ be the measurement of $\angle QOP$. This way, angle θ is in standard position and the triangle OPQ is obtained by **projecting** point P perpendicularly onto the x-axis. Thus in this setting, the position of point P actually determines both the angle θ and the ΔOPQ. Observe that the coordinates of point P ($\boldsymbol{x}$ and $\boldsymbol{y}$) really represent the length of the **adjacent** and the **opposite** side, correspondingly. Since the length of the **hypotenuse** represents the distance of the point P from the origin, it is often denoted by $\boldsymbol{r}$ (from *radius*.)

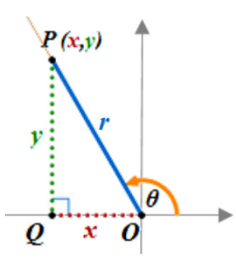


Figure 2.4

By rotating the radius $\boldsymbol{r}$ and projecting the point P perpendicularly onto x-axis (follow the green dotted line from P to Q in *Figure 2.4*), we can obtain a right triangle corresponding to any angle θ, not only an acute angle. Since the coordinates of a point in a plane can be negative, to establish a correcpondence between the coordinates x and y of the point P, and the distances OQ and QP, it is convenient to think of **directed distances** rather than just distances. Distance becomes directed if we assign a sign to it. So, lets assign a positive sign to horizontal or vertical distances that follow the directions of the corresponding number lines, and a negative sign otherwise. For example, the directed distance $OQ = x$ in *Figure 2.3* is positive because the direction from O to Q follows the order on the x-axis while the directed distance $OQ = x$ in *Figure 2.4* is negative because the direction from O to Q is against the order on the x-axis. Likewise, the directed distance $QP = y$ is positive for angles in the first and second quadrant (as in *Figure 2.3* and *2.4*), and it is negative for angles in the third and fourth quadrant (convince yourself by drawing a diagram).

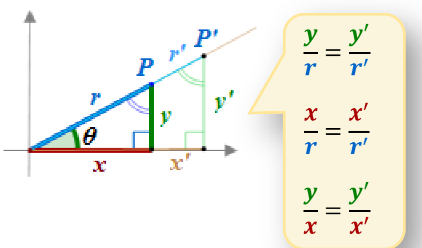
Definition 2.2 ► Let $P(\boldsymbol{x},\boldsymbol{y})$ be any point, different than the origin, on the terminal side of an angle $\boldsymbol{\theta}$ in standard position. Also, let $\boldsymbol{r}=\sqrt{\boldsymbol{x}^2+\boldsymbol{y}^2}$ be the distance of the point P from the origin. We define

$$\sin\boldsymbol{\theta}=\frac{\boldsymbol{y}}{\boldsymbol{r}},\qquad \cos\boldsymbol{\theta}=\frac{\boldsymbol{x}}{\boldsymbol{r}},\qquad \tan\boldsymbol{\theta}=\frac{\boldsymbol{y}}{\boldsymbol{x}}\ \text{(for } x\neq 0)$$

$$\csc\boldsymbol{\theta}=\frac{\boldsymbol{r}}{\boldsymbol{y}}\ \text{(for } y\neq 0),\qquad \sec\boldsymbol{\theta}=\frac{\boldsymbol{r}}{\boldsymbol{x}}\ \text{(for } x\neq 0),\qquad \cot\boldsymbol{\theta}=\frac{\boldsymbol{x}}{\boldsymbol{y}}\ \text{(for } y\neq 0)$$

Observations:

- For acute angles, *Definition 2.2* agrees with the "**SOH – CAH – TOA**" *Definition 2.1*.



- Proportionality of similar triangles guarantees that each point of the same terminal ray defines the same trigonometric ratio. This means that the above definition assigns a unique value to each trigonometric ratio for any given angle regardless of the point chosen on the terminal side of this angle. Thus, the above trigonometric ratios are in fact **functions of any real angle** and these functions are properly defined in terms of x, y, and r.

- Since $r>0$, the first two trigonometric functions, **sine** and **cosine**, are defined for any real angle θ.

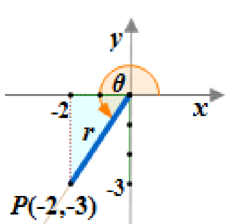
- The reamining trigonometric functions, **tangent**, **cosecant**, **secant**, and **cotangent**, are defined for all real angles θ, except for angles that create a zero in the ratio's denominator. For example, tangent is defined for all angles except those with terminal sides on the y-axis. This is because the x-coordinate of any point on the y-axis equals zero, which cannot be used to create the ratio $\frac{y}{x}$. Thus, tangent is a function of all real angles, except for $90°, 270°$, and so on (generally, except for angles of the form $\mathbf{90°}+\boldsymbol{k}\cdot\mathbf{180°}$, where $\boldsymbol{k}$ is an integer).

Example 3 ► **Evaluating Trigonometric Functions of any Angle in Standard Position**

Find the exact value of the six trigonometric functions of an angle θ in standard position whose terminal side contains the point

a. $P(-2,-3)$ **b.** $P(0,1)$

Solution ► **a.** To ilustrate the situation, lets sketch the least positive angle θ in standard position with the point $P(-2,-3)$ on its terminal side.



To find values of the trigonometric functions, first, we will determine the length of r:

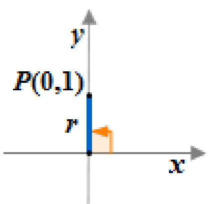
$$r=\sqrt{(-2)^2+(-3)^2}=\sqrt{4+9}=\sqrt{13}$$

Now, we can state the exact values of the trigonometric functions:

$$\sin\theta=\frac{y}{r}=\frac{-3}{\sqrt{13}}=\frac{\mathbf{-3\sqrt{13}}}{\mathbf{13}},\qquad \csc\theta=\frac{r}{y}=\frac{\sqrt{13}}{-3}=-\frac{\mathbf{\sqrt{13}}}{\mathbf{3}}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} = \frac{\mathbf{-2\sqrt{13}}}{\mathbf{13}} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{13}}{-2} = -\frac{\mathbf{\sqrt{13}}}{\mathbf{2}},$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{-2} = \frac{\mathbf{3}}{\mathbf{2}}, \qquad \cot\theta = \frac{x}{y} = \frac{-2}{-3} = \frac{\mathbf{2}}{\mathbf{3}}.$$



b. Since $x = 0,\ y = 1,\ r = \sqrt{0^2 + 1^2} = 1,$ then

$$\sin\theta = \frac{y}{r} = \frac{1}{1} = \mathbf{1}, \qquad \csc\theta = \frac{r}{y} = \frac{1}{1} = \mathbf{1},$$

we can't divide by zero!

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = \mathbf{0}, \qquad \sec\theta = \frac{r}{x} = \frac{1}{0} = \textbf{undefined},$$

$$\tan\theta = \frac{y}{x} = \frac{1}{0} = \textbf{undefined}, \qquad \cot\theta = \frac{x}{y} = \frac{0}{1} = \mathbf{0}.$$

Notice that the measure of the least positive angle θ in standard position with the point $P(0,1)$ on its terminal side is 90°. Therefore, we have

$$\sin 90° = 1, \quad \cos 90° = 0, \quad \tan 90° = \text{undefined}$$
$$\csc 90° = 1, \quad \sec 90° = DNE, \quad \cot 90° = 0$$

The values of trigonometric functions of other commonly used quadrantal angles, such as 0°, 180°, 270°, and 360°, can be found similarly as in *Example 3b*. These values for the primary functions are summarized in the table below. The reader is encouraged to extend the table for the reciprocal functions.

Table 2.1 Function Values of Quadrantal Angles

θ / function	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	−1	0
$\cos\theta$	1	0	−1	0	1
$\tan\theta$	0	undefined	0	undefined	0

Example 4 ► **Evaluating Trigonometric Functions Using Basic Identities**

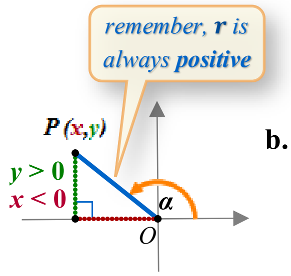
Knowing that $\cos\alpha = -\frac{3}{4}$ and the angle α is in quadrant II, find

a. $\sin\alpha$ **b.** $\tan\alpha$

Solution ►

a. We know that $\cos\alpha = -\frac{3}{4} = \frac{x}{r}$. Hence, the terminal side of angle $\alpha \in QII$ contains a point $P(x,y)$ satisfying the condition $\frac{x}{r} = -\frac{3}{4}$. Since r must be positive, we will assign $x = -3$ and $r = 4$, to model the situation. Using the Pythagorean equation and the fact that the y-coordinate of any point in the second quadrant is positive, we determine the corresponding y-value to be

$$y = \sqrt{r^2 - x^2} = \sqrt{4^2 - (-3)^2} = \sqrt{16 - 9} = \sqrt{7}.$$



Now, we are ready to use *Definition 2.2* to state the sine value of angle α:

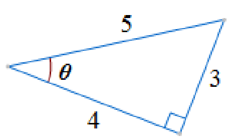
$$\sin\alpha = \frac{y}{r} = \frac{\sqrt{7}}{4}.$$

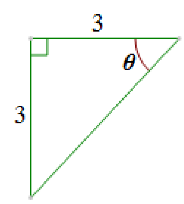
b. To find the value of $\tan\alpha$, since we already know the values of x, y, and r, we can again use *Definition 2.2*:

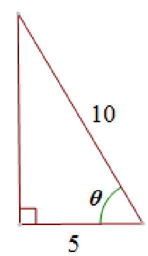
$$\tan\alpha = \frac{y}{x} = \frac{\sqrt{7}}{-3} = -\frac{\sqrt{7}}{3}.$$

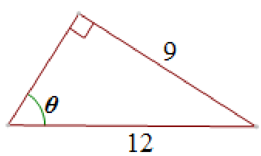
T.2 Exercises

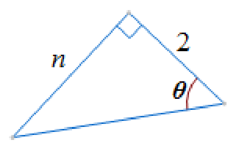
*Find the **exact values** of the six trigonometric functions for the indicated angle θ. Rationalize denominators when applicable.*

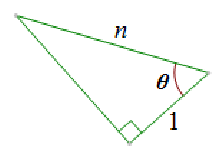
1. 

2. 

3. 

4. 

5. 

6. 

Sketch an angle θ in standard position such that θ has the least positive measure, and the given point is on the terminal side of θ. Then find the values of the three primary trigonometric functions for each angle. Rationalize denominators when applicable.

7. $(-3,4)$ **8.** $(-4,-3)$ **9.** $(5,-12)$ **10.** $(0,3)$ **11.** $(-4,0)$

12. $(1,\sqrt{3})$ **13.** $(3,5)$ **14.** $(0,-8)$ **15.** $(-2\sqrt{3},-2)$ **16.** $(5,0)$

17. If the terminal side of an angle θ is in quadrant III, what is the sign of each of the trigonometric function values of θ?

*Suppose that the point (x, y) is in the indicated quadrant. Decide whether the given ratio is **positive** or **negative**.*

18. $QI,\ \frac{y}{x}$ **19.** $QII,\ \frac{y}{x}$ **20.** $QII,\ \frac{y}{r}$ **21.** $QIII,\ \frac{x}{r}$ **22.** $QIV,\ \frac{y}{x}$

23. $QIII,\ \frac{y}{x}$ **24.** $QIV,\ \frac{y}{r}$ **25.** $QI,\ \frac{y}{r}$ **26.** $QIV,\ \frac{x}{r}$ **27.** $QII,\ \frac{x}{r}$

Use the definition of trigonometric functions in terms of ***x***, ***y***, *and* ***r*** *to determine each value. If it is undefined, say so.*

28. $\sin 90°$ **29.** $\cos 0°$ **30.** $\tan 180°$ **31.** $\cos 180°$ **32.** $\cot 270°$

33. $\cos 270°$ **34.** $\csc 270°$ **35.** $\sec 90°$ **36.** $\sin 0°$ **37.** $\cot 90°$

Determine the values of the remaining two primary trigonometric functions of the angle satisfying the given conditions. Rationalize denominators when applicable.

38. $\sin\alpha = \frac{\sqrt{2}}{4};\ \alpha \in Q\mathrm{II}$ **39.** $\sin\beta = -\frac{2}{3};\ \beta \in Q\mathrm{III}$ **40.** $\cos\theta = \frac{2}{5};\ \theta \in Q\mathrm{IV}$

T3 Evaluation of Trigonometric Functions

In the previous section, we defined sine, cosine, tangent, secant, cosecant, and cotangent as functions of real angles. In this section, we will take interest in finding values of these functions for angles $\theta \in [0°, 360°)$. As shown before, one can find exact values of trigonometric functions of an angle θ with the aid of a right triangle with the acute angle θ and given side lengths, or by using coordinates of a given point on the terminal side of the angle θ in standard position. What if such data is not given? Then, one could consider approximating trigonometric function values by measuring sides of a right triangle with the desired angle θ and calculating corresponding ratios. However, this could easily prove to be a cumbersome process, with inaccurate results. Luckily, we can rely on calculators, which are programmed to return approximated values of the three primary trigonometric functions for any angle.

Attention: *In this section, any calculator instruction will refer to scientific calculators.*

Example 1 ► **Evaluating Trigonometric Functions Using a Calculator**

Find each function value up to four decimal places.

When evaluating functions of angles in degrees, the calculator must be set to the **degree mode**.

a. $\sin 39°12'\,10''$ **b.** $\tan 102.6°$

Solution ► **a.** Before entering the expression into the calculator, we need to check if the calculator is in degree mode by pressing the DRG key until DEG appears at the top of the screen. Now we can enter $\sin 39°12'10''$ by pressing

sin 39 $D°M'S$ 12 $D°M'S$ 10 =

Thus $\sin 39°12'10'' \approx \mathbf{0.6321}$ when rounded to four decimal places.

b. When evaluating trigonometric functions of angles in decimal degrees, it is not necessary to write the degree (°) sign when in degree mode. We simply key in

tan 102.6 =

to obtain $\tan 102.6° \approx \mathbf{-4.4737}$ when rounded to four decimal places.

Special Angles

It has already been discussed how to find the **exact values** of trigonometric functions of **quadrantal angles** using the definitions in terms of x, y, and r. *See Section T2, Example 3b, and Table 2.1.*

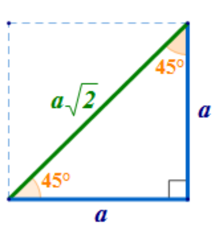


Figure 3.1

Are there any other angles for which the trigonometric functions can be evaluated exactly? Yes, we can find the exact values of trigonometric functions of any angle that can be modelled by a right triangle with known sides. For example, angles such as 30°, 45°, or 60° can be modeled by half of a square or half of an equilateral triangle. In each triangle, the relations between the lengths of sides are easy to establish.

In the case of half a square *(see Figure 3.1)*, we obtain a right triangle with two acute angles of **45°**, and two equal sides of certain length $\boldsymbol{a}$.

Hence, by The Pythagorean Theorem, the diagonal $d = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \boldsymbol{a\sqrt{2}}$.

Summary: The sides of any $\mathbf{45° - 45° - 90°}$ triangle are in the relation $\boldsymbol{a - a - a\sqrt{2}}$.

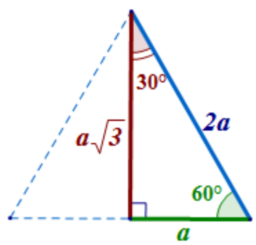


Figure 3.2

By dividing an equilateral triangle *(see Figure 3.1)* along its height, we obtain a right triangle with acute angles of $\mathbf{30°}$ and $\mathbf{60°}$. If the length of the side of the original triangle is denoted by $\boldsymbol{2a}$, then the length of half a side is $\boldsymbol{a}$, and the length of the height can be calculated by applying The Pythagorean Theorem, $h = \sqrt{(2a)^2 - a^2} = \sqrt{3a^2} = \boldsymbol{a\sqrt{3}}$.

Summary: The sides of any $\mathbf{30° - 60° - 90°}$ triangle are in the relation $\boldsymbol{a - 2a - a\sqrt{3}}$.

Since the trigonometric ratios do not depend on the size of a triangle, for simplicity, we can assume that $a = 1$ and work with the following **special triangles**:

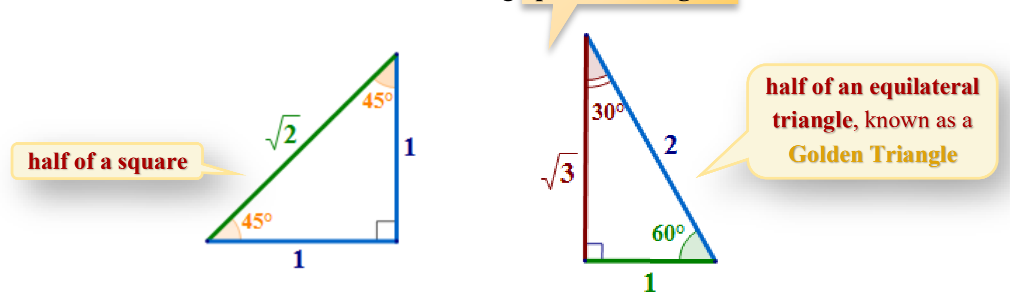


Figure 3.3

Special angles such as $\mathbf{30°}, \mathbf{45°}$, and $\mathbf{60°}$ are frequently seen in applications. We will often refer to the exact values of trigonometric functions of these angles. Special triangles give us a tool for finding those values.

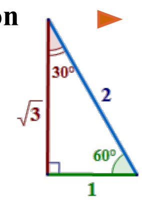
Advice: *Make sure that you can **recreate the special triangles** by taking half of a square or half of an equilateral triangle anytime you wish to **recall the relations** between their sides.*

Example 2 ► **Finding Exact Values of Trigonometric Functions of Special Angles**

Find the *exact* value of each expression.

a. $\cos 60°$ **b.** $\tan 30°$ **c.** $\sin 45°$ **d.** $\tan 45°$

Solution ►

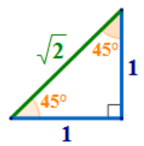


a. Refer to the $30° - 60° - 90°$ triangle and follow the SOH-CAH-TOA definition of sine:

$$\cos 60° = \frac{adj.}{hyp.} = \frac{1}{2}$$

b. Refer to the same triangle as above:

$$\tan 30° = \frac{opp.}{adj.} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$



c. Refer to the $45° - 45° - 90°$ triangle:

$$\sin 45° = \frac{opp.}{hyp.} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

d. Refer to the $45° - 45° - 90°$ triangle:

$$\tan 45° = \frac{opp.}{adj.} = \frac{1}{1} = 1$$

The exact values of trigonometric functions of special angles are summarized in the table below.

Table 3.1 Function Values of Special Angles

function \ θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Observations:

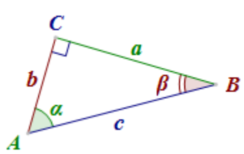


Figure 3.4

- Notice that $\sin 30° = \cos 60°$, $\sin 60° = \cos 30°$, and $\sin 45° = \cos 45°$. Is there any general rule to explain this fact? Lets look at a right triangle with acute angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (*see Figure 3.4*). Since the sum of angles in any triangle is $180°$ and $\angle C = 90°$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} = 90°$, therefore they are **complementary angles**. From the definition, we have $\sin\boldsymbol{\alpha} = \frac{\boldsymbol{a}}{\boldsymbol{b}} = \cos\boldsymbol{\beta}$. Since angle $\boldsymbol{\alpha}$ was chosen arbitrarily, this rule applies to any pair of acute complementary angles. It happens that this rule actually applies to all complementary angles. So we have the following claim:

sin $\boldsymbol{\alpha}$ **= cos** $(90° - \alpha)$
sec $\boldsymbol{\alpha}$ **= csc** $(90° - \alpha)$
tan $\boldsymbol{\alpha}$ **= cot** $(90° - \alpha)$

The **cofunctions** (like sine and cosine, secant and cosecant, or tangent and cotangent) of **complementary** angles are equal.

Example 3 ► **Using the Cofunction Relationship**

Rewrite $\cos 75°$ in terms of the cofunction of the complementary angle.

Solution ► Since the complement of $75°$ is $90° - 75° = 15°$, then $\cos 75° = \mathbf{\sin 15°}$.

Reference Angles

Can we determine exact values of trigonometric functions of nonquadrantal angles that are larger than 90°?

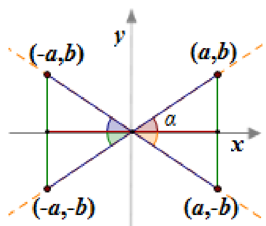
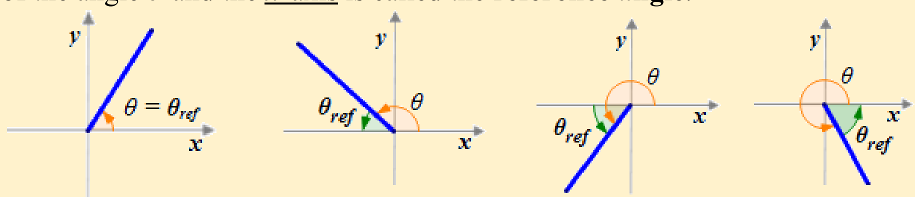


Figure 3.5

Assume that point (a, b) lies on the terminal side of acute angle α. By *Definition 2.2*, the values of trigonometric functions of angles with terminals containing points $(-a, b), (-a, -b)$, and $(a, -b)$ are the same as the values of corresponding functions of the angle α, except for their signs.

Therefore, to find the value of a trigonometric function of any angle θ, it is enough to evaluate this function at the corresponding acute angle θ_{ref}, called the **reference angle**, and apply the sign appropriate to the quadrant of the terminal side of θ.

Definition 3.1 ► Let θ be an angle in standard position. The acute angle $\boldsymbol{\theta_{ref}}$ formed by the <u>terminal side</u> of the angle θ and the <u>x-axis</u> is called the **reference angle**.



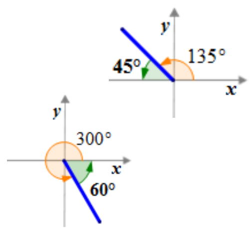
Attention: *Think of a **reference angle** as the smallest rotation of the terminal arm required to line it up with the **x-axis**.*

Example 4 ► **Finding the Reference Angle**

Find the **reference angle** for each of the given angles.

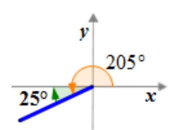
a. 40° **b.** 135° **c.** 210° **d.** 300°

Solution ►



a. Since $\mathbf{40°} \in QI$, this is already the reference angle.

b. Since $135° \in QII$, the reference angle equals $180° - 135° = \mathbf{45°}$.

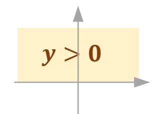


c. Since $205° \in QIII$, the reference angle equals $205° - 180° = \mathbf{25°}$.

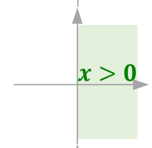
d. Since $300° \in QIV$, the reference angle equals $360° - 300° = \mathbf{60°}$.

CAST Rule

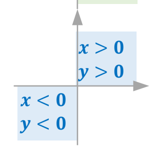
Using the x, y, r definition of trigonometric functions, we can determine and summarize the signs of those functions in each of the quadrants.



Since $\sin\theta = \frac{y}{r}$ and r is positive, then the sign of the sine ratio is the same as the sign of the y-value. This means that the values of sine are positive only in quadrants where y is positive, thus in QI and QII.



Since $\cos\theta = \frac{x}{r}$ and r is positive, then the sign of the cosine ratio is the same as the sign of the x-value. This means that the values of cosine are positive only in quadrants where x is positive, thus in QI and QIV.



Since $\tan\theta = \frac{y}{x}$, then the values of the tangent ratio are positive only in quadrants where both x and y have the same signs, thus in QI and QIII.

Table 3.2 Signs of Trigonometric Functions in Quadrants

function \ $\theta \in$	QI	QII	QIII	QIV
$\sin\theta$	$+$	$+$	$-$	$-$
$\cos\theta$	$+$	$-$	$-$	$+$
$\tan\theta$	$+$	$-$	$+$	$-$

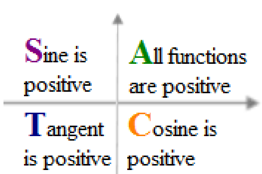


Figure 3.6

Since we will be making frequent decisions about signs of trigonometric function values, it is convenient to have an acronym helping us memorizing these signs in different quadrants. The first letters of the names of functions that are positive in particular quadrants, starting from the fourth quadrant and going counterclockwise, spells **CAST**, which is very helpful when working with trigonometric functions of any angles.

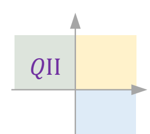
Example 5 ► **Identifying the Quadrant of an Angle**

Identify the quadrant or quadrants for each angle satisfying the given conditions.

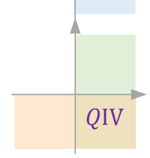
a. $\sin\theta > 0;\ \tan\theta < 0$ **b.** $\cos\theta > 0;\ \sin\theta < 0$

Solution ►

a. Using **CAST**, we have $\sin\theta > 0$ in QI(**A**ll) and QII(**S**ine) and $\tan\theta < 0$ in QII and QIV. Therefore both conditions are met only in **quadrant II**.



b. $\cos\theta > 0$ in QI(**A**ll) and QIV(**C**osine) and $\sin\theta < 0$ in QIII and QIV. Therefore both conditions are met only in **quadrant IV**.



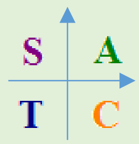
Example 6 ► **Identifying Signs of Trigonometric Functions of Any Angle**

Using the **CAST rule**, identify the sign of each function value.

a. $\cos 150°$ **b.** $\tan 225°$

Solution ► **a.** Since $150° \in QII$ and cosine is negative in QII, then $\cos 150°$ is **negative**.

b. Since $225° \in QIII$ and tangent is positive in $QIII$, then $\tan 225°$ is **positive**.



To find the exact value of a trigonometric function $\boldsymbol{T}$ of an angle $\boldsymbol{\theta}$ with the reference angle $\boldsymbol{\theta_{ref}}$ being a special angle, we follow the rule:

$$\boldsymbol{T(\theta) = \pm T(\theta_{ref})},$$

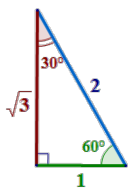
where the final sign is determined according to the quadrant of angle θ and the **CAST** rule.

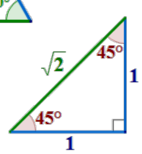
Example 7 ► **Finding Exact Function Values Using Reference Angles**

Find the exact values of the following expressions.

a. $\sin 240°$ **b.** $\cos 315°$

Solution ►





a. The reference angle of $240°$ is $240° - 180° = 60°$. Since $240° \in QIII$ and sine in the third quadrant is **negative**, we have

$$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}$$

b. The reference angle of $315°$ is $360° - 315° = 45°$. Since $315° \in QIV$ and cosine in the fourth quadrant is **positive**, we have

$$\cos 315° = \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Finding Special Angles in Various Quadrants when Given Trigonometric Function Value

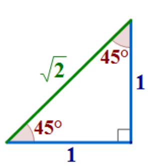
Now that it has been shown how to find exact values of trigonometric functions of angles that have a reference angle of one of the special angles (**30°**, **45°**, or **60°**), we can work at reversing this process. Familiarity with values of trigonometric functions of the special angles, in combination with the ideas of reference angles and quadrantal sign analysis, should help us in solving equations of the type $\boldsymbol{T(\theta) = exact\ value}$, where $\boldsymbol{T}$ represents any trigonometric function.

Example 8 ► **Finding Angles with a Given Exact Function Value, in Various Quadrants**

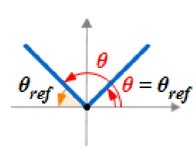
Find all angles θ satisfying the following conditions.

a. $\sin\theta = \frac{\sqrt{2}}{2};\ \theta \in [0°, 180°)$ **b.** $\cos\theta = -\frac{1}{2};\ \theta \in [0°, 360°)$

Solution ►



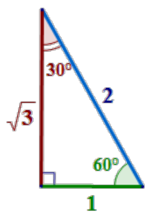
a. Refering to the half of a square triangle, we recognize that $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ represents the ratio of sine of $45°$. Thus, the reference angle $\theta_{ref} = 45°$. Moreover, we are searching for an angle θ from the interval $[0°, 180°)$ and we know that $\sin\theta > 0$. Therefore, θ must lie in the first or second quadrant and have the reference angle of $45°$. Each quadrant gives us one solution, as shown in the figure on the right.



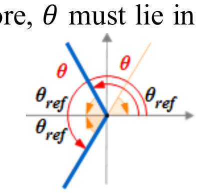
If θ is in the first quadrant, then $\theta = \theta_{ref} = 45°$.
If θ is in the second quadrant, then $\theta = 180° - 45° = 135°$.

So the solution set of the above problem is $\{\mathbf{45°}, \mathbf{135°}\}$.

here we can disregard the sign of the given value as we are interested in the reference angle only



b. Refering to the half of an equlateral triangle, we recognize that $\frac{1}{2}$ represents the ratio of cosine of 60°. Thus, the reference angle $\theta_{ref} = 60°$. We are searching for an angle θ from the interval $[0°, 360°)$ and we know that $\cos\theta < 0$. Therefore, θ must lie in the second or third quadrant and have the reference angle of 60°.



If θ is in the second quadrant, then $\theta = 180° - 60° = 120°$.
If θ is in the third quadrant, then $\theta = 180° + 60° = 240°$.

So the solution set of the above problem is $\{\mathbf{120°}, \mathbf{240°}\}$.

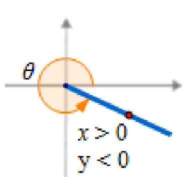
Finding Other Trigonometric Function Values

Example 9 ► **Finding Other Function Values Using a Known Value, Quadrant Analysis, and the $\boldsymbol{x, y, r}$ Definition of Trigonometric Ratios**

Find values of the remaining primary trigonometric functions of the angle satisfying the given conditions.

a. $\sin\theta = -\frac{7}{13}$; $\theta \in QIV$ **b.** $\tan\theta = \frac{15}{8}$; $\theta \in$ QIII

Solution ►



a. We know that $\sin\theta = -\frac{7}{13} = \frac{y}{r}$. Hence, the terminal side of angle $\theta \in QIV$ contains a point $P(x, y)$ satisfying the condition $\frac{y}{r} = -\frac{7}{13}$. Since r must be positive, we will assign $y = -7$ and $r = 13$, to model the situation. Using the Pythagorean equation and the fact that the x-coordinate of any point in the fourth quadrant is positive, we determine the corresponding x-value to be

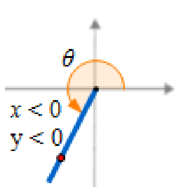
$$x = \sqrt{r^2 - y^2} = \sqrt{13^2 - (-7)^2} = \sqrt{169 - 49} = \sqrt{120} = 2\sqrt{30}.$$

Now, we are ready to state the remaining function values of angle θ:

$$\cos\theta = \frac{x}{r} = \frac{\mathbf{2\sqrt{30}}}{\mathbf{13}}$$

and

$$\tan\theta = \frac{y}{x} = \frac{-7}{2\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = \frac{\mathbf{-7\sqrt{30}}}{\mathbf{60}}.$$



b. We know that $\tan\theta = \frac{15}{8} = \frac{y}{x}$. Similarly as above, we would like to determine x, y, and r values that would model the situation. Since angle $\theta \in Q$III, both x and y values must be negative. So we assign $y = -15$ and $x = -8$. Therefore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-15)^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Now, we are ready to state the remaining function values of angle θ:

$$\sin\theta = \frac{y}{r} = \frac{\mathbf{-15}}{\mathbf{17}}$$

and

$$\cos\theta = \frac{x}{r} = \frac{\mathbf{-8}}{\mathbf{17}}.$$

T.3 Exercises

Use a calculator to ***approximate*** *each value to* ***four*** *decimal places.*

1. $\sin 36°52'\,05''$ **2.** $\tan 57.125°$ **3.** $\cos 204°25'$

Give the ***exact*** *function value,* ***without*** *the aid of a calculator. Rationalize denominators when applicable.*

4. $\cos 30°$ **5.** $\sin 45°$ **6.** $\tan 60°$ **7.** $\sin 60°$

8. $\tan 30°$ **9.** $\cos 60°$ **10.** $\sin 30°$ **11.** $\tan 45°$

Give the equivalent expression using the ***cofunction*** *relationship.*

12. $\cos 50°$ **13.** $\sin 22.5°$ **14.** $\sin 10°$

For each angle, find the ***reference angle****.*

15. $98°$ **16.** $212°$ **17.** $13°$ **18.** $297°$ **19.** $186°$

Identify the ***quadrant*** *or* ***quadrants*** *for each angle satisfying the given conditions.*

20. $\cos\alpha > 0$ **21.** $\sin\beta < 0$ **22.** $\tan\gamma > 0$

23. $\sin\theta > 0$; $\cos\theta < 0$ **24.** $\cos\alpha < 0$; $\tan\alpha > 0$ **25.** $\sin\alpha < 0$; $\tan\alpha < 0$

Identify the ***sign*** *of each function value by quadrantal analysis.*

26. $\cos 74°$ **27.** $\sin 245°$ **28.** $\tan 129°$ **29.** $\sin 183°$

30. $\tan 298°$ **31.** $\cos 317°$ **32.** $\sin 285°$ **33.** $\tan 215°$

Using reference angles, quadrantal analysis, and special triangles, find the ***exact values*** *of the expressions. Rationalize denominators when applicable.*

34. $\cos 225°$ **35.** $\sin 120°$ **36.** $\tan 150°$ **37.** $\sin 150°$

38. $\tan 240°$ **39.** $\cos 210°$ **40.** $\sin 330°$ **41.** $\tan 225°$

Find all values of $\theta \in [0°, 360°)$ *satisfying the given condition.*

42. $\sin\theta = -\frac{1}{2}$ **43.** $\cos\theta = \frac{1}{2}$ **44.** $\tan\theta = -1$ **45.** $\sin\theta = \frac{\sqrt{3}}{2}$

46. $\tan\theta = \sqrt{3}$ **47.** $\cos\theta = -\frac{\sqrt{2}}{2}$ **48.** $\sin\theta = 0$ **49.** $\tan\theta = -\frac{\sqrt{3}}{3}$

Find values of the remaining primary trigonometric functions of the angle satisfying the given conditions.

50. $\sin\theta = \frac{\sqrt{5}}{7};\ \theta \in Q\mathrm{II}$ **51.** $\cos\alpha = \frac{3}{5};\ \alpha \in Q\mathrm{IV}$ **52.** $\tan\beta = \sqrt{3};\ \beta \in Q\mathrm{III}$

T4 Applications of Right Angle Trigonometry

Solving Right Triangles

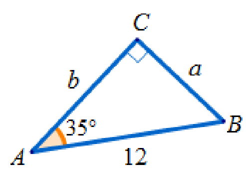
Geometry of right triangles has many applications in the real world. It is often used by carpenters, surveyors, engineers, navigators, scientists, astronomers, etc. Since many application problems can be modelled by a right triangle and trigonometric ratios allow us to find different parts of a right triangle, it is essential that we learn how to apply trigonometry to solve such triangles first.

Definition 4.1 ► To **solve a triangle** means to find the measures of all the unknown **sides** and **angles** of the triangle.

Example 1 ► **Solving a Right Triangle Given an Angle and a Side**

Given the information, solve triangle ABC, assuming that $\angle C = 90°$.

a.



b. $\angle B = 11.4°,\ b = 6\ cm$

Solution ► **a.** To find the length a, we want to relate it to the given length of 12 and the angle of 35°. Since a is opposite angle 35° and 12 is the length of the hypotenuse, we can use the ratio of sine:

$$\frac{a}{12} = \sin 35°$$

Then, after multiplying by 12, we have

$$a = 12 \sin 35° \simeq \mathbf{6.9}$$

round **lengths** to **one decimal place**

Attention: To be more accurate, if possible, **use the given data** rather than the previously calculated ones, which are most likely already rounded.

Since we already have the value of a, the length b can be determined in two ways: by applying the Pythagorean Theorem, or by using the cosine ratio. For better accuracy, we will apply the cosine ratio:

$$\frac{b}{12} = \cos 35°$$

which gives

$$b = 12 \cos 35° \simeq \mathbf{9.8}$$

Finally, since the two acute angles are complementary, $\angle B = 90° - 35° = \mathbf{55}°$.

We have found the three missing measurements, $a \simeq 6.9$, $b \simeq 9.8$, and $\angle B = 55°$, so the triangle is solved.

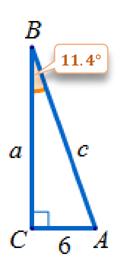


Figure 1

b. To visualize the situation, let's sketch a right triangle with $\angle B = 11.4°$ and $b = 6$ (see *Figure 1*). To find side a, we we would like to set up an equation that relates 6, a, and 11.4°. Since $b = 6$ is the opposite and a is the adjacent with respect to $\angle B = 11.4°$, we will use the ratio of tangent:

$$\tan 11.4° = \frac{6}{a}$$

To solve for a, we may want to multiply both sides of the equation by a and divide by $\tan 11.4°$. Observe that this will cause a and $\tan 11.4°$ to interchange (swap) their positions. So, we obtain

$$a = \frac{6}{\tan 11.4°} \simeq \mathbf{29.8}$$

To find side c, we will set up an equation that relates 6, c, and 11.4°. Since $b = 6$ is the opposite to $\angle B = 11.4°$ and c is the hypothenuse, the ratio of sine applies. So, we have

$$\sin 11.4° = \frac{6}{c}$$

Similarly as before, to solve for c, we can simply interchange the position of $\sin 11.4°$ and c to obtain

$$c = \frac{6}{\sin 11.4°} \simeq \mathbf{30.4}$$

Finally, $\angle A = 90° - 11.4° = \mathbf{78.6°}$, which completes the solution.

In summary, $\angle A = 78.6°$, $a \simeq 29.8$, and $c \simeq 30.4$.

Observation: Notice that after approximated length a was found, we could have used the Pythagorean Theorem to find length c. However, this could decrease the accuracy of the result. For this reason, it is advised that we use the given rather than approximated data, if possible.

Finding an Angle Given a Trigonometric Function Value

So far we have been evaluating trigonometric functions for a given angle. Now, what if we wish to reverse this process and try to recover an angle that corresponds to a given trigonometric function value?

Example 2 ► **Finding an Angle Given a Trigonometric Function Value**

Find an angle θ, satisfying the given equation. *Round to one decimal place, if needed.*

a. $\sin\theta = 0.7508$ **b.** $\cos\theta = -0.5$

Solution ► **a.** Since 0.7508 is not a special value, we will not be able to find θ by relating the equation to a special triangle as we did in *Section T3, Example 8*. This time, we will need to rely on a calculator. To find θ, we want to "undo" the sine. The function that can "undo" the sine is called **arcsine**, or **inverse sine**, and it is often abbreviated by $\mathbf{sin^{-1}}$. By applying the $\sin^{-1}$ to both sides of the equation

$$\sin\theta = 0.7508,$$

we have

$$\cancel{\sin^{-1}}(\cancel{\sin}\,\theta) = \sin^{-1}(0.7508)$$

Since $\sin^{-1}$ "undoes" the sine function, we obtain

$$\theta = \sin^{-1} 0.7508 \simeq \mathbf{48.7°}$$

round **angles** to **one decimal place**

On most calculators, to find this value, we follow the sequence of keys:

2nd or INV or Shift , SIN , 0.7508, ENTER or =

b. In this example, the absolute value of cosine is a special value. This means that θ can be found by referring to the **golden triangle** properties and the **CAST** rule of signs as in *Section T3, Example 8b*. The other way of finding θ is via a calculator

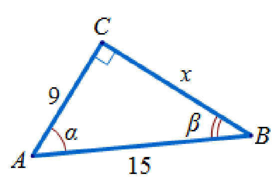
$$\theta = \cos^{-1}(-0.5) = \mathbf{120°}$$

Note: *Calculators are programed to return*
$\mathbf{sin^{-1}}$ *and* $\mathbf{tan^{-1}}$ *as angles from the interval* $[\mathbf{-90°}, \mathbf{90°}]$ *and*
$\mathbf{cos^{-1}}$ *as angles from the interval* $[\mathbf{0°}, \mathbf{180°}]$.
That implies that when looking for an obtuse angle, it is easier to work with $\mathbf{cos^{-1}}$, *if possible, as our calculator will return the actual angle. When using* $\mathbf{sin^{-1}}$ *or* $\mathbf{tan^{-1}}$, *we might need to search for a corresponding angle in the second quadrant on our own.*

More on Solving Right Triangles

Example 3 ► **Solving a Right Triangle Given Two Sides**

Solve the triangle.



Solution ► Since $\triangle ABC$ is a right triangle, to find the length x, we can use the Pythagorean Theorem.

$$x^2 + 9^2 = 15^2$$

so

$$\boldsymbol{x} = \sqrt{225 - 81} = \sqrt{144} = \mathbf{12}$$

To find the angle α, we can relate either $x = 12$, 9, and α, or 12, 15, and α. We will use the second triple and the ratio of sine. Thus, we have

$$\sin\alpha = \frac{12}{15},$$

therefore

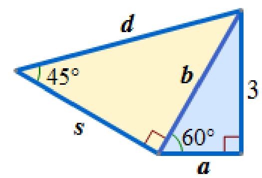
$$\boldsymbol{\alpha} = \sin^{-1}\frac{12}{15} \simeq \mathbf{53.1°}$$

Finally, $\boldsymbol{\beta} = 90° - \alpha \simeq 90° - 53.1° = \mathbf{36.9°}$.

In summary, $\alpha = 53.1°$, $\beta \simeq 36.9°$, and $x = 12$.

Example 4 ► **Using Relationships Between Sides of Special Triangles**

Find the **exact** value of each unknown in the figure.



Solution ► First, consider the blue right triangle. Since one of the acute angles is 60°, the other must be 30°. Thus the blue triangle represents half of an equilateral triangle with the side $\boldsymbol{b}$ and the height of 3 units. Using the relation $h = a\sqrt{3}$ between the height h and half a side a of an equilateral triangle, we obtain

$$\boldsymbol{a}\sqrt{3} = 3,$$

which gives us $\boldsymbol{a} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \boldsymbol{\sqrt{3}}$. Consequently, $\boldsymbol{b} = 2\boldsymbol{a} = \boldsymbol{2\sqrt{3}}$.

Now, considering the yellow right triangle, we observe that both acute angles are equal to 45° and therefore the triangle represents half of a square with the side $\boldsymbol{s} = \boldsymbol{b} = \boldsymbol{2\sqrt{3}}$.

Finally, using the relation between the diagonal and a side of a square, we have

$$\boldsymbol{d} = \boldsymbol{s}\sqrt{2} = 2\sqrt{3}\sqrt{2} = \boldsymbol{2\sqrt{6}}.$$

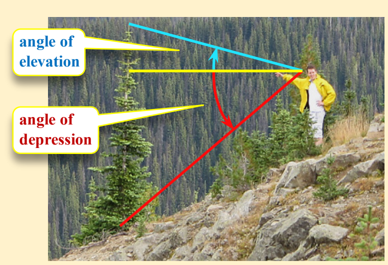
Angles of Elevation or Depression in Applications

The method of solving right triangles is widely adopted in solving many applied problems. One of the critical steps in the solution process is sketching a triangle that models the situation, and labeling the parts of this triangle correctly.

In trigonometry, many applied problems refer to angles of **elevation** or **depression**, or include some navigation terminology, such as **direction** or **bearing**.

Definition 4.2 ► **Angle of elevation** (or **inclination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **above** the horizontal line.

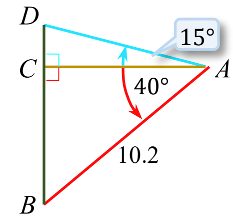
Angle of depression (or **declination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **below** the horizontal line.



Example 5 ► Applying Angles of Elevation or Depression

Find the height of the tree in the picture given next to *Definition 4.2*, assuming that the observer sees the top of the tree at an angle of elevation of 15°, the base of the tree at an angle of depression of 40°, and the distance from the base of the tree to the observer's eyes is 10.2 meters.

Solution ► First, let's draw a diagram to model the situation, label the vertices, and place the given data. Then, observe that the height of the tree BD can be obtained as the sum of distances BC and CD.



BC can be found from $\triangle ABC$, by using the ratio of sine of 40°.
From the equation

$$\frac{BC}{10.2} = \sin 40°,$$

we have

$$BC = 10.2 \sin 40° \simeq \mathbf{6.56}$$

To calculate the length DC, we would need to have another piece of information about $\triangle ADC$ first. Notice that the side AC is common for the two triangles. This means that we can find it from $\triangle ABC$, and use it for $\triangle ADC$ in subsequent calculations.
From the equation

$$\frac{CA}{10.2} = \cos 40°,$$

we have

$$CA = 10.2 \cos 40° \simeq 7.8137$$

since we use this result in further calculations, four decimals of accuracy is advised

Now, employing tangent of 15° in $\triangle ADC$, we have

$$\frac{CD}{7.8137} = \tan 15°$$

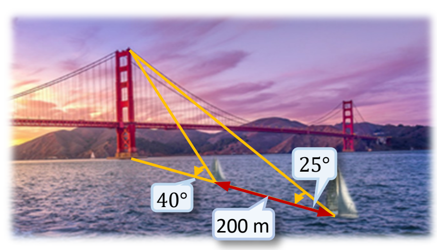
which gives us

$$CD = 7.8137 \cdot \tan 15° \simeq \mathbf{2.09}$$

Hence the height of the tree is $BC \simeq 6.56 + 2.09 = 8.65 \simeq \mathbf{8.7}$ meters.

Example 6 ► Using Two Angles of Elevation at a Given Distance to Determine the Height

When Ricky and Sonia were sailing their boat on a river, they observed the tip of a bridge tower at a 25° elevation angle. After sailing 200 meters closer to the tower, they noticed that the tip of the tower was visible at 40° elevation angle. Approximate the height of the tower to the nearest meter.



Solution ► To model the situation, let us draw the diagram and adopt the notation as in *Figure 2*. We look for height $\boldsymbol{h}$, which is a part of the two right triangles $\triangle ABC$ and $\triangle DBC$.

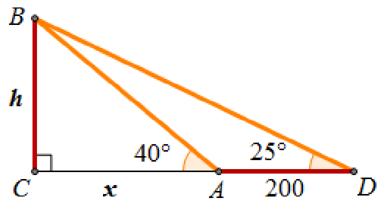


Figure 2

Since trigonometric ratios involve two sides of a triangle, and we already have length AD, a part of the side CD, it is reasonable to introduce another unknown, call it $\boldsymbol{x}$, to represent the remaining part CA. Then, applying the ratio of tangent to each of the right triangles, we produce the following system of equations:

$$\begin{cases} \dfrac{h}{x} = \tan 40° \\ \dfrac{h}{x+200} = \tan 25° \end{cases}$$

To solve the above system, we first solve each equation for h

$$\begin{cases} h \simeq 0.8391x \\ h \simeq 0.4663(x+200)\,, \end{cases}$$

substitute to the top equation

and then by equating the right sides, we obtain

$$0.8391x = 0.4663(x+200)$$

$$0.8391x - 0.4663x = 93.26$$

$$0.3728x = 93.26$$

$$x = \frac{93.26}{0.3728} \simeq 250.16$$

Therefore, $\boldsymbol{h} \simeq 0.8391 \cdot 250.16 \simeq \mathbf{210}\ \boldsymbol{m}$.

The height of the tower is approximately **210** meters.

Direction or Bearing in Applications

A large group of applied problems in trigonometry refer to **direction** or **bearing** to describe the location of an object, usually a plane or a ship. The idea comes from following the behaviour of a compass. The magnetic needle in a compass points North. Therefore, the location of an object is described as a clockwise deviation from the SOUTH-NORTH line.

There are two main ways of describing directions:

- One way is by stating the angle θ that starts from the North and opens clockwise until the line of sight of an object. For example, we can say that the point $\boldsymbol{B}$ is seen in the **direction** of 108° from the point $\boldsymbol{A}$, as in *Figure 2a*.
- Another way is by stating the acute angle formed by the South-North line and the line of sight. Such an angle starts either from the North (**N**) or the South (**S**) and opens either towards the East (**E**) or the West (**W**). For instance, the position of the point $\boldsymbol{B}$ in *Figure 2b* would be described as being at a **bearing** of **S72°E** (*read:* South 72° towards the East) from the point $\boldsymbol{A}$.

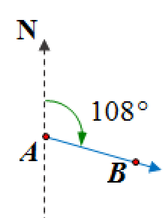


Figure 2a

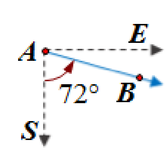


Figure 2b

This, for example, means that:

the direction of $195°$ can be seen as the bearing **S15°W** and the direction of $290°$ means the same as **N70°W**.

Example 7 ► **Using Direction in Applications Involving Navigation**

An airplane flying at a speed of 400 mi/hr flies from a point A in the direction of $153°$ for one hour and then flies in the direction of $63°$ for another hour.

a. How long will it take the plane to get back to the point A?

b. What is the direction that the plane needs to fly in order to get back to the point A?

Solution ►

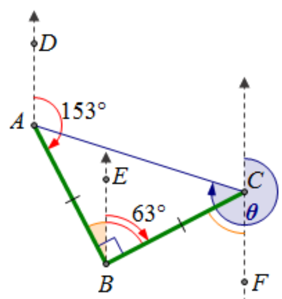


Figure 3

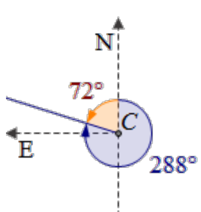
a. First, let's draw a diagram modeling the situation. Assume the notation as in *Figure 3*. Since the plane flies at $153°$ and the South-North lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ are parallel, by the property of interior angles, we have $\angle ABE = 180° - 153° = 27°$. This in turn gives us $\angle ABC = \angle ABE + \angle EBC = 27° + 63° = 90°$. So the ΔABC is right angled with $\angle B = 90°$ and the two legs of length $AB = BC = 400\ mi$. This means that the ΔABC is in fact a special triangle of the type $45° - 45° - 90°$.

Therefore $AC = AB\sqrt{2} = 400\sqrt{2} \simeq 565.7\ mi$.

Now, solving the well-known motion formula $R \cdot T = D$ for the time T, we have

$$T = \frac{D}{R} \simeq \frac{\cancel{400}\sqrt{2}}{\cancel{400}} = \sqrt{2} \simeq 1.4142\ hr \simeq \mathbf{1\ \textit{hr}\ 25\ \textit{min}}$$

Thus, it will take the plane approximately 1 hour and 25 minutes to return to the starting point A.



b. To direct the plane back to the starting point, we need to find angle θ, marked in blue, rotating clockwise from the North to the ray $\overrightarrow{CA}$. By the property of alternating angles, we know that $\angle FCB = 63°$. We also know that $\angle BCA = 45°$, as $\angle ABD$ is the "half of a square" special triangle. Therefore,

$$\theta = 180° + 63° + 45° = \mathbf{288°}.$$

Thus, to get back to the point A, the plane should fly in the direction of $288°$. Notice that this direction can also be stated as **N72°W**.

T.4 Exercises

Using a calculator, find an angle θ satisfying the given equation. Leave your answer in decimal degrees rounded to the nearest tenth of a degree if needed.

1. $\sin\theta = 0.7906$

2. $\cos\theta = 0.7906$

3. $\tan\theta = 2.5302$

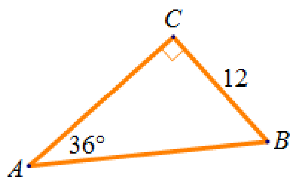
4. $\cos\theta = -0.75$

5. $\tan\theta = \sqrt{3}$

6. $\sin\theta = \frac{3}{4}$

Given the data, solve each triangle ABC with $\angle C = 90°$.

7.



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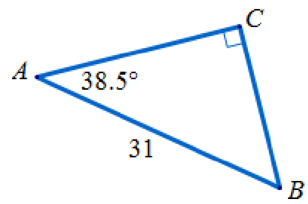

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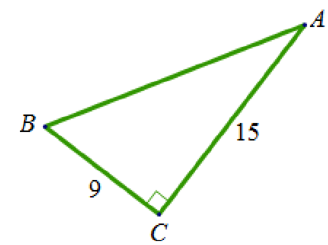


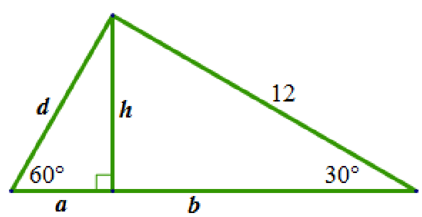
10. $\angle A = 42°,\ b = 17$

11. $a = 9.45,\ c = 9.81$

12. $\angle B = 63°12',\ b = 19.1$

Find the ***exact*** *value of each unknown in the figure.*

13.



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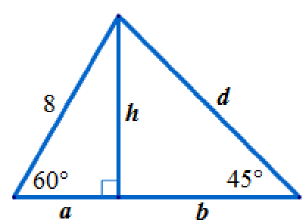

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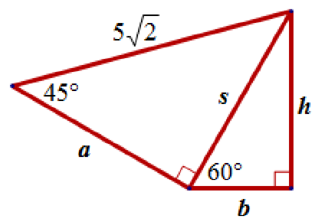
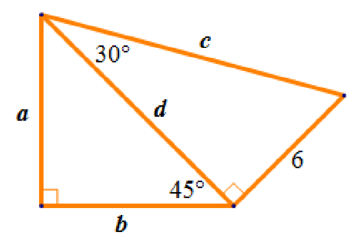


16.



17. A circle of radius 8 centimeters is inscribed in a regular hexagon. Find the exact perimeter of the hexagon.

18. A regular pentagon is inscribed in a circle with 10 meters diameter. To the nearest centimeter, find the perimeter of the pentagon.

19. A 25 meters long supporting rope connects the top of a 23 meters high mast of a sailboat with the deck of the boat. To the nearest degree, find the angle between the rope and the mast.

20. A 16 meters long guy wire is attached to the top of a utility pole. The angle between the guy wire and the ground is 54°. To the nearest tenth of a meter, how tall is the pole?

21. From the top of a 52 m high cliff, the angle of depression to a boat is $4°15'$. To the nearest meter, how far is the boat from the base of the cliff?

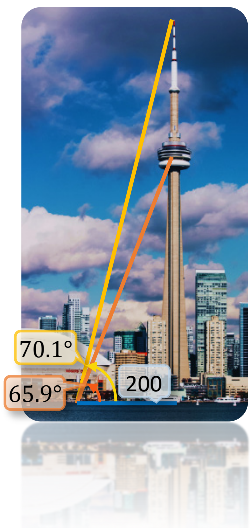
22. A spotlight reflector mounted to a ceiling of a 3.5 meters high hall is directed onto a piece of art displayed 1.5 meters above the floor. To the nearest degree, what angle of depression should be used to direct the light onto the piece of art if the reflector is 3.8 meters away from it?

23. To determine the height of the Eiffel Tower, a 1.8 meters tall tourist standing 50 meters from the center of the base of the tower measures the angle of elevation to the top of the tower to be 81°. Using this information, determine the height of the Eiffel Tower to the nearest meter.

24. To the nearest meter, find the height of an isosceles triangle with 25.2 meters long base and $35°40'$ angle by the base.

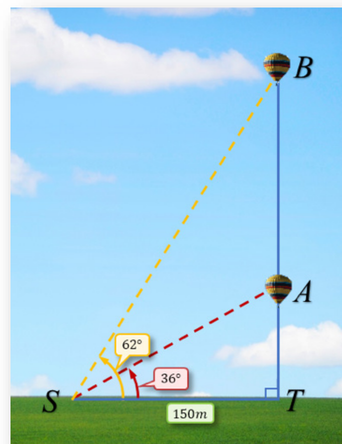
25. A plane flies 700 kilometers at a bearing of **N**56°**E** and then 850 kilometers at a bearing of **S**34°**E**. How far and in what direction is the plane from the starting point? Round the answers to the nearest kilometer and the nearest degree.

26. A plane flies at 420 km/h for 30 minutes in the direction of 142°. Then, it changes its direction to 232° and flies for 45 minutes. To the nearest kilometer, how far is the plane at that time from the starting point? To the nearest degree, in what direction should the plane fly to come back to the starting point?

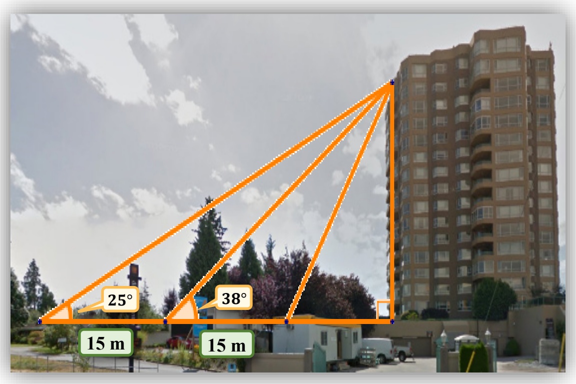


27. Standing 200 meters from the base of the CN Tower, a tourist sees the pinnacle of the tower at 70.1° elevation angle. The tower has a built-in restaurant as in the accompanying picture. The tourist can see this restaurant at 65.9° elevation angle. To the nearest meter, how tall is the CN Tower, including its pinnacle? How high above the ground is the restaurant?

28. A hot air balloon rises vertically at a constant rate, as shown in the accompanying figure. A hundred fifty meters away from the balloon's lift-off place, a spectator notices the balloon at 36° angle of elevation. A minute later, the spectator records that the angle of elevation of the balloon is 62°. To the nearest meter per second, determine the rate of the balloon.



29. Two people observe an eagle nest on a tall tree in a park. One person sees the nest at the angle of elevation of 60° while the other at the angle of elevation of 75°. If the people are 25 meters apart from each other and the tree is between them, determine the altitude at which the nest is situated. Round your answer to the nearest tenth of a meter.



30. A person approaching a tall building records the angle of elevation to the top of the building to be 32°. Fifteen meters closer to the building, this angle becomes 40°. To the nearest meter, how tall is the building? What would the angle of elevation be in another 15 meters?

31. Suppose that the length of the shadow of The Palace of Culture and Science in Warsaw increases by 15.5 meters when the angle of elevation of the sun decreases from 48° to 46°. Based on this information, determine the height of the palace. Round your answer to the nearest meter.

32. A police officer observes a road from 150 meters distance as in the accompanying diagram. A car moving on the road covers the distance between two chosen by the officer points, A and B, in 1.5 seconds. If the angles between the lines of sight to points A and B and the line perpendicular to the observed road are respectively 34.1° and 20.3°, what was the speed of the car? State your answer in kilometers per hour rounded up to one decimal.

T5 The Laws of Sines and Cosines and Their Applications

The concepts of solving triangles developed in *Section T4* can be extended to all triangles. A triangle that is not right-angled is called an **oblique triangle**. Many application problems involve solving oblique triangles, yet we can not use the SOH-CAH-TOA rules when solving those triangles since **SOH-CAH-TOA** definitions **apply only to right triangles!** So, we need to search for other rules that will allow us to solve oblique triangles.

The Sine Law

Observe that all triangles can be classified with respect to the size of their angles as **acute** (with all acute angles), **right** (with one right angle), or **obtuse** (with one obtuse angle). Therefore, oblique triangles are either acute or obtuse.

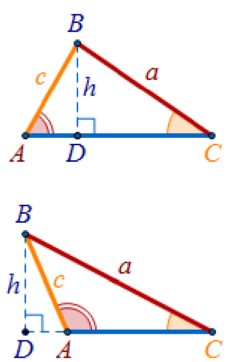


Figure 1

Let's consider both cases of an oblique $\triangle ABC$, as in *Figure 1*. In each case, let's drop the height h from vertex B onto the line $\overleftrightarrow{AC}$, meeting this line at point D. This way, we obtain two more right triangles, $\triangle ADB$ with hypotenuse c, and $\triangle BDC$ with hypotenuse a. Applying the ratio of sine to both of these triangles, we have:

$$\sin\angle A = \frac{h}{c}, \text{ so } h = c\sin\angle A$$

and

$$\sin\angle C = \frac{h}{a}, \text{ so } h = a\sin\angle C.$$

Thus,

$$a\sin\angle C = c\sin\angle A,$$

and we obtain

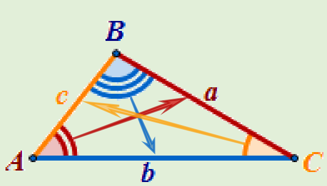
$$\frac{\boldsymbol{a}}{\boldsymbol{\sin\angle A}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin\angle C}}.$$

Similarly, by dropping heights from the other two vertices, we can show that

$$\frac{\boldsymbol{a}}{\boldsymbol{\sin\angle A}} = \frac{\boldsymbol{b}}{\boldsymbol{\sin\angle B}} \text{ and } \frac{\boldsymbol{b}}{\boldsymbol{\sin\angle B}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin\angle C}}.$$

This result is known as the Law of Sines.

The Sine Law ► In any triangle ABC, the lengths of the **sides are proportional to the sines of the opposite angles**. This fact can be expressed in any of the following, equivalent forms:



$$\frac{\boldsymbol{a}}{\boldsymbol{b}} = \frac{\boldsymbol{\sin\angle A}}{\boldsymbol{\sin\angle B}}, \quad \frac{\boldsymbol{b}}{\boldsymbol{c}} = \frac{\boldsymbol{\sin\angle B}}{\boldsymbol{\sin\angle C}}, \quad \frac{\boldsymbol{c}}{\boldsymbol{a}} = \frac{\boldsymbol{\sin\angle C}}{\boldsymbol{\sin\angle A}}$$

or

$$\frac{\boldsymbol{a}}{\boldsymbol{\sin\angle A}} = \frac{\boldsymbol{b}}{\boldsymbol{\sin\angle B}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin\angle C}}$$

or

$$\frac{\boldsymbol{\sin\angle A}}{\boldsymbol{a}} = \frac{\boldsymbol{\sin\angle B}}{\boldsymbol{b}} = \frac{\boldsymbol{\sin\angle C}}{\boldsymbol{c}}$$

Observation: *As with any other proportion, to solve for one variable, we need to know the three remaining values. Notice that when using the Sine Law proportions, the three known values must include **one pair of opposite data**: a side and its opposite angle.*

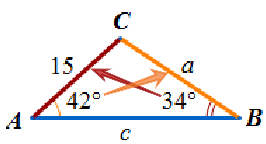
Example 1 ► **Solving Oblique Triangles with the Aid of The Sine Law**

Given the information, solve each triangle ABC.

a. $\angle A = 42°,\ \angle B = 34°,\ b = 15$ **b.** $\angle A = 35°,\ a = 12,\ b = 9$

Solution ►

a. First, we will sketch a triangle ABC that models the given data. Since the sum of angles in any triangle equals 180°, we have



$$\angle \boldsymbol{C} = 180° - 42° - 34° = \mathbf{104°}.$$

Then, to find length a, we will use the pair $(a, \angle A)$ of opposite data, side a and $\angle A$, and the given pair $(b, \angle B)$. From the Sine Law proportion, we have

$$\frac{a}{\sin 42°} = \frac{15}{\sin 34°},$$

for easier calculations, keep the unknown in the numerator

which gives

$$\boldsymbol{a} = \frac{15 \cdot \sin 42°}{\sin 34°} \simeq \mathbf{17.9}$$

To find length c, we will use the pair $(c, \angle C)$ and the given pair of opposite data $(b, \angle B)$. From the Sine Law proportion, we have

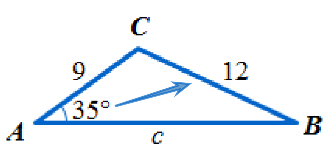
$$\frac{c}{\sin 104°} = \frac{15}{\sin 34°},$$

which gives

$$\boldsymbol{c} = \frac{15 \cdot \sin 104°}{\sin 34°} \simeq \mathbf{26}$$

So the triangle is solved.

b. As before, we will start by sketching a triangle ABC that models the given data. Using the pair $(9, \angle B)$ and the given pair of opposite data $(12, 35°)$, we can set up a proportion



$$\frac{\sin \angle B}{9} = \frac{\sin 35°}{12}.$$

Then, solving it for $\sin \angle B$, we have

$$\sin \angle B = \frac{9 \cdot \sin 35°}{12} \simeq 0.4302,$$

which, after applying the inverse sine function, gives us

$$\angle \boldsymbol{B} \simeq \mathbf{25.5°}$$

Now, we are ready to find $\angle \boldsymbol{C} = 180° - 35° - 25.5° = \mathbf{119.5°}$, and finally, from the proportion

$$\frac{c}{\sin 119.5°} = \frac{12}{\sin 35°},$$

we have

$$\boldsymbol{c} = \frac{12 \cdot \sin 119.5°}{\sin 35°} \simeq \mathbf{18.2}$$

Thus, the triangle is solved.

Ambiguous Case

Observe that the size of one angle and the length of two sides does not always determine a unique triangle. For example, there are two different triangles that can be constructed with $\angle A = 35°$, $a = 9$, $b = 12$.

Such a situation is called an **ambiguous case**. It occurs when the opposite side to the given angle is shorter than the other given side but long enough to complete the construction of an oblique triangle, as illustrated in *Figure 2*.

In application problems, if the given information does not determine a unique triangle, both possibilities should be considered in order for the solution to be complete.

On the other hand, not every set of data allows for the construction of a triangle. For example (see *Figure 3*), if $\angle A = 35°$, $a = 5$, $b = 12$, the side a is too short to complete a triangle, or if $a = 2$, $b = 3$, $c = 6$, the sum of lengths of a and b is smaller than the length of c, which makes impossible to construct a triangle fitting the data.

Note that in any triangle, the **sum of lengths of any two sides** is always **bigger than the length of the third side**.

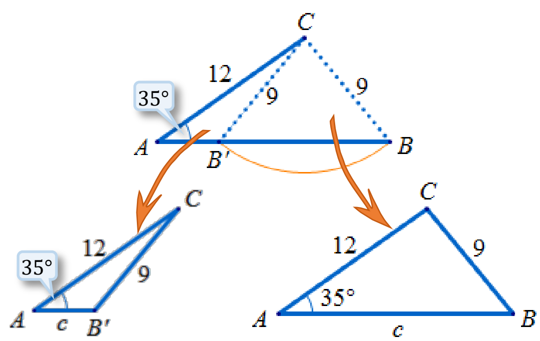


Figure 2

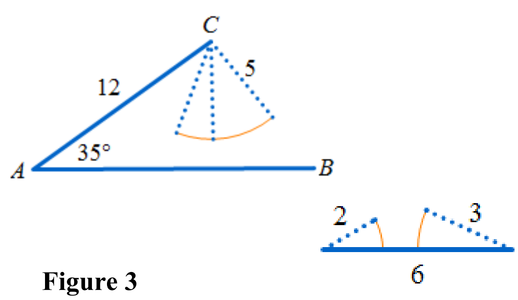
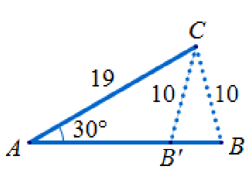


Figure 3

Example 2 ► **Using the Sine Law in an Ambiguous Case**

Solve triangle ABC, knowing that $\angle A = 30°$, $a = 10$, $b = 19$.

Solution ►



When sketching a diagram, we notice that there are two possible triangles, ΔABC and $\Delta AB'C$, complying with the given information. ΔABC can be solved in the same way as the triangle in *Example 1b*. In particular, one can calculate that in ΔABC, we have $\angle \boldsymbol{B} \simeq \mathbf{71.8}°$, $\angle \boldsymbol{C} \simeq \mathbf{78.2}°$, and $\boldsymbol{c} \simeq \mathbf{19.6}$.

Let's see how to solve $\Delta AB'C$ then. As before, to find $\angle B'$, we will use the proportion

$$\frac{\sin \angle B'}{19} = \frac{\sin 30°}{10},$$

which gives us $\sin \angle B' = \frac{19 \cdot \sin 30°}{10} = 0.95$. However, when applying the inverse sine function to the number 0.95, a calculator returns the approximate angle of 71.8°. Yet, we know that angle B' is obtuse. So, we should look for an angle in the second quadrant, with the reference angle of 71.8°. Therefore, $\angle \boldsymbol{B'} = 180° - 71.8° = \mathbf{108.2}°$.

Now, $\angle \boldsymbol{C} = 180° - 30° - 108.2° = \mathbf{41.8}°$

and finally, from the proportion

$$\frac{c}{\sin 41.8°} = \frac{10}{\sin 30°},$$

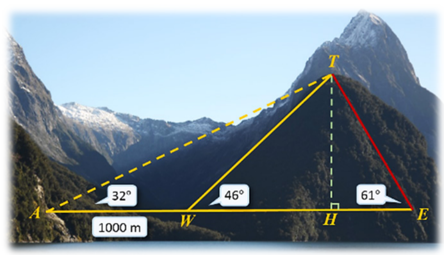
we have

$$\boldsymbol{c} = \frac{10 \cdot \sin 41.8°}{\sin 30°} \simeq \mathbf{13.3}$$

Thus, $\triangle AB'C$ is solved.

Example 3 ► **Solving an Application Problem Using the Sine Law**

Refer to the accompanying diagram. Round all your answers to the nearest tenth of a meter.



From a distance of 1000 meters from the west base of a mountain, the top of the mountain is visible at a 32° angle of elevation. At the west base, the average slope of the mountain is estimated to be 46°.

a. Determine the distance WT from the west base to the top of the mountain.
b. What is the distance ET from the east base to the top of the mountain, if the average slope of the mountain there is 61°?
c. Find the height HT of the mountain.

Solution ► **a.** To find distance WT, consider $\triangle AWT$. Observe that one can easily find the remaining angles of this triangle, as shown below:

$$\angle AWT = 180° - 46° = 134°$$ supplementary angles

and

$$\angle ATW = 180° - 32° - 134° = 14°$$ sum of angles in a ⊿

Therefore, applying the Law of Sines, we have

$$\frac{WT}{\sin 32°} = \frac{1000}{\sin 14°},$$

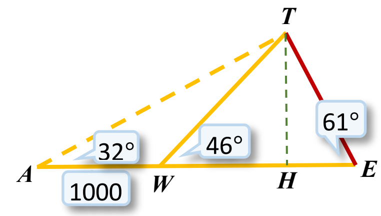
which gives

$$\boldsymbol{WT} = \frac{1000 \sin 32°}{\sin 14°} \simeq \mathbf{2190.5}\ \boldsymbol{m}.$$

b. To find distance ET, we can apply the Law of Sines to $\triangle WET$ using the pair $(2190.5, 61°)$. From the equation

$$\frac{ET}{\sin 46°} = \frac{2190.5}{\sin 61°},$$

we have



$$\boldsymbol{ET} = \frac{2190.5 \sin 46°}{\sin 61°} \simeq \mathbf{1801.6}\ \boldsymbol{m}.$$

c. To find the height HT of the mountain, we can use the right triangle WHT. By the definition of sine, we have

$$\frac{HT}{2190.5} = \sin 46°,$$

so $\boldsymbol{HT} = 2190.5 \sin 46° \simeq \mathbf{1575.7}\ \boldsymbol{m}$.

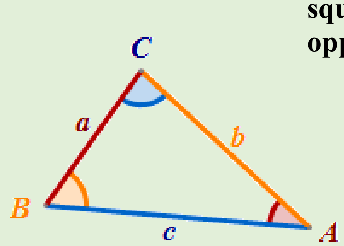
The Cosine Law

The above examples show how the **Sine Law** can help in solving oblique triangles when one **pair of opposite data** is given. However, the Sine Law is not enough to solve a triangle if the given information is

- the length of the **three sides** (but no angles), or
- the length of **two sides** and the **enclosed angle**.

Both of the above cases can be solved with the use of another property of a triangle, called the Cosine Law.

The Cosine Law ► In any triangle ABC, **the square of a side of a triangle is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the opposite angle**.



$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos \angle A}$$

$$\boldsymbol{b^2 = a^2 + c^2 - 2ac \cos \angle B}$$

$$\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos \angle C}$$

note the opposite side and angle

Observation: *If the angle of interest in any of the above equations is right, since* $\cos 90° = 0$, *the equation becomes Pythagorean. So the* ***Cosine Law*** *can be seen as an* ***extension of the Pythagorean Theorem****.*

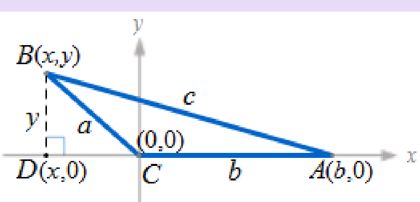


Figure 3

To derive this law, let's place an oblique triangle ABC in the system of coordinates so that vertex C is at the origin, side AC lies along the positive x-axis, and vertex B is above the x-axis, as in *Figure 3*.

Thus $C = (0,0)$ and $A = (b, 0)$. Suppose point B has coordinates (x, y). By *Definition 2.2*, we have

$$\cos \angle C = \frac{x}{a},$$

which gives us

$$x = a \cos \angle C\,.$$

Let $D = (x, 0)$ be the perpendicular projection of the vertex B onto the x-axis. After applying the Pythagorean equation to the right triangle ABD, with $\angle D = 90°$, we obtain

here we apply the Pythagorean equation to ΔBDC and replace $y^2 + x^2$ with a^2

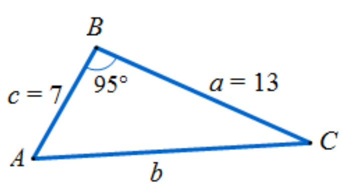
$$\begin{aligned} \boldsymbol{c^2} &= y^2 + (b - x)^2 \\ &= y^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx \\ &= a^2 + b^2 - 2b(a \cos \angle C) \\ &= \boldsymbol{a^2 + b^2 - 2ab \cos \angle C} \end{aligned}$$

Similarly, by placing the vertices A or B at the origin, one can develop the remaining two forms of the Cosine Law.

Example 4 ► **Solving Oblique Triangles Given Two Sides and the Enclosed Angle**

Solve triangle ABC, given that $\angle B = 95°$, $a = 13$, and $c = 7$.

Solution ►



First, we will sketch an oblique triangle ABC to model the situation. Since there is no pair of opposite data given, we cannot use the Law of Sines. However, applying the Law of Cosines with respect to side b and $\angle B$ allows for finding the length b. From

$$b^2 = a^2 + c^2 - 2ac \cos \angle B = 13^2 + 7^2 - 2 \cdot 13 \cdot 7 \cos 95° \simeq 233.86,$$

we have $\boldsymbol{b \simeq 15.3}$.

watch the order of operations here!

Now, since we already have the pair of opposite data $(15.3, 95°)$, we can apply the Law of Sines to find, for example, $\angle C$. From the proportion

$$\frac{\sin \angle C}{7} = \frac{\sin 95°}{15.3},$$

we have

$$\sin \angle C = \frac{7 \cdot \sin 95°}{15.3} \simeq 0.4558,$$

thus $\angle \boldsymbol{C} = \sin^{-1} 0.4558 \simeq \mathbf{27.1}°$.
Finally, $\angle \boldsymbol{A} = 180° - 95° - 27.1° = \mathbf{57.9}°$ and the triangle is solved.

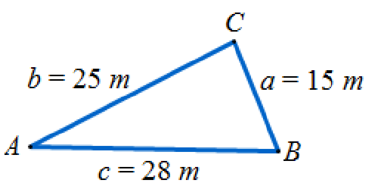
When applying the Law of Cosines in the above example, there was no other choice but to start with the pair of opposite data $(b, \angle B)$. However, in the case of three given sides, one could apply the Law of Cosines corresponding to any pair of opposite data. Is there any preference as to which pair to start with? Actually, yes. Observe that after using the Law of Cosines, we often use the **Law of Sines** to complete the solution since the **calculations are usually easier** to perform this way. Unfortunately, when solving a sine proportion for an obtuse angle, one would need to change the angle obtained from a calculator to its supplementary one. This is because calculators are programmed to return angles from the first quadrant when applying $\sin^{-1}$ to positive ratios. If we look for an obtuse angle, we need to employ the fact that $\sin \alpha = \sin(180° - \alpha)$ and take the supplement of the calculator's answer. To avoid this ambiguity, it is recommended to **apply the Cosine Law** to the pair of the **longest side and largest angle** first. This will guarantee that the Law of Sines will be used to find only acute angles and thus it will not cause ambiguity.

Recommendations:
- ***apply the Cosine Law only when it is absolutely necessary*** *(SAS or SSS)*
- ***apply the Cosine Law to find the largest angle first****, if applicable*

Example 5 ► **Solving Oblique Triangles Given Three Sides**

Solve triangle ABC, given that $a = 15\ m$, $b = 25\ m$, and $c = 28\ m$.

Solution ►



First, we will sketch a triangle ABC to model the situation. As before, there is no pair of opposite data given, so we cannot use the Law of Sines. So, we will apply the Law of Cosines with respect to the pair $(28, \angle C)$, as the side $c = 28$ is the longest. To solve the equation

$$28^2 = 15^2 + 25^2 - 2 \cdot 15 \cdot 25 \cos \angle C$$

for $\angle C$, we will first solve it for $\cos \angle C$, and have

watch the order of operations when solving for cosine

$$\cos \angle C = \frac{28^2 - 15^2 - 25^2}{-2 \cdot 15 \cdot 25} = \frac{-66}{-750} = 0.088,$$

which, after applying $\cos^{-1}$, gives $\angle \boldsymbol{C} \simeq \mathbf{85}°$.

Since now we have the pair of opposite data $(28, 85°)$, we can apply the Law of Sines to find, for example, $\angle A$. From the proportion

$$\frac{\sin \angle A}{15} = \frac{\sin 85°}{28},$$

we have

$$\sin \angle A = \frac{15 \cdot \sin 85°}{28} \simeq 0.5337,$$

thus $\angle \boldsymbol{A} = \sin^{-1} 0.5337 \simeq \mathbf{32.3}°$.
Finally, $\angle \boldsymbol{B} = 180° - 85° - 32.3° = \mathbf{62.7}°$ and the triangle is solved.

Example 6 ► **Solving an Application Problem Using the Cosine Law**

Two planes leave an airport at the same time and fly in different directions. Plane A flies in the direction of 155° at 390 km/h and plane B flies in the direction of 260° at 415 km/h. To the nearest kilometer, how far apart are the planes after two hours?

Solution ► As usual, we start the solution by sketching a diagram appropriate to the situation. Assume the notation as in *Figure 3*.

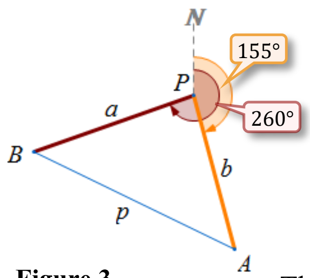


Figure 3

Since plane A flies at 390 km/h for two hours, we can find the distance

$$b = 2 \cdot 390 = 780\ km.$$

Similarly, since plane B flies at 415 km/h for two hours, we have

$$a = 2 \cdot 415 = 830\ km.$$

The measure of the enclosed angle APB can be obtained as a difference between the given directions. So we have

$$\angle APB = 260° - 155° = 105°.$$

Now, we are ready to apply the Law of Cosines in reference to the pair $(p, 105°)$. From the equation

$$p^2 = 830^2 + 780^2 - 2 \cdot 830 \cdot 780 \cos 105° \simeq 1632418.9,$$

we have $p \simeq \sqrt{1632418.9} \simeq 1278\ km$.

So we know that after two hours, the two planes are about **1278 kilometers** apart.

Area of a Triangle

The method used to derive the Law of Sines can also be used to derive a handy formula for finding the area of a triangle, without knowing its height.

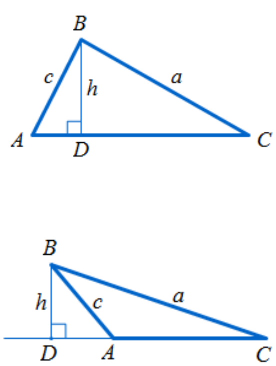


Figure 4

Let ABC be a triangle with height h dropped from the vertex B onto the line $\overleftrightarrow{AC}$, meeting $\overleftrightarrow{AC}$ at the point D, as shown in *Figure 4*. Using the right $\triangle ABD$, we have

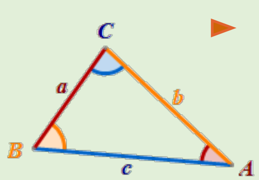
$$\sin \angle A = \frac{h}{c},$$

and equivalently $h = c \sin \angle A$, which after substituting into the well known formula for area of a triangle $[\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{bh}$, gives us

$$[\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{bc}\ \mathbf{sin}\ \angle \boldsymbol{A}$$

Starting the proof with dropping a height from a different vertex would produce two more versions of this formula, as stated below.

The Sine Formula for Area of a Triangle

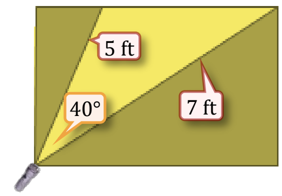


The **area** $[\boldsymbol{ABC}]$ of a triangle ABC can be calculated by taking **half of a product of the lengths of two sides and the sine of the enclosed angle.** We have

$$[\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{bc}\ \mathbf{sin}\ \angle \boldsymbol{A}, \quad [\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{ac}\ \mathbf{sin}\ \angle \boldsymbol{B}, \quad \text{or} \quad [\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{ab}\ \mathbf{sin}\ \angle \boldsymbol{C}.$$

Example 7 ► **Finding Area of a Triangle Given Two Sides and the Enclosed Angle**

In a search for her lost earring, Irene used a flashlight to illuminate part of the floor under her bed. If the flashlight emitted the light at 40° angle and the length of the outside rays of light was 5 ft and 7 ft as indicated in the accompanying diagram, how many square feet of the floor were illuminated?



Solution ► We start with sketching an appropriate diagram. Assume the notation as in *Figure 5*.

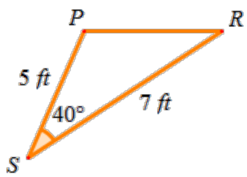


Figure 5

From the sine formula for area of a triangle, we have

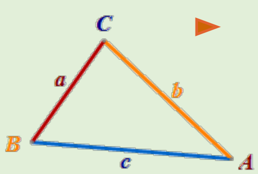
$$[PRS] = \frac{1}{2} \cdot 5 \cdot 7 \sin 40° \simeq 11.2 \text{ ft}^2.$$

The area of the illuminated part of the floor under the bed was about **11 square feet**.

Heron's Formula

The Law of Cosines can be used to derive a formula for the area of a triangle when only the lengths of the three sides are known. This formula is known as Heron's formula (as mentioned in *Section RD1*), named after the Greek mathematician Heron of Alexandria.

Heron's Formula for Area of a Triangle



► The **area** $[\boldsymbol{ABC}]$ of a triangle ABC with sides a, b, c, and **semiperimeter** $\boldsymbol{s} = \frac{\boldsymbol{a+b+c}}{\boldsymbol{2}}$ can be calculated using the formula

$$[\boldsymbol{ABC}] = \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}$$

Example 8 ► **Finding Area of a Triangle Given Three Sides**

The city of Abbotsford plans to convert a triangular lot into public parking. In square meters, what would the area of the parking be if the three sides of the lot are 45 m, 57 m, and 60 m long?

Solution ► To find the area of the triangular lot with given sides, we would like to use Heron's Formula. For this reason, we first calculate the semiperimeter

$$s = \frac{45 + 57 + 60}{2} = 81.$$

Then, the area equals

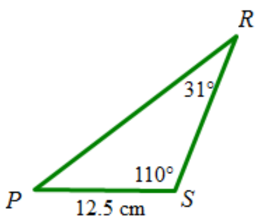
$$\sqrt{81(81-45)(81-57)(81-60)} = \sqrt{1469664} \simeq 1212.3\ m^2.$$

Thus, the area of the parking lot would be approximately **1212 square meters**.

T.5 Exercises

Use the Law of Sines to solve each triangle.

1.



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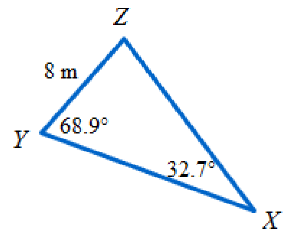

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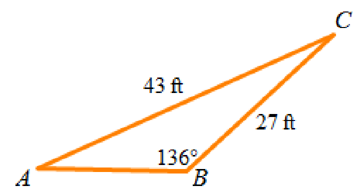

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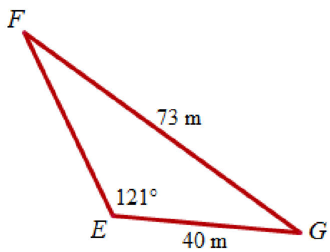

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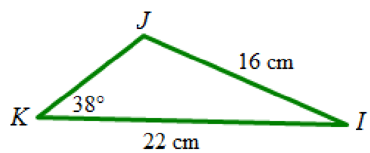

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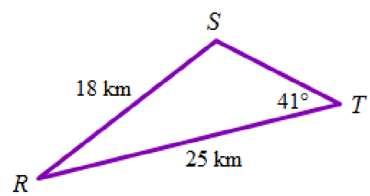


7. $\angle A = 30°,\ \angle B = 30°,\ a = 10$

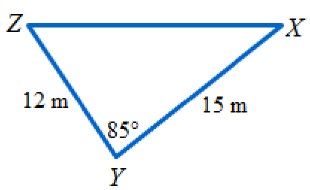
8. $\angle A = 150°,\ \angle C = 20°,\ a = 200$

9. $\angle C = 145°,\ b = 4,\ c = 14$

10. $\angle A = 110°15',\ a = 48,\ b = 16$

Use the Law of Cosines to solve each triangle.

11.



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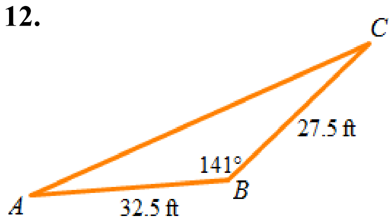

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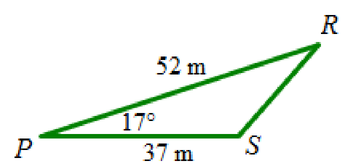

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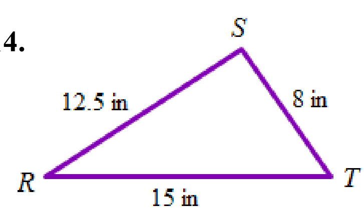

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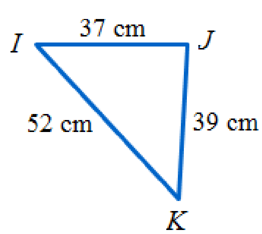

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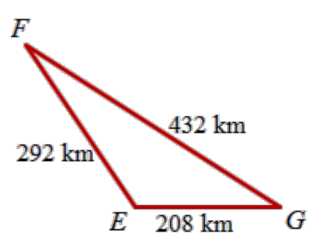


17. $\angle C = 60°,\ a = 3,\ b = 10$

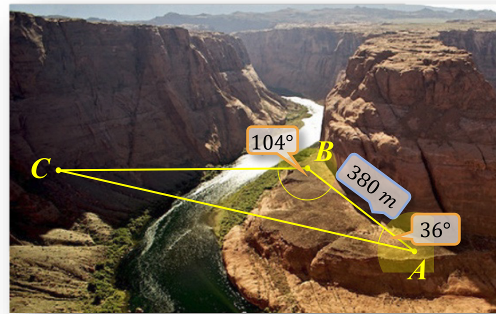
18. $\angle B = 112°,\ a = 23,\ c = 31$

19. $a = 2,\ b = 3,\ c = 4$

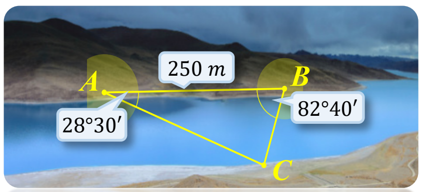
20. $a = 34,\ b = 12,\ c = 17.5$

21. In a triangle ABC, $\angle A$ is twice as large as $\angle B$. Does this mean that side a is twice as long as side b?

Use the appropriate law to solve each application problem.



22. To approximate the distance across the Colorado River Canyon at the Horseshoe Bend, a hiker designates three points, A, B, and C, as in the accompanying figure. Then, he records the following measurements: $AB = 380$ meters, $\angle CAB = 36°$ and $\angle ABC = 104°$. How far is from B to C?



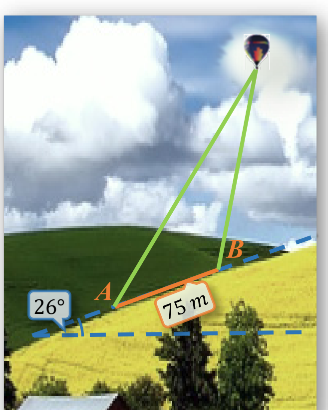
23. To find the width of a river, Peter designates three spots: A and B along one side of the river 250 meters apart from each other, and C, on the opposite side of the river (*see the accompanying figure*). Then, he finds that $\angle A = 28°30'$, and $\angle B = 82°40'$. To the nearest meter, what is the width of the river?

24. The captain of a ship sailing south spotted a castle tower at the distance of approximately 8 kilometers and the bearing of $\boldsymbol{S}47.5°\boldsymbol{E}$. In half an hour, the bearing of the tower was $\boldsymbol{N}35.7°\boldsymbol{E}$. What was the speed of the ship in km/h?

25. The captain of a ship sailing south saw a lighthouse at the bearing of $\boldsymbol{N}52.5°\boldsymbol{W}$. In 4 kilometers, the bearing of the lighthouse was $\boldsymbol{N}35.8°\boldsymbol{E}$. To the nearest tenth of a kilometer, how far was the ship from the lighthouse at each location?

26. Sam and Dan started sailing their boats at the same time and from the same spot. Sam followed the bearing of N12°W while Dan directed his boat at N5°E. After 3 hours, Sam was exactly west of Dan. If both sailors were 4 kilometers away from each other at that time, determine the distance sailed by Sam. *Round your answer to the nearest meter.*

27. A pole is anchored to the ground by two metal cables, as shown in the accompanying figure. The angles of inclination of the two cables are 51° and 60° respectively. Approximately how long is the top cable if the bottom one is attached to the pole 1.6 meters lower than the top one? *Round your answer to the nearest tenth of a meter.*



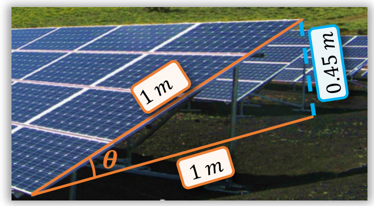
28. Two forest rangers were observing the forest from different lookout towers. At a certain moment, they spotted a group of lost hikers. The ranger on tower A saw the hikers at the direction of 46.7° and ranger on tower B saw the hikers at the direction of 315.8°. If tower A was 3.25 kilometers west of tower B, how far were the hikers from tower A? *Round your answer to the nearest hundredth of a kilometer.*

29. A hot-air balloon rises above a hill that inclines at 26°, as indicated in the accompanying diagram. Two spectators positioned on the hill at points A and B (*refer to the diagram*) observe the movement of the balloon. They notice that at a particular moment, the angle of elevation of the balloon from point A is 64° and from point B is 73°. If the spectators are 75 meters from each other, how far is the balloon from each of them? *Round your answers to the nearest meter.*

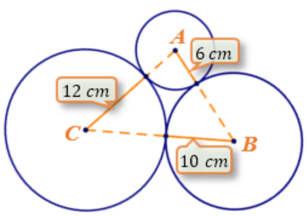
30. To the nearest centimeter, how long is the chord subtending a central angle of 25° in a circle of radius 30 cm?

31. An airplane takes off from city A and flies in the direction of $32°15'$ to city B, which is 500 km from A. After an hour of layover, the plane is heading in the direction of $137°25'$ to reach city C, which is 740 km from A. How far and in what direction should the plane fly to go back to city A?

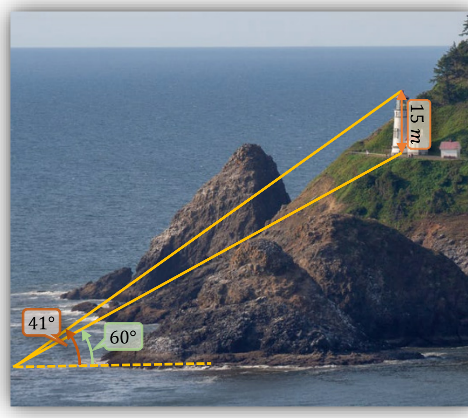
32. Find the area of a triangular hang-glider with two 7.5-meter sides that enclose the angle of 142°. *Round your answer to the nearest tenth of a square meter.*



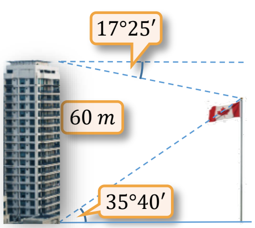
33. One-meter-wide solar panels were installed on a flat surface by tilting them up at an angle θ, as shown in the accompanying figure. If the distance between the top corner of a panel in the flat and tilted position is 0.45 meters, determine the measure of angle θ.



34. Three pipes with centres at points A, B, and C are tangent to each other. A perpendicular cross-section of the arrangement is shown in the accompanying figure. To the nearest tenth of a degree, determine the angles of triangle ABC, if the radii of the pipes are 6 cm, 10 cm, and 12 cm, respectively.



35. A 15-meters tall lighthouse is standing on a cliff. A person observing the lighthouse from a boat approaching the shore notices that the angle of elevation to the top of the lighthouse is 41° and to the bottom is 36°. Disregarding the person's height, estimate the height of the cliff.

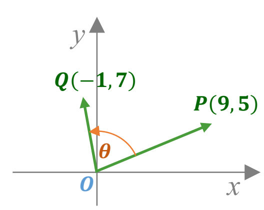


36. The top of a flag pole is visible from the top of a 60 meters high building at $17°25'$ angle of depression. From the bottom of this building, the tip of the flag pole can be seen at $35°40'$ angle of elevation. To the nearest centimeter, how tall is the flag pole?

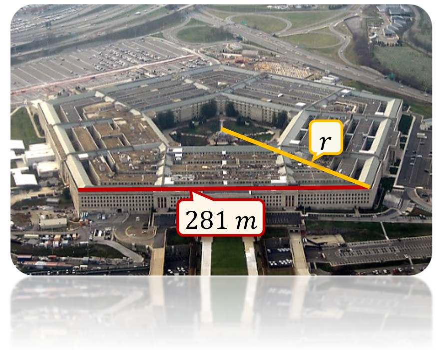
37. Find the area of a triangular parcel having two sides of lengths 51.4 m and 62.1 m, and 48.7° angle between them.

38. A city plans to pave a triangular area with sides of length 82 meters, 78 meters, and 112 meters. A pallet of bricks chosen for the job can cover 10 square meters of area. How many pallets should be ordered?

39. Suppose points P and Q are located respectively at $(9,5)$ and $(-1,7)$. If point O is the origin of the Cartesian coordinate system, determine the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. *Round your answer to the nearest degree.*



40. The building of The Pentagon in Washington D.C. is in a shape of a regular pentagon with about 281 meters long side, as shown in the accompanying figure. To the nearest meter, determine the radius of the **circumcircle** of this pentagon (*the circle that passes through all the vertices of the polygon*).



41. The locations A, B, and C of three FM radio transmitters form a triangle with sides $AB = 75$ m, $BC = 85$ m, and $AC = 90$ m. The transmitters at A, B, and C have a circular range of radius 35 m, 40 m, and 50 m, correspondingly. Assuming that no area can receive a signal from more than one transmitter, determine the area of the ABC triangle that does not receive any signal from any of the three FM radio transmitters. Round your answer to the nearest tenth of a square meter.

Attributions