G4 Linear Inequalities in Two Variables

In many real-life situations, we are interested in a range of values satisfying certain conditions rather than in one specific value. For example, when exercising, we like to keep the heart rate between 120 and 140 beats per minute. The systolic blood pressure of a healthy person is usually between 100 and 120 mmHg (millimeters of mercury). Such conditions can be described using inequalities. Solving systems of inequalities has its applications in many practical business problems, such as how to allocate resources to achieve a maximum profit or a minimum cost. In this section, we study graphical solutions of linear inequalities.

Linear Inequalities in Two Variables

Definition 4.1 ► Any inequality that can be written as

$$\boldsymbol{Ax + By < C},\ \boldsymbol{Ax + By \le C},\ \boldsymbol{Ax + By > C},\ \boldsymbol{Ax + By \ge C}, \text{ or } \boldsymbol{Ax + By \ne C},$$

where $A, B, C \in \mathbb{R}$ and A and B are not both 0, is a **linear inequality in two variables**.

To **solve** an inequality in two variables, x and y, means to **find <u>all</u> ordered pairs** $(\boldsymbol{x}, \boldsymbol{y})$ satisfying the inequality.

Inequalities in two variables arise from many situations. For example, suppose that the number of full-time students, f, and part-time students, p, enrolled in upgrading courses at the University of the Fraser Valley is at most 1200. This situation can be represented by the inequality

$$f + p \le 1200.$$

Some of the solutions (f, p) of this inequality are: $(1000, 200)$, $(1000, 199)$, $(1000, 198)$, $(600, 600)$. $(550, 600)$, $(1100, 0)$, and many others.

The solution sets of inequalities in two variables contain infinitely many ordered pairs of numbers which, when graphed in a system of coordinates, fulfill specific regions of the coordinate plane. That is why it is more beneficial to present such solutions in the form of a graph rather than using set notation. To graph the region of points satisfying the inequality $f + p \le 1200$, we may want to solve it first for p,

$$p \le -f + 1200,$$

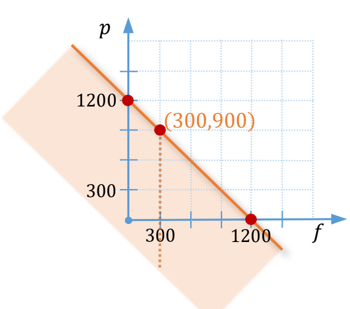


Figure 1a

and then graph the related equation, $p = -f + 1200$, called the **boundary line**. Notice, that setting f to, for instance, 300 results in the inequality

$$p \le -300 + 1200 = 900.$$

So, any point with the first coordinate of 300 and the second coordinate of 900 or less satisfies the inequality (see the dotted half-line in *Figure 1a*). Generally, observe that any point with the first coordinate f and the second coordinate $-f + 1200$ or less satisfies the inequality. Since the union of all half-lines that start from the boundary line and go down is the whole half-plane below the boundary line, we shade it as the solution set to the discussed inequality (see *Figure 1a*). The solution set also includes the points of the boundary line, as the inequality includes equation.

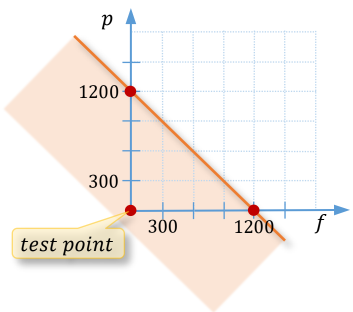


Figure 1b

The above strategy can be applied to any linear inequality in two variables. Hence, one can conclude that the solution set to a given linear inequality in two variables consists of **all points of one of the half-planes** obtained by cutting the coordinate plane by the corresponding boundary line. This fact allows us to find the solution region even faster. After graphing the boundary line, to know which half-plane to shade as the solution set, it is enough to check just one point, called a **test point**, chosen outside of the boundary line. In our example, it was enough to test for example point $(0,0)$. Since $0 \leq -0 + 1200$ is a true statement, then the point $(0,0)$ belongs to the solution set. This means that the half-plane containing this test point must be the solution set to the given inequality, so we shade it.

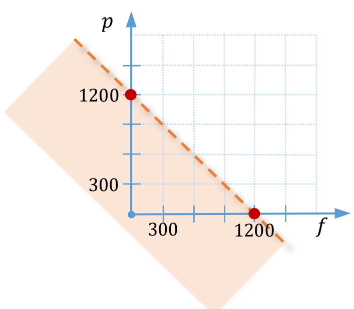


Figure 1c

The solution set of the strong inequality $p < -f + 1200$ consists of the same region as in *Figure 1b*, except for the points on the boundary line. This is because the points of the boundary line satisfy the equation $p = -f + 1200$, but not the inequality $p < -f + 1200$. To indicate this on the graph, we draw the boundary line using a dashed line (see *Figure 1c*).

In summary, to graph the solution set of a linear inequality in two variables, follow the steps:

1. Draw the graph of the corresponding **boundary line**.
   Make the line **solid** if the inequality involves $\leq$ or $\geq$.
   Make the line **dashed** if the inequality involves $<$ or $>$.
2. Choose a **test point** outside of the line and substitute the coordinates of that point into the inequality.
3. If the test point satisfies the original inequality, **shade the half-plane containing the point.**
   If the test point does not satisfy the original inequality, **shade the other half-plane** (the one that does not contain the point).

Example 1 ► **Determining if a Given Ordered Pair of Numbers is a Solution to a Given Inequality**

Determine if the points $(3,1)$ and $(2,1)$ are solutions to the inequality $5x - 2y > 8$.

Solution ► An ordered pair is a solution to the inequality $5x - 2y > 8$ if its coordinates satisfy this inequality. So, to determine whether the pair $(3,1)$ is a solution, we substitute 3 for x and 1 for y. The inequality becomes

$$5 \cdot 3 - 2 \cdot 1 > 8,$$

which simplifies to the true inequality $13 > 8$.
Thus, $(3,1)$ is a solution to $5x - 2y > 8$.

However, replacing x by 2 and y by 1 results in $5 \cdot 2 - 2 \cdot 1 > 8$, or equivalently $8 > 8$. Since 8 is not larger than 8, the point $(2,1)$ does not satisfy the inequality. Thus, $(2,1)$ is not a solution to $5x - 2y > 8$.

Example 2 ► **Graphing Linear Inequalities in Two Variables**

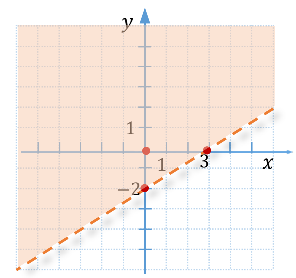
Graph the solution set of each inequality in two variables.

a. $2x - 3y < 6$ **b.** $y \le 3x - 1$

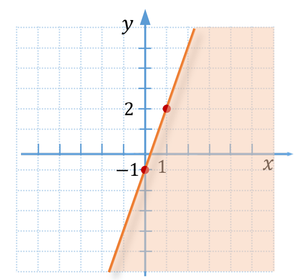
c. $x \ge -3$ **d.** $y \ne x$

Solution ►

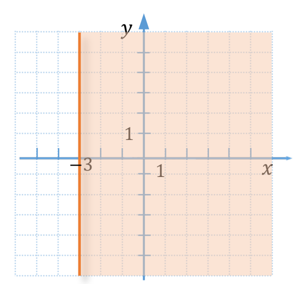
a. First, we graph the boundary line $2x - 3y = 6$, using the x- and y-intercepts: $(3,0)$ and $(0,-2)$. Since the inequality $<$ does not involve an equation, the line is marked as **dashed**, which indicates that the points on the line are not the solutions of the inequality. Then, we choose the point $(0,0)$ for the test point. Since $2 \cdot 0 - 3 \cdot 0 < 6$ is a true statement, then we shade the half-plane containing $(0,0)$ as the solution set.



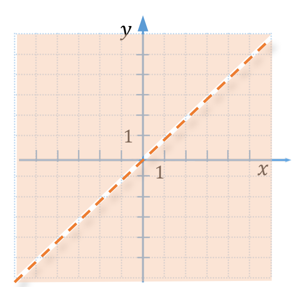
b. First, we graph the boundary line $y \le 3x - 1$, using the slope and y-intercept. Since the inequality $\le$ contains an equation, the line is marked as **solid**. This indicates that the points on the line belong to solutions of the inequality. To decide which half-plane to shade as the solution region, we observe that y is lower than or equal to the $3x - 1$, which tells us that the solution points lie below or on the boundary line. So, we shade the half-plane below the line.



c. As before, to graph $x \ge -3$, first, we graph the solid vertical line $x = -3$, and then we shade the half-plane consisting of points with x-coordinates larger or equal to -3. So the solution set is the half-plane to the right of the boundary line, including this line.



d. The solution set of the inequality $y \ne x$ consists of all points that do not satisfy the equation $y = x$. This means that we mark the boundary line as dashed and shade the rest of the points of the coordinate plane.



G.4 Exercises

For each inequality, determine if the given points belong to the solution set of the inequality.

1. $y \geq -4x + 3$; $(1, -1)$, $(1,0)$

2. $2x - 3y < 6$; $(3,0)$, $(2, -1)$

3. $y > -2$; $(0,0)$, $(-1, -1)$

4. $x \geq -2$; $(-2,1)$, $(-3,1)$

5. Match the given inequalities with the graphs of their solution sets.

a. $y \geq x + 2$

b. $y < -x + 2$

c. $y \leq x + 2$

d. $y > -x + 2$

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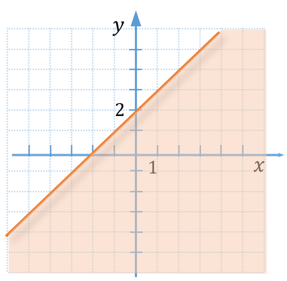

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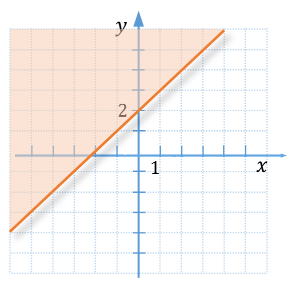

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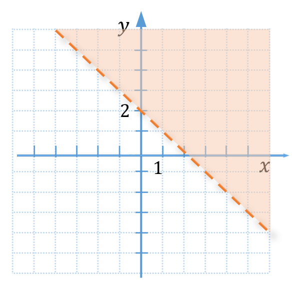

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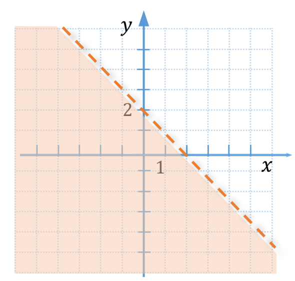


Graph each linear inequality in two variables.

6. $y \geq -\frac{1}{2}x + 3$

7. $y \leq \frac{1}{3}x - 2$

8. $y < 2x - 4$

9. $y > -x + 3$

10. $y \geq -3$

11. $y < 4.5$

12. $x > 1$

13. $x \leq -2.5$

14. $x + 3y > -3$

15. $5x - 3y \leq 15$

16. $y - 3x \geq 0$

17. $3x - 2y < -6$

18. $3x \leq 2y$

19. $3y \neq 4x$

20. $y \neq 2$

G5 Concept of Function, Domain, and Range

In mathematics, we often investigate relationships between two quantities. For example, we might be interested in the average daily temperature in Abbotsford, BC, over the last few years, the amount of water wasted by a leaking tap over a certain period of time, or particular connections among a group of bloggers. The relations can be described in many different ways: in words, by a formula, through graphs or arrow diagrams, or simply by listing the ordered pairs of elements that are in the relation. A group of relations, called *functions*, will be of special importance in further studies. In this section, we will define functions, examine various ways of determining whether a relation is a function, and study related concepts such as *domain* and *range*.

Relations, Domains, and Ranges

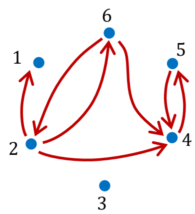


Figure 1

Consider a relation of knowing each other in a group of 6 people, represented by the arrow diagram shown in *Figure 1*. In this diagram, the points 1 through 6 represent the six people and an arrow from point x to point y tells us that the person x knows the person y. This correspondence could also be represented by listing the ordered pairs (x, y) whenever person x knows person y. So, our relation can be shown as the set of points

$$\{(2,1), (2,4), (2,6), (4,5), (5,4), (6,2), (6,4)\}$$

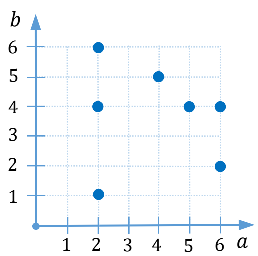
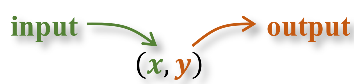


Figure 2a

The x-coordinate of each pair (x, y) is called the **input**, and the y-coordinate is called the **output**.



The ordered pairs of numbers can be plotted in a system of coordinates, as in *Figure 2a*. The obtained graph shows that some inputs are in a relation with many outputs. For example, input 2 is in a relation with output 1, and 4, and 6. Also, the same output, 4, is assigned to many inputs. For example, the output 4 is assigned to the input 2, and 5, and 6.

The set of all the inputs of a relation is its **domain**. Thus, the domain of the above relation consists of all first coordinates

$$\{2, 4, 5, 6\}$$

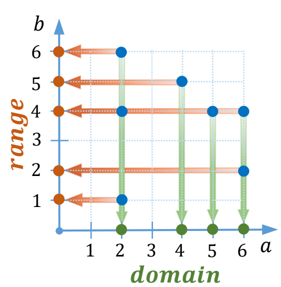


Figure 2b

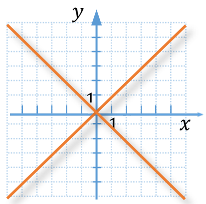
The set of all the outputs of a relation is its **range**. Thus, the range of our relation consists of all second coordinates

$$\{1, 2, 4, 5, 6\}$$

The domain and range of a relation can be seen on its graph through the **perpendicular projection** of the graph **onto the horizontal axis**, for the **domain**, and **onto the vertical axis**, for the **range**. See *Figure 2b*.

In summary, we have the following definition of a relation and its domain and range:

Definition 5.1 ► A **relation** is any **set of ordered pairs**. Such a set establishes a **correspondence** between the **input** and **output** values. In particular, any subset of a coordinate plane represents a relation.
The **domain** of a relation consists of all **inputs** (**first coordinates**).
The **range** of a relation consists of all **outputs** (**second coordinates**).



Relations can also be given by an equation or an inequality. For example, the equation

$$|y| = |x|$$

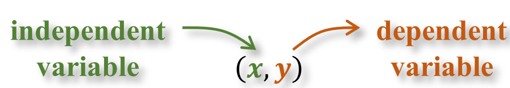
describes the set of points in the xy-plane that lie on two diagonals, $y = x$ and $y = -x$. In this case, the domain and range for this relation are both the set of real numbers because the projection of the graph onto each axis covers the entire axis.

Functions, Domains, and Ranges

Relations that have exactly one output for every input are of special importance in mathematics. This is because as long as we know the rule of a correspondence defining the relation, the output can be uniquely determined for every input. Such relations are called **functions**. For example, the linear equation $y = 2x + 1$ defines a function, as for every input x, one can calculate the corresponding y-value in a unique way. Since both the input and the output can be any real number, the domain and range of this function are both the set of real numbers.

Definition 5.2 ► A **function** is a relation that assigns **exactly one** **output** value in the **range** to each **input** value of the **domain**.

If (x, y) is an ordered pair that belongs to a function, then x can be any arbitrarily chosen input value of the domain of this function, while y must be the uniquely determined value that is assigned to x by this function. That is why x is referred to as an **independent** variable while y is referred to as the **dependent** variable (because the y-value depends on the chosen x-value).



How can we recognize if a relation is a function?

If the relation is given as a set of ordered pairs, it is enough to check if there are no two pairs with the same inputs. For example:

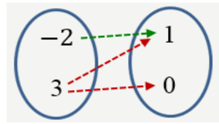
$\{(2,1), (2,4), (1,3)\}$
relation

The pairs $(2,1)$ and $(2,4)$ have the same inputs. So, there are **two y-values** assigned to the x-value 2, which makes it not a function.

$\{(2,1), (1,3), (4,1)\}$
function

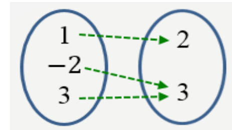
There are no pairs with the same inputs, so each x-value is associated with exactly one pair and consequently with exactly one y-value. This makes it a function.

If the relation is given by a diagram, we want to check if no point from the domain is assigned to two points in the range. For example:



relation

There are **two arrows** starting from 3. So, there are two y-values assigned to 3, which makes it not a function.



function

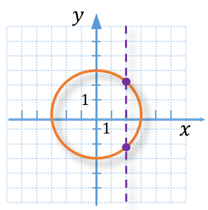
Only one arrow starts from each point of the domain, so each x-value is associated with exactly one y-value. Thus this is a function.

VERTICAL LINE TEST

If the relation is given by a graph, we use **The Vertical Line Test**:

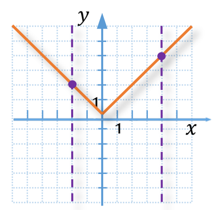
A relation is a **function** if **no vertical line** intersects the graph more than once.

For example:



relation

There is a vertical line that intersects the graph **twice**. So, there are two y-values assigned to an x-value, which makes it not a function.



function

Any vertical line intersects the graph only **once**. So, by The Vertical Line Test, this is a function.

If the relation is given by an equation, we check whether the y-value can be determined uniquely. For example:

$x^2 + y^2 = 1$

relation

Both points $(0,1)$ and $(0,-1)$ belong to the relation. So, there are **two y-values** assigned to 0, which makes it not a function.

$y = \sqrt{x}$

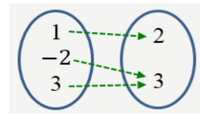
function

The y-value is uniquely defined as the square root of the x-value, for $x \geq 0$. So, this is a function.

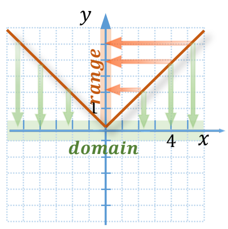
In general, to determine if a given relation is a function, we analyse the relation to see whether it assigns two different y-values to the same x-value. If it does, it is just a relation, not a function. If it doesn't, it is a function.

Since functions are a special type of relation, the **domain and range of a function** can be determined the same way as in the case of a relation.
Let us look at domains and ranges of the above examples of functions.

The domain of the function $\{(2,1),(1,3),(4,1)\}$ is the set of the first coordinates of the ordered pairs, which is $\{1,2,4\}$. The range of this function is the set of second coordinates of the ordered pairs, which is $\{1,3\}$.

The domain of the function defined by the diagram 

is the first set of points, particularly $\{1,-2,3\}$.
The range of this function is the second set of points, which is $\{2,3\}$.



The domain of the function given by the accompanying graph is the projection of the graph onto the x-axis, which is the set of all real numbers $\mathbb{R}$.
The range of this function is the projection of the graph onto the y-axis, which is the interval of points larger or equal to zero, $[0,\infty)$.

The domain of the function given by the equation $y=\sqrt{x}$ is the set of nonnegative real numbers, $[0,\infty)$, since the square root of a negative number is not real.
The range of this function is also the set of nonnegative real numbers, $[0,\infty)$, as the value of a square root is never negative.

Example 1 ► **Determining Whether a Relation is a Function and Finding Its Domain and Range**

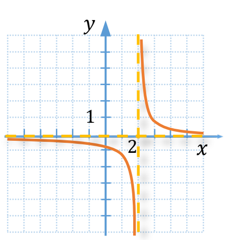
Decide whether each relation defines a function, and give the domain and range.

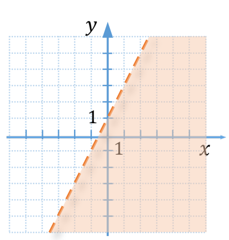
a. $y=\frac{1}{x-2}$ **b.** $y<2x+1$

c. $x=y^2$ **d.** $y=\sqrt{2x-1}$

Solution ►

a. Since $\frac{1}{x-2}$ can be calculated uniquely for every x from its domain, the relation $y=\frac{1}{x-2}$ is a function.
The domain consists of all real numbers that make the denominator, $x-2$, different than zero. Since $x-2=0$ for $x=2$, then the domain, D, is the set of all real numbers except for 2. We write $D=\mathbb{R}\setminus\{2\}$.
Since a fraction with nonzero numerator cannot be equal to zero, the range of $y=\frac{1}{x-2}$ is the set of all real numbers except for 0. We write $range=\mathbb{R}\setminus\{0\}$.

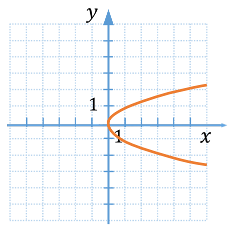




b. The inequality $y<2x+1$ is not a function as for every x-value there are many y-values that are lower than $2x+1$. Particularly, points $(0,0)$ and $(0,-1)$ satisfy the inequality and show that the y-value is not unique for $x=0$.

In general, because of the many possible y-values, no inequality defines a function.

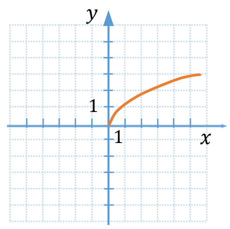
Since there are no restrictions on x-values, the domain of this relation is the set of all real numbers, $\mathbb{R}$. The range is also the set of all real numbers, $\mathbb{R}$, as observed in the accompanying graph.



c. Here, we can show two points, $(1,1)$ and $(1,-1)$, that satisfy the equation, which contradicts the requirement of a single y-value assigned to each x-value. So, this relation is not a function.

Since x is a square of a real number, it cannot be a negative number. So the domain consists of all nonnegative real numbers. We write, $D = [0, \infty)$. However, y can be any real number, so $range = \mathbb{R}$.

d. The equation $y = \sqrt{2x-1}$ represents a function, as for every x-value from the domain, the y-value can be calculated in a unique way.



The domain of this function consists of all real numbers that would make the radicand $2x-1$ nonnegative. So, to find the domain, we solve the inequality:

$$2x - 1 \geq 0$$
$$2x \geq 1$$
$$x \geq \frac{1}{2}$$

Thus, $D = \left[\frac{1}{2}, \infty\right)$. As for the range, since the values of a square root are nonnegative, we have $range = [0, \infty)$

G.5 Exercises

Decide whether each relation defines a function, and give its ***domain*** *and* ***range****.*

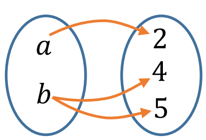
1. $\{(2,4), (0,2), (2,3)\}$

2. $\{(3,4), (1,2), (2,3)\}$

3. $\{(2,3), (3,4), (4,5), (5,2)\}$

4. $\{(1,1), (1,-1), (2,5), (2,-5)\}$

5.



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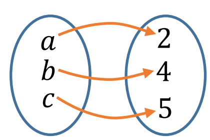

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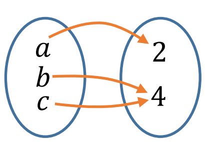

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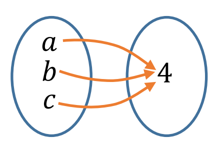


9.

x	y
0	1
0	−1
1	2
1	−2

10.

x	y
−1	4
0	2
1	0
2	−2

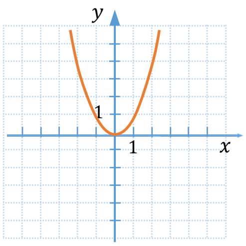
11.

x	y
3	1
6	2
9	1
12	2

12.

x	y
−2	3
−2	0
−2	−3
−2	−6

13.



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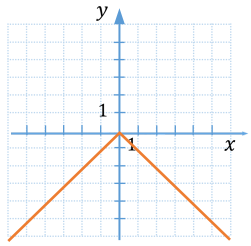

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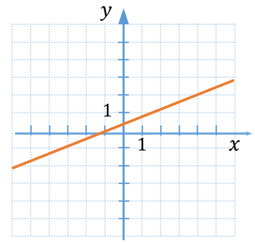

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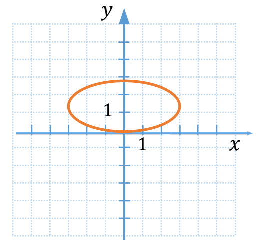

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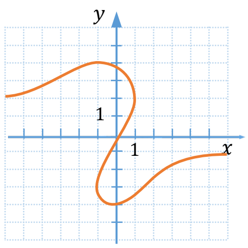

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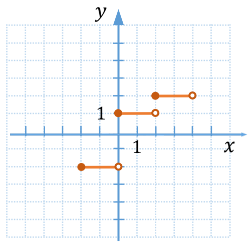

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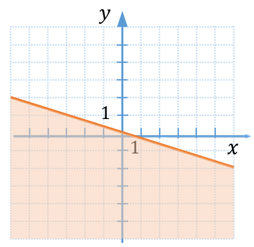

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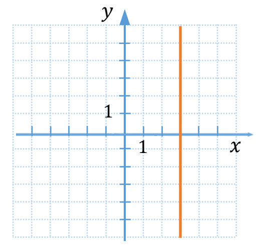


*Find the **domain** of each relation and decide whether the relation defines y as a function of x.*

21. $y = 3x + 2$

22. $y = 5 - 2x$

23. $y = |x| - 3$

24. $x = |y| + 1$

25. $y^2 = x^2$

26. $y^2 = x^4$

27. $x = y^4$

28. $y = x^3$

29. $y = -\sqrt{x}$

30. $y = \sqrt{2x - 5}$

31. $y = \frac{1}{x+5}$

32. $y = \frac{1}{2x-3}$

33. $y = \frac{x-3}{x+2}$

34. $y = \frac{1}{|2x-3|}$

35. $y \leq 2x$

36. $y - 3x \geq 0$

37. $y \neq 2$

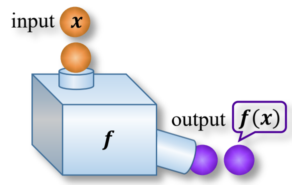
38. $x = -1$

39. $y = x^2 + 2x + 1$

40. $xy = -1$

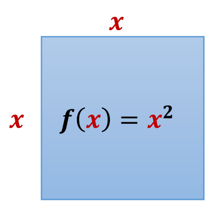
41. $x^2 + y^2 = 4$

G6 Function Notation and Evaluating Functions



A function is a correspondence that assigns a single value of the range to each value of the domain. Thus, a function can be seen as an input-output machine, where the input is taken independently from the domain, and the output is the corresponding value of the range. The rule that defines a function is often written as an equation, with the use of x and y for the independent and dependent variables, for instance, $y = 2x$ or $y = x^2$. To emphasize that y depends on x, we write $\boldsymbol{y = f(x)}$, where f is the name of the function. The expression $\boldsymbol{f(x)}$, read as "$\boldsymbol{f}$ ***of*** $\boldsymbol{x}$", represents the dependent variable assigned to the particular x. Such notation shows the dependence of the variables as well as allows for using different names for various functions. It is also handy when evaluating functions. In this section, we introduce and use *function notation*, and show how to evaluate functions at specific input-values.

Function Notation



Consider the equation $y = x^2$, which relates the length of a side of a square, x, and its area, y. In this equation, the y-value depends on the value x, and it is uniquely defined. So, we say that y is a function of x. Using function notation, we write

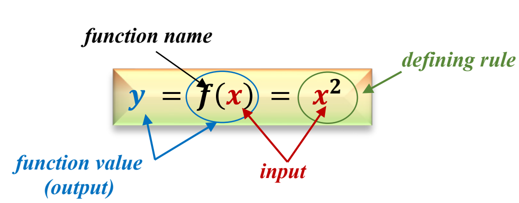
$$f(x) = x^2$$

The expression $f(x)$ is just another name for the dependent variable y, and it shouldn't be confused with a product of f and x. Even though $f(x)$ is really the same as y, we often write $f(x)$ rather than just y, because the notation $f(x)$ carries more information. Particularly, it tells us the name of the function so that it is easier to refer to the particular one when working with many functions. It also indicates the independent value for which the dependent value is calculated. For example, using function notation, we find the area of a square with a side length of 2 by evaluating $f(2) = 2^2 = 4$. So, 4 is the area of a square with a side length of 2.

The statement $f(2) = 4$ tells us that the pair $(2,4)$ belongs to function f, or equivalently, that 4 is assigned to the input of 2 by the function f. We could also say that function f attains the value 4 at 2.

If we calculate the value of function f for $x = 3$, we obtain $f(3) = 3^2 = 9$. So the pair $(3,9)$ also belongs to function f. This way, we may produce many ordered pairs that belong to f and consequently, make a graph of f.

Here is what each part of **function notation** represents:



Note: Functions are customarily denoted by a single letter, such as f, g, h, but also by abbreviations, such as sin, cos, or tan.

Function Values

Function notation is handy when evaluating functions for several input values. To evaluate a function given by an equation at a specific x-value from the domain, we substitute the x-value into the defining equation. For example, to evaluate $f(x) = \frac{1}{x-1}$ at $x = 3$, we calculate

$$f(3) = \frac{1}{3-1} = \frac{1}{2}$$

So $f(3) = \frac{1}{2}$, which tells us that when $x = 3$, the y-value is $\frac{1}{2}$, or equivalently, that the point $\left(3, \frac{1}{2}\right)$ belongs to the graph of the function f.

Notice that function f cannot be evaluated at $x = 1$, as it would make the denominator $(x-1)$ equal to zero, which is not allowed. We say that $f(1) = DNE$ (read: *Does Not Exist*). Because of this, the domain of function f, denoted D_f, is $\mathbb{R}\setminus\{1\}$.

Graphing a function usually requires evaluating it for several x-values and then plotting the obtained points. For example, evaluating $f(x) = \frac{1}{x-1}$ for $x = \frac{3}{2}, 2, 5, \frac{1}{2}, 0, -1$, gives us

$$f\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}-1} = \frac{1}{\frac{1}{2}} = 2$$

$$f(2) = \frac{1}{2-1} = \frac{1}{1} = 1$$

$$f(5) = \frac{1}{5-1} = \frac{1}{4}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}-1} = \frac{1}{-\frac{1}{2}} = -2$$

$$f(0) = \frac{1}{0-1} = -1$$

$$f(-1) = \frac{1}{-1-1} = -\frac{1}{2}$$

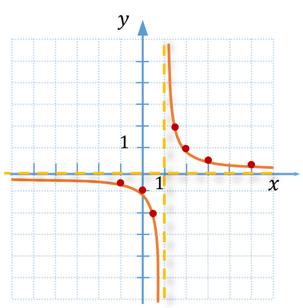


Figure 1

Thus, the points $\left(\frac{3}{2}, 2\right)$, $(2, 1)$, $\left(3, \frac{1}{2}\right)$, $\left(5, \frac{1}{4}\right)$, $\left(\frac{1}{2}, -2\right)$, $(0, -1)$, $\left(-1, -\frac{1}{2}\right)$ belong to the graph of f. After plotting them in a system of coordinates and predicting the pattern for other x-values, we produce the graph of function f, as in *Figure 1*.

Observe that the graph seems to be approaching the vertical line $x = 1$ as well as the horizontal line $y = 0$. These two lines are called ***asymptotes*** and are not a part of the graph of function f; however, they shape the graph. Asymptotes are customarily graphed by dashed lines.

Sometimes a function is given not by an equation but by a graph, a set of ordered pairs, a word description, etc. To evaluate such a function at a given input, we simply apply the function rule to the input.

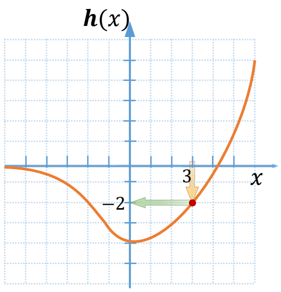


Figure 2a

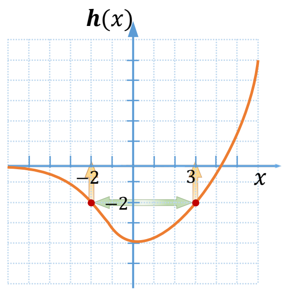


Figure 2b

For example, to find the value of function h, given by the graph in *Figure 2a*, for $x = 3$, we read the second coordinate of the intersection point of the vertical line $x = 3$ with the graph of h. Following the arrows in *Figure 2*, we conclude that $h(3) = -2$.

Notice that to find the x-value(s) for which $h(x) = -2$, we reverse the above process. This means: we read the first coordinate of the intersection point(s) of the horizontal line $y = -2$ with the graph of h. By following the reversed arrows in *Figure 2b*, we conclude that $h(x) = -2$ for $x = 3$ and for $x = -2$.

Example 1 ► **Evaluating Functions**

Evaluate each function at $x = 2$ and write the answer using function notation.

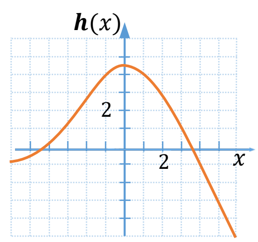
a. $f(x) = 3 - 2x$

b. function f squares the input

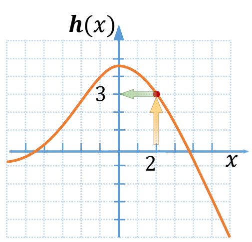
c.

x	$g(x)$
−1	2
2	5
3	−1

d.



Solution ►



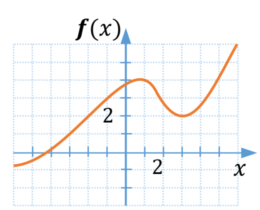
a. Following the formula, we have $f(2) = 3 - 2(2) = 3 - 4 = \mathbf{-1}$

b. Following the word description, we have $f(2) = 2^2 = \mathbf{4}$

c. $g(2)$ is the value in the second column of the table that corresponds to 2 from the first column. Thus, $g(2) = \mathbf{5}$.

d. As shown in the graph, $h(2) = \mathbf{3}$.

Example 2 ► **Finding from a Graph the x-value for a Given $f(x)$-value**

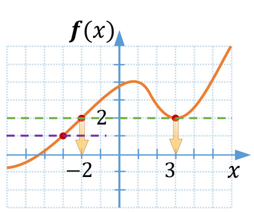


Given the graph, find all x-values for which

a. $f(x) = 1$

b. $f(x) = 2$

Solution ►



a. The purple line $y = 1$ cuts the graph at $x = -3$, so $f(x) = 1$ for $x = -3$.

b. The green line $y = 2$ cuts the graph at $x = -2$ and $x = 3$, so $f(x) = 2$ for $x \in \{-2, 3\}$.

Example 3 ► **Evaluating Functions and Expressions Involving Function Values**

Suppose $f(x) = \frac{1}{2}x - 1$ and $g(x) = x^2 - 5$. Evaluate each expression.

a. $f(4)$ **b.** $g(-2)$ **c.** $g(a)$ **d.** $f(2a)$

e. $g(a-1)$ **f.** $3f(-2)$ **g.** $g(2+h)$ **h.** $f(2+h) - f(2)$

Solution ►

a. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the value 4. So,

$f(4) = \frac{1}{2}(4) - 1 = 2 - 1 = \mathbf{1}$.

b. Replace x in the equation $g(x) = x^2 - 5$ by the value -2, using parentheses around the -2. So, $g(-2) = (-2)^2 - 5 = 4 - 5 = \mathbf{-1}$.

c. Replace x in the equation $g(x) = x^2 - 5$ by the input a. So, $g(a) = \boldsymbol{a^2 - 5}$.

d. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the input $2a$. So,

$f(2a) = \frac{1}{\not{2}}(\not{2}a) - 1 = \boldsymbol{a - 1}$.

$(a-1)^2$
$= (a-1)(a-1)$
$= a^2 - a - a + 1$
$= a^2 - 2a + 1$

e. Replace x in the equation $g(x) = x^2 - 5$ by the input $(a-1)$, using parentheses around the input. So, $g(a-1) = (a-1)^2 - 5 = a^2 - 2a + 1 - 5 = \boldsymbol{a^2 - 2a - 4}$.

f. The expression $3f(-2)$ means three times the value of $f(-2)$, so we calculate

$3f(-2) = 3 \cdot \left(\frac{1}{\not{2}}(-\not{2}) - 1\right) = 3(-1-1) = 3(-2) = \mathbf{-6}$.

$(2+h)^2$
$= (2+h)(2+h)$
$= 4 + 2h + 2h + h^2$
$= 4 + 4h + h^2$

g. Replace x in the equation $g(x) = x^2 - 5$ by the input $(2+h)$, using parentheses around the input. So, $g(2+h) = (2+h)^2 - 5 = 4 + 4h + h^2 - 5 = \boldsymbol{h^2 + 4h - 1}$.

h. When evaluating $f(2+h) - f(2)$, focus on evaluating $f(2+h)$ first and then, to subtract the expression $f(2)$, use a bracket just after the subtraction sign. So,

$$f(2+h) - f(2) = \underbrace{\frac{1}{2}(2+h) - 1}_{f(2+h)} - \underbrace{\left[\frac{1}{2}(2) - 1\right]}_{f(2)} = 1 + \frac{1}{2}h - 1 - [1-1] = \boldsymbol{\frac{1}{2}h}$$

Note: To perform the perfect squares in the solution to *Example 3e* and *3g*, we follow the **perfect square formula** $\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$ or $\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2}$. One can check that this formula can be obtained as a result of applying the distributive law, often referred to as the *FOIL* method, when multiplying two binomials (see the examples in callouts in the left margin). However, we prefer to use the perfect square formula rather than the *FOIL* method, as it makes the calculation process more efficient. See Section P2 for more details.

Function Notation in Graphing and Application Problems

By *Definition 1.1* in *Section G1*, a linear equation is an equation of the form $Ax + By = C$. The graph of any linear equation is a line, and any nonvertical line satisfies the Vertical Line Test. Thus, any linear equation $Ax + By = C$ with $B \neq 0$ defines a linear function.

How can we write this function using function notation?

Since $y = f(x)$, we can replace the variable y in the equation $Ax + By = C$ with $f(x)$ and then solve for $f(x)$. So, we obtain

Alternatively, we can solve the original equation for y and then replace y with $f(x)$.

$$Ax + B \cdot f(x) = C$$
$$B \cdot f(x) = -Ax + C$$
$$f(x) = -\frac{A}{B}x + \frac{C}{B}$$

must assume that $B \neq 0$

Definition 6.1 ► Any function that can be written in the form

$$\boldsymbol{f(x) = mx + b},$$

where m and b are real numbers, is called a **linear function**. The value $\boldsymbol{m}$ represents the **slope** of the graph, and the point $(0, \boldsymbol{b})$ represents the $\boldsymbol{y}$**-intercept** of this function. The **domain** of any linear function is the set of all real numbers, $\mathbb{R}$.

In particular:

Definition 6.2 ► A linear function with slope $\boldsymbol{m = 0}$ takes the form

$$\boldsymbol{f(x) = b},$$

where b is a real number, and is called a **constant function**.

Note: Similarly as the domain of any linear function, the **domain** of a constant function is the set $\mathbb{R}$. However, the **range** of a constant function is the one element set $\{\boldsymbol{b}\}$, while the range of any nonconstant linear function is the set $\mathbb{R}$.

Generally, any equation in two variables, x and y, that defines a function can be written using function notation by solving the equation for y and then letting $y = f(x)$. For example, to rewrite the equation $-4x^2 + 2y = 5$ **explicitly** as a function f of x, we solve for y,

implicit form

explicit form

$$2y = 4x^2 + 5$$
$$y = 2x^2 + \frac{5}{2},$$

and then replace y by $f(x)$. So, $f(x) = 2x^2 + \frac{5}{2}$.

Using function notation, the graph of a function is defined as follows:

Definition 6.3 ► The graph of a function $\boldsymbol{f}$ of $\boldsymbol{x}$ is the set of ordered pairs $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$ for every input x form the domain $\boldsymbol{D_f}$ of the function. This can be stated as

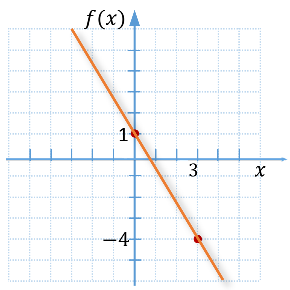
$$\boldsymbol{graph}\ of\ \boldsymbol{f} = \{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) | \boldsymbol{x} \in \boldsymbol{D_f}\}$$

Example 4 ► **Function Notation in Writing and Graphing Functions**

Each of the given equations represents a function of x. Rewrite the formula in explicit form, using function notation. Then graph this function and state its domain and range.

a. $5x + 3 \cdot f(x) = 3$ **b.** $|x| - y = -3$

Solution ► **a.** After solving the equation for $f(x)$,



$$5x + 3 \cdot f(x) = 3$$
$$3 \cdot f(x) = -5x + 3$$
$$f(x) = -\tfrac{5}{3}x + 1,$$

we observe that the function f is linear. So, we can graph it using the slope-intercept method. The graph confirms that the domain and range of this function are both the set of all real numbers, $\mathbb{R}$.

b. After solving the equation for y,

$$|x| - y = 2$$
$$|x| - 2 = y,$$

we obtain the function $f(x) = |x| - 2$.

If we are not sure what the graph of this function looks like, we may evaluate $f(x)$ for several x-values, plot the obtained points, and observe the pattern. For example, let $x = -2, -1, 0, 1, 2$. We fill in the table of values,

x	$\|x\| - 2 = f(x)$	$(x, f(x))$
−2	$\|-2\| - 2 = 0$	$(-2,0)$
−1	$\|-1\| - 2 = -1$	$(-1,-1)$
0	$\|0\| - 2 = -2$	$(0,-2)$
1	$\|1\| - 2 = -1$	$(1,-1)$
2	$\|2\| - 2 = 0$	$(2,0)$

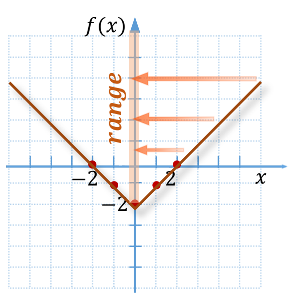


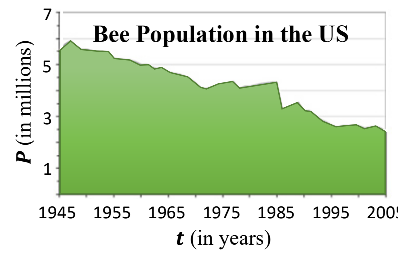
Figure 3

and plot the points listed in the third column. One may evaluate $f(x)$ for several more x-values, plot the additional points, and observe that these points form a V-shape with a vertex at $(0, -2)$. By connecting the points as in *Figure 3*, we obtain the graph of function $f(x) = |x| - 2$.

Since one can evaluate the function $f(x) = |x| - 2$ for any real x, the domain of f is the set $\mathbb{R}$. The range can be observed by projecting the graph perpendicularly onto the vertical axis. So, the range is the interval $[-2, \infty)$, as shown in *Figure 3*.

Example 5 ► **A Function in Applied Situations**

The bee population in the US was declining during the years 1945-2005, as shown in the accompanying graph.



a. Based on the graph what was the approximate value of $P(1960)$ and $P(2000)$ and what does it tell us about the bee population?

b. Estimate the average rate of change in the bee population over the years $1960 - 2000$, and interpret the result in the context of the problem.

c. Approximate the year(s) in which $P(t)$ was 4 million bees.

d. What is the general tendency of the function $P(t)$ over the years $1945 - 2005$?

e. Assuming that function P continue declining at the same rate, predict the year in which the bees in the US would become extinct.

Solution ►

a. One may read from the graph that $P(1960) \approx 5$ and $P(2000) \approx 2.6$ (see the orange line in *Figure 4a*). The first equation tells us that in 1960 there were approximately 5 million bees in the US. The second equation indicates that in the year 2000 there were approximately 2.6 million bees in the US.

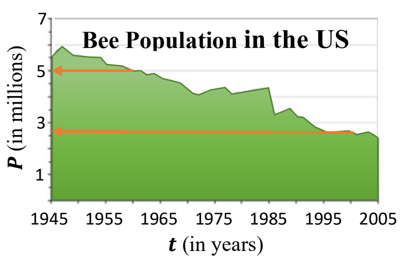


Figure 4a

b. The average rate of change is represented by the slope of a straight line between $(1960, 5)$ and $(2000, 2.6)$. Since the change in bee population over the years $1960 - 2000$ is $2.6 - 5 = -2.4$ million, and the change in time $1960 - 2000 = 40$ years, then the slope is $-\frac{2.4}{40} = -0.06$ million per year. This means that, in the US, the population of bees decreased an average of 60,000 each year between 1960 and 2000.

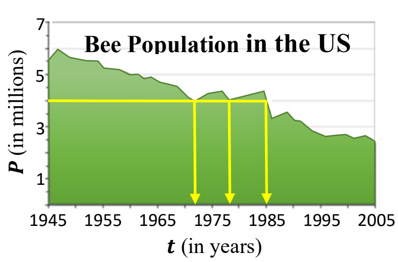


Figure 4b

c. As indicated by yellow arrows in *Figure 4b*, $P(t) = 4$ for $t \approx 1972$, $t \approx 1978$, and $t \approx 1985$.

d. The general tendency of function $P(t)$ over the years $1945 - 2005$ is declining.

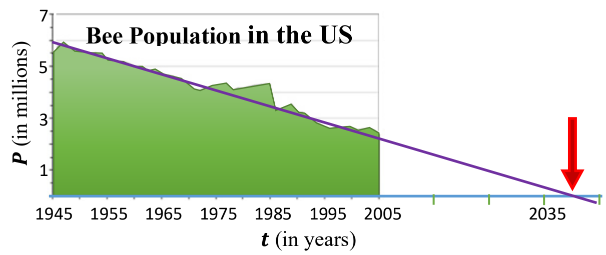
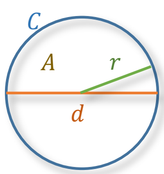


Figure 4c

e. Assuming the same declining tendency, to estimate the year in which the bee population in the US will disappear, we extend the t-axis and the approximate line of tendency (see the purple line in *Figure 4c*) to see where they intersect. After extending of the scale on the t-axis, we predict that the bee population will disappear around the year 2040.

Example 6 ► **Constructing Functions**



Consider a circle with area A, circumference C, radius r, and diameter d.

a. Write A as a function of r.
b. Write r as a function of d.
c. Write A as a function of d.
d. Write r as a function of C.
e. Write A as a function of C.

Solution ►

a. Using the formula for the area of a circle, $A = \pi r^2$, the function A of r is $\boldsymbol{A(r) = \pi r^2}$.

b. To express r as a function of d, we solve the formula $d = 2r$ for r. This gives us $r = \frac{d}{2}$. So, the function r of d is $\boldsymbol{r(d) = \frac{d}{2}}$.

c. To write A as a function of d, we start by connecting the formula for the area A in terms of r and the formula that expresses r in terms of d. Since

$$A = \pi r^2 \quad \text{and} \quad r = \frac{d}{2},$$

then using substitution, we have

$$A = \pi r^2 = \pi \cdot \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}.$$

Hence, our function A of d is $\boldsymbol{A(d) = \frac{1}{4}\pi d^2}$.

d. The relation between circumference C and radius r is $C = 2\pi r$. After solving this formula for r, we have $r = \frac{C}{2\pi}$. So, our function is $\boldsymbol{r(C) = \frac{C}{2\pi}}$.

e. To write A as a function of C, we use the formula $r = \frac{C}{2\pi}$ to replace r in the area formula $A = \pi r^2$ by the expression $\frac{C}{2\pi}$. This gives us

$$A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{\cancel{\pi} C^2}{4\pi^{\cancel{2}}} = \frac{C^2}{4\pi}.$$

So, our function is $\boldsymbol{A(C) = \frac{C^2}{4\pi}}$.

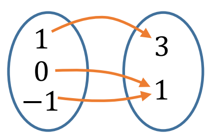
G.6 Exercises

*For each function, find **a)** $f(-1)$ and **b)** all x-values such that $f(x) = 1$.*

1. $\{(2,4), (-1,2), (3,1)\}$

2. $\{(-1,1), (1,2), (2,1)\}$

3.



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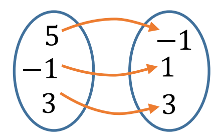


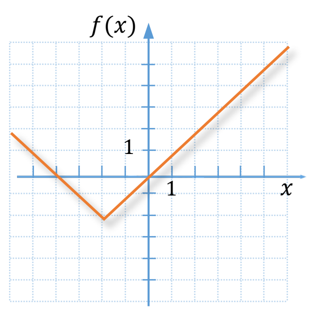
5.

x	$f(x)$
−1	4
0	2
2	1
4	−1

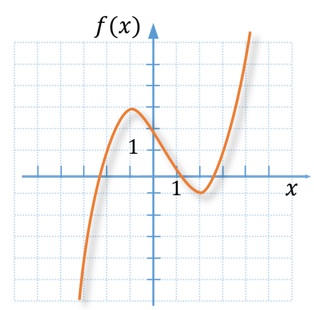
6.

x	$f(x)$
−3	1
−1	2
1	2
3	1

7.



8.



Let $f(x) = -3x + 5$ *and* $g(x) = -x^2 + 2x - 1$. *Find the following.*

9. $f(1)$

10. $g(0)$

11. $g(-1)$

12. $f(-2)$

13. $f(p)$

14. $g(a)$

15. $g(-x)$

16. $f(-x)$

17. $f(a+1)$

18. $g(a+2)$

19. $g(x-1)$

20. $f(x-2)$

21. $f(2+h)$

22. $g(1+h)$

23. $g(a+h)$

24. $f(a+h)$

25. $f(3) - g(3)$

26. $g(a) - f(a)$

27. $3g(x) + f(x)$

28. $f(x+h) - f(x)$

Fill in each blank.

29. The graph of the equation $2x + y = 6$ is a ___________. The point (1,___) lies on the graph of this line. Using function notation, the above equation can be written as $f(x) =$ __________ . Since $f(1) =$ _____ , the point (___ ,___) lies on the graph of function f.

Graph each function. Give the domain and range.

30. $f(x) = -2x + 5$

31. $g(x) = \frac{1}{3}x + 2$

32. $h(x) = -3x$

33. $F(x) = x$

34. $G(x) = 0$

35. $H(x) = 2$

36. $x - h(x) = 4$

37. $-3x + f(x) = -5$

38. $2 \cdot g(x) - 2 = x$

39. $k(x) = |x - 3|$

40. $m(x) = 3 - |x|$

41. $q(x) = x^2$

42. $Q(x) = x^2 - 2x$

43. $p(x) = x^3 + 1$

44. $s(x) = \sqrt{x}$

Solve each problem.

45. A taxi driver charges $1.50 per kilometer.

a. Complete the table by writting the charge $f(x)$ for a trip of x kilometers.

b. Find the linear function that calculates the charge $f(x) =$ ________ for a trip of x kilometers.

c. Graph $f(x)$ for the domain $\{0, 2, 4\}$.

x	$f(x)$
0	
2	
4	

46. Given the information about the linear function f, find the following:

a. $f(1)$

b. x-value such that $f(x) = -0.4$

c. slope of f

d. y-intercept of f

e. an equation for $f(x)$

x	$f(x)$
-2	3.2
-1	2.3
0	1.4
1	0.5
2	-0.4
3	-1.3

47. Suppose the cost of renting a car at Los Angeles International Airport consists of the initial fee of $18.80 and $24.60 per day. Let $C(d)$ represent the total cost of renting the car for d days.

a. Write a linear function that models this situation.

b. Find $C(4)$ and interpret your answer in the context of the problem.

c. Find the value of d satisfying the equation $C(d) = 191$ and interpret it in the context of this problem.

48. Suppose a house cleaning service charges $20 per visit plus $32 per hour.

a. Express the total charge, C, as a function of the number of hours worked, n.

b. Find $C(3)$ and interpret your answer in the context of this problem.

c. If Stacy was charged $244 for a one-visit work, how long it took to clean her house?

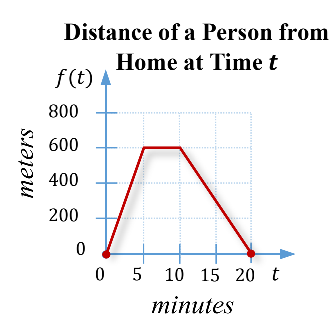
49. *Refer to the given graph of function f to answer the questions below.*

a. What is the range of possible values for the independent variable? What is the range of possible values for the independent variable?

b. For how long is the person going away from home? Coming closer to home?

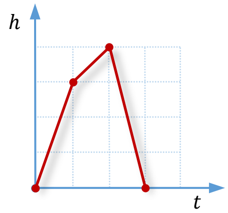
c. How far away from home is the person after 10 minutes?

d. Call this function f. What is $f(15)$ and what does this mean in the context of the problem?

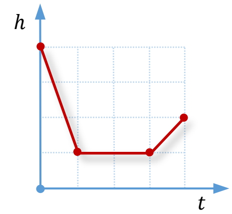


Questions 51 and 52 show graphs of the height of water in a bathtub. The t-axis represents time, and the h-axis represents height. Interpret the graph by describing the rate of change of the height of water in the bathtub.

50.



51.



52. Consider a square with area A, side s, perimeter P, and diagonal d.

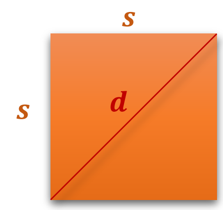
a. Write A as a function of s.

b. Write s as a function of P.

c. Write A as a function of P.

d. Write A as a function of d.

(*Hint*: in part (**d**) apply the Pythagorean equation $a^2 + b^2 = c^2$, where c is the hypotenuse of a right angle triangle with arms a and b.)



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