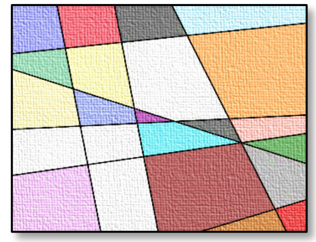
Systems of Linear Equations



As stated in *Section G1, Definition 1.1*, a linear equation in two variables is an equation of the form $Ax + By = C$, where A and B are not both zero. Such an equation has a line as its graph. Each point of this line is a solution to the equation in the sense that the coordinates of such a point satisfy the equation. So, there are infinitely many ordered pairs (x, y) satisfying the equation. Although analysis of the relation between x and y is instrumental in some problems, many application problems call for a particular, single solution. This occurs when, for example, x and y are required to satisfy an additional linear equation whose graph intersects the original line. In such a case, the solution to both equations is the point at which the lines intersect. Generally, to find unique values for **two** given **variables**, we need a system of **two equations** in these variables. In this section, we discuss several methods for solving systems of two linear equations.

E1 Systems of Linear Equations in Two Variables

Any collection of equations considered together is called a **system of equations**. For example, a system consisting of two equations, $x + y = 5$ and $4x - y = 10$, is written as

$$\begin{cases} x + y = 5 \\ 4x - y = 10 \end{cases}$$

Since the equations in the system are linear, the system is called a **linear system of equations**.

Definition 1.1 ► A **solution** of a system of two equations in two variables, x and y, is any ordered pair (x, y) satisfying both equations of the system.

A **solution set** of a system of two linear equations in two variables, x and y, is the set of all possible solutions (x, y).

Note: The two variables used in a system of two equations can be denoted by any two different letters. In such case, to construct an ordered pair, we follow an alphabetical order. For example, if the variables are p and q, the corresponding ordered pair is (p, q), as p appears in the alphabet before q. This also means that a corresponding system of coordinates has the horizontal axis denoted as p-axis and the vertical axis denoted as q-axis.

Example 1 ► **Deciding Whether an Ordered Pair Is a Solution**

Decide whether the ordered pair $(3, 2)$ is a solution of the given system.

a. $\begin{cases} x + y = 5 \\ 4x - y = 10 \end{cases}$ **b.** $\begin{cases} m + 2n = 7 \\ 3m - n = 6 \end{cases}$

Solution ► **a.** To check whether the pair $(3, 2)$ is a solution, we let $x = 3$ and $y = 2$ in both equations of the system and check whether these equations are true. Since both equations,

$$3 + 2 = 5 \qquad\qquad 4 \cdot 3 - 2 = 10$$
$$5 = 5 \; ✓ \qquad \text{and} \qquad 10 = 10, \; ✓$$

are true, then the pair $(3, 2)$ is a solution to the system.

b. First, we notice that alphabetically, m is before n. So, we let $m = 3$ and $n = 2$ and substitute these values into both equations.

$3 + 2 \cdot 2 = 7$ $3 \cdot 3 - 2 = 6$

$7 = 7$ ✓   but $7 = 6$ ✗

Since the pair $(3, 2)$ is not a solution of the second equation, it is not a solution to the whole system.

Solving Systems of Linear Equations by Graphing

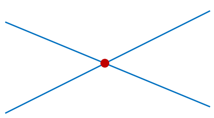


Figure 1a

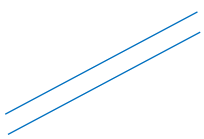


Figure 1b

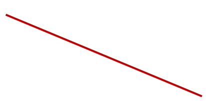


Figure 1c

Solutions to a system of two linear equations are all the ordered pairs that satisfy both equations. If an ordered pair satisfies an equation, then such a pair belongs to the graph of this equation. This means that the solutions to a system of two linear equations are the points that belong to both graphs of these lines. So, to solve such system, we can graph each line and take the common points as solutions.

How many solutions can a linear system of two equations have?

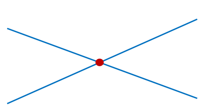
There are three possible arrangements of two lines in a plane. The lines can **intersect** each other, be **parallel**, or be the **same**.

1. If a system of equations corresponds to a pair of **intersecting** lines, it has exactly **one solution**. The solution set consists of the **intersection point**, as shown in *Figure 1a*.
2. If a system of equations corresponds to a pair of **parallel** lines, it has **no solutions**. The solution set is empty, as shown in *Figure 1b*.
3. If a system of equations corresponds to a pair of the **same** lines, it has **infinitely many solutions**. The solution set consists of all the **points of the line**, as shown in *Figure 1c*.

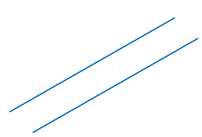
Definition 1.2 ► A linear system is called **consistent** if it has **at least one solution**. Otherwise, the system is **inconsistent**.

A linear system of two equations is called **independent** if the two lines are different. Otherwise, the system is **dependent**.

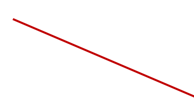
Here is the classification of systems corresponding to the following graphs:



consistent
independent



inconsistent
independent



consistent
dependent

Example 2 ► **Solving Systems of Linear Equations by Graphing**

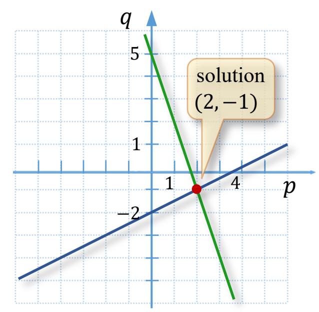
Solve each system by graphing and classify it as *consistent or inconsistent* and *dependent or independent*.

a. $\begin{cases} 3p + q = 5 \\ p - 2q = 4 \end{cases}$ **b.** $\begin{cases} 3y - 2x = 6 \\ 4x - 6y = -12 \end{cases}$ **c.** $\begin{cases} f(x) = -\frac{1}{2}x + 3 \\ 2g(x) + x = -4 \end{cases}$

Solution ► **a.** To graph the first equation, it is convenient to use the slope-intercept form,

$$q = -3p + 5.$$

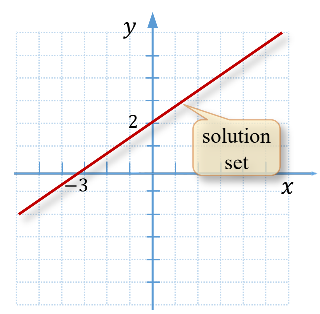
To graph the second equation, it is convenient to use the p- and q-intercepts, $(4, 0)$ and $(0, -2)$.



The first equation is graphed in green and the second – in blue. The intersection point is at $(\mathbf{2}, \mathbf{-1})$, which is the **only solution** of the system.

The system is **consistent**, as it has a solution, and **independent**, as the lines are different.

b. Notice that when using the x- and y-intercept method of graphing, both equations have x-intercepts equal to $(-3, 0)$ and y-intercepts equal to $(0, 2)$. So, both equations represent the same line. Therefore the solution set to this system consists of all points of the line $3y - 2x = 6$. We can record this set of points with the use of set-builder notation as $\{(\boldsymbol{x}, \boldsymbol{y}) | \mathbf{3}\boldsymbol{y} - \mathbf{2}\boldsymbol{x} = \mathbf{6}\}$, and state that the system has **infinitely many solutions**.

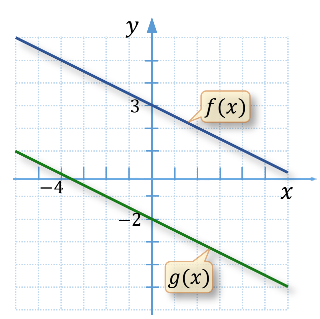


The system is **consistent**, as it has solutions, and **dependent**, as both lines are the same.

c. We plan to graph both functions, f and g, on the same grid. Function f is already given in the slope-intercept form, which is convenient for graphing. To graph function g, we can either use the x- and y-intercept method or solve the equation for $g(x)$ and use the slope-intercept method. So, we have

$$2g(x) = -x - 4$$

$$g(x) = -\frac{1}{2}x - 2$$



After graphing both functions, we observe that the two lines are parallel because they have the same slope. Having different y-intercepts, the lines do not have any common points. Therefore, the system has **no solutions**.

Such a system is **inconsistent**, as it has no solutions, and **independent**, as the lines are different.

Solving a system of equations by graphing, although useful, is not always a reliable method. For example, if the solution is an ordered pair of fractional numbers, we may have a hard time to read the exact values of these numbers from the graph. Luckily, a system of equations can be solved for exact values using algebra. Below, two algebraic methods for solving systems of two equations, referred to as substitution and elimination, are shown.

Solving Systems of Linear Equations by the Substitution Method

In the substitution method, as shown in *Example 3* below, we eliminate a variable from one equation by substituting an expression for that variable from the other equation. This method is particularly suitable for solving systems in which one variable is either already isolated or is easy enough to isolate.

Example 3 ► **Solving Systems of Linear Equations by Substitution**

Solve each system by substitution.

a. $\begin{cases} x = y + 1 \\ x + 2y = 4 \end{cases}$ **b.** $\begin{cases} 3a - 2b = 6 \\ 6a + 4b = -20 \end{cases}$

Solution ►

a. Since x is already isolated in the first equation, $x = y + 1$, we replace x by $y + 1$ in the second equation, $x + 2y = 4$. Thus,

$$(y + 1) + 2y = 4$$

which after solving for y, gives us

$$3y = 3$$
$$\boldsymbol{y} = \mathbf{1}$$

Then, we substitute the value $y = 1$ back into the first equation, and solve for x. This gives us $\boldsymbol{x} = 1 + 1 = \mathbf{2}$.

One can check that $x = 2$ and $y = 1$ satisfy both of the original equations. So, the solution set of this system is $\{(\mathbf{2}, \mathbf{1})\}$.

b. To use the substitution method, we need to solve one of the equations for one of the variables, whichever is easier. Out of the coefficients by the variables, -2 seems to be the easiest coefficient to work with. So, let us solve the first equation, $3a - 2b = 6$, for b.

$$3a - 2b = 6$$
$$-2b = -3a + 6$$
$$b = \frac{3}{2}a - 3 \qquad (*)$$

substitution equation

Then, substitute the expression $\frac{3}{2}a - 3$ to the second equation, $6a + 4b = -20$, for b. So, we obtain

$$6b + 4\left(\frac{3}{2}a - 3\right) = -20,$$

which can be solved for a:

$$6a + 6a - 12 = -20$$
$$12a = -8$$

equation in one variable

$$a = -\frac{8}{12}$$
$$\boldsymbol{a} = -\frac{2}{3}$$

Then, we plug $a = -\frac{2}{3}$ back into the substitution equation $(*)$ $b = \frac{3}{2}a - 3$ to find the b-value. This gives us

$$\boldsymbol{b} = \frac{3}{2}\left(-\frac{2}{3}\right) - 3 = -1 - 3 = \boldsymbol{-4}$$

To check that the values $a = -\frac{2}{3}$ and $b = -4$ satisfy both equations of the system, we substitute them into each equation, and simplify each side. Since both equations,

$$3\left(-\frac{2}{3}\right) - 2(-4) = 6$$
$$-2 + 8 = 6$$
$$6 = 6 \quad ✓$$

and

$$6\left(-\frac{2}{3}\right) + 4(-4) = -20$$
$$-4 - 16 = -20$$
$$-20 = -20, \quad ✓$$

are satisfied, the solution set of this system is $\left\{\left(-\frac{2}{3}, -4\right)\right\}$.

Summary of Solving Systems of Linear Equations by Substitution

Step 1 **Solve one of the equations for one of the variables.** Choose to solve for the variable with the easiest coefficient to work with. The obtained equation will be referred to as the **substitution equation** $(*)$.

Step 2 **Plug the substitution equation into the other equation.** The result should be an equation with just one variable.

Step 3 **Solve** the resulting equation to find the value of the variable.

Step 4 **Find the value of the other variable** by substituting the result from Step 3 into the substitution equation from Step 1.

Step 5 **Check** if the variable values satisfy both of the original equations. Then **state the solution set** by listing the ordered pair(s) of numbers.

Solving Systems of Linear Equations by the Elimination Method

Another algebraic method, the **elimination method**, involves combining the two equations in a system so that one variable is eliminated. This is done using the addition property of equations.

Recall: If $\boldsymbol{a = b}$ and $\boldsymbol{c = d}$, then $\boldsymbol{a + c = b + d}$.

Example 4 ► **Solving Systems of Linear Equations by Elimination**

Solve each system by elimination.

a. $\begin{cases} r + 2s = 3 \\ 3r - 2s = 5 \end{cases}$

b. $\begin{cases} 2x + 3y = 6 \\ 3x + 5y = -2 \end{cases}$

Solution ► **a.** Notice that the equations contain opposite terms, $2s$ and $-2s$. Therefore, if we add these equations, side by side, the s-variable will be eliminated. So, we obtain

$$+\begin{cases} r + 2s = 3 \\ 3r - 2s = 5 \end{cases}$$

$$4r = 8$$
$$r = 2$$

Now, since the r-value is already known, we can substitute it to one of the equations of the system to find the s-value. Using the first equation, we obtain

$$2 + 2s = 3$$
$$2s = 1$$
$$s = \frac{1}{2}$$

One can check that the values $r = 2$ and $s = \frac{1}{2}$ make both equations of the original system true. Therefore, the pair $\left(\mathbf{2}, \frac{1}{2}\right)$ is the solution of this system. We say that the solution set is $\left\{\left(\mathbf{2}, \frac{1}{2}\right)\right\}$.

b. First, we choose which variable to eliminate. Suppose we plan to remove the x-variable. To do this, we need to transform the equations in such a way that the coefficients in the x-terms become opposite. This can be achieved by multiplying, for example, the first equation by 3 and the second equation, by -2.

$$\begin{cases} 2x + 3y = 6 \\ 3x + 5y = -2 \end{cases}$$

multiply 1st eq by 3
multiply 2nd eq by -2

$$+\begin{cases} 6x + 9y = 18 \\ -6x - 10y = 4 \end{cases}$$

Then, we add the two equations, side by side,

$$-y = 22$$

and solve the resulting equation for y,

$$y = -22.$$

To find the x-value, we substitute $y = -22$ to one of the original equations. Using the first equation, we obtain

$$2x + 3(-22) = 6$$
$$2x - 66 = 6$$
$$2x = 72$$
$$x = 36$$

One can check that the values $x = 36$ and $y = -22$ make both equations of the original system true. Therefore, the solution of this system is the pair $(\mathbf{36}, \mathbf{-22})$. We say that the solution set is $\{(\mathbf{36}, \mathbf{-22})\}$.

Summary of Solving Systems of Linear Equations by Elimination

- **Write both equations in standard form $Ax + By = C$.** Keep A and B as integers by clearing any fractions, if needed.
- **Choose a variable to eliminate.**
- **Make the chosen variable's terms opposites** by multiplying one or both equations by appropriate numbers if necessary.
- **Eliminate a variable by adding the respective sides of the equations** and then solve for the remaining variable.
- **Find the value of the other variable** by substituting the result from Step 4 into either of the original equations and solve for the other variable.
- **Check** if the variable values satisfy both of the original equations. Then **state the solution set** by listing the ordered pair(s) of numbers.

Comparing Methods of Solving Systems of Equations

When deciding which method to use, consider the suggestions in the table below.

Method	Strengths	Weaknesses
Graphical	• **Visualization.** The solutions can be "seen" and approximated.	• **Inaccuracy.** When solutions involve numbers that are not integers, they can only be approximated. • **Grid limitations.** Solutions may not appear on the part of the graph drawn.
Substitution	• **Exact solutions.** • Most convenient to use when a variable has a **coefficient of 1**.	• **Computations.** Often requires extensive computations with fractions.
Elimination	• **Exact solutions.** • Most convenient to use when all **coefficients** by variables are **different than 1**.	• **Preparation.** The method requires that the coefficients by one of the variables are opposite.

Solving Systems of Linear Equations in Special Cases

As it was shown in solving linear systems of equations by graphing, some systems have no solution or infinitely many solutions. The next example demonstrates how to solve such systems algebraically.

Example 5 ► Solving Inconsistent or Dependent Systems of Linear Equations

Solve each system algebraically.

a. $\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$ **b.** $\begin{cases} 2x - y = 3 \\ 6x - 3y = 9 \end{cases}$

Solution ►

a. When trying to eliminate one of the variables, we might want to multiply the first equation by 2. This, however, causes both variables to be eliminated, resulting in

$$+\begin{cases} 2x + 6y = 8 \\ -2x - 6y = 3 \end{cases}$$

$$0 = 11,$$

parallel lines

which is *never true*. This means that there is no ordered pair (x, y) that would make this equation true. Therefore, there is **no solution** to this system. The solution set is $\emptyset$. The system is **inconsistent**, so the equations must describe **parallel lines**.

b. When trying to eliminate one of the variables, we might want to multiply the first equation by -3. This, however, causes both variables to be eliminated and we obtain

$$+\begin{cases} -6x + 3y = -9 \\ 6x - 3y = 9 \end{cases}$$

$$0 = 0,$$

same line

which is *always true*. This means that any x-value together with its corresponding y-value satisfy the system. Therefore, there are **infinitely many solutions** to this system. These solutions are all points of one of the equations. Therefore, the solution set can be recorded in set-builder notation, as

$$\{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{2x - y = 3}\}$$

Read: the set of all ordered pairs (x, y), such that $2x - y = 3$

Since the equations of the system are equivalent, they represent the same line. So, the system is **dependent**.

Summary of Special Cases of Linear Systems

If both variables are eliminated when solving a linear system of two equations, then the solution sets are determined as follows.

Case 1 If the resulting statement is **true**, there are **infinitely many solutions**. The system is ***consistent***, and the equations are ***dependent***.

Case 2 If the resulting statement is **false**, there is **no solution**. The system is ***inconsistent***, and the equations are ***independent***.

Another way of determining whether a system of two linear equations is inconsistent or dependent is by examining slopes and y-intercepts in the two equations.

Example 6 ► **Using Slope-Intercept Form to Determine the Number of Solutions and the Type of System**

For each system, determine the number of solutions and classify the system without actually solving it.

a. $\begin{cases} \frac{1}{2}x = \frac{1}{8}y + \frac{1}{4} \\ 4x - y = -2 \end{cases}$ **b.** $\begin{cases} 2x + 5y = 6 \\ 0.4x + y = 1.2 \end{cases}$

Solution ► **a.** First, let us clear the fractions in the first equation by multiplying it by 8,

$$\begin{cases} 4x = y + 2 \\ 4x - y = -2 \end{cases},$$

and then solve each equation for y.

parallel lines

$$\begin{cases} 4x - 2 = y \\ 4x + 2 = y \end{cases}$$

Then, observe that the slopes in both equations are the same and equal to 4, but the y-intercepts are different, -2 and 2. The same slopes tell us that the corresponding lines are **parallel** while different y-intercepts tell us that the two lines are **different**. So, the system has **no solution**, which means it is **inconsistent**, and the lines are **independent**.

b. We will start by solving each equation for y. So, we have

$$\begin{cases} 2x + 5y = 6 \\ 0.4x + y = 1.2 \end{cases}$$

$$\begin{cases} 5y = -2x + 6 \\ y = -0.4x + 1.2 \end{cases}$$

same line

$$\begin{cases} y = -\frac{2}{5}x + \frac{6}{5} \\ y = -0.4x + 1.2 \end{cases}$$

Notice that $-\frac{2}{5} = -0.4$ and $\frac{6}{5} = 1.2$. Since the resulting equations have the same slopes and the same y-intercepts, they represent the same line. Therefore, the system has **infinitely many solutions**, which means it is **consistent**, and the lines are **dependent**.

E.1 Exercises

1. Describe the graph of a system of equations that has no solution.

Decide whether the given ordered pair is a solution of the given system.

2. $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$; $(5, 2)$

3. $\begin{cases} x + y = 1 \\ 2x - 3y = -8 \end{cases}$; $(-1, 2)$

4. $\begin{cases} p + 3q = 1 \\ 5p - q = -9 \end{cases}$; $(-2, 1)$

5. $\begin{cases} 2a + b = 3 \\ a - 2b = -9 \end{cases}$; $(-1, 5)$

Solve each system of equations ***graphically****. Then, classify the system as* ***consistent*** *or* ***inconsistent*** *and* ***dependent*** *or* ***independent****.*

6. $\begin{cases} 3x + y = 5 \\ x - 2y = 4 \end{cases}$

7. $\begin{cases} 3x + 4y = 8 \\ x + 2y = 6 \end{cases}$

8. $\begin{cases} f(x) = x - 1 \\ g(x) = -2x + 5 \end{cases}$

9. $\begin{cases} f(x) = -\frac{1}{4}x + 1 \\ g(x) = \frac{1}{2}x - 2 \end{cases}$

10. $\begin{cases} y - x = 5 \\ 2x - 2y = 10 \end{cases}$

11. $\begin{cases} 6x - 2y = 2 \\ 9x - 3y = -1 \end{cases}$

12. $\begin{cases} y = 3 - x \\ 2x + 2y = 6 \end{cases}$

13. $\begin{cases} 2x - 3y = 6 \\ 3y - 2x = -6 \end{cases}$

14. $\begin{cases} 2u + v = 3 \\ 2u = v + 7 \end{cases}$

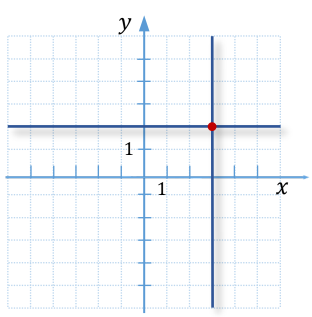
15. $\begin{cases} 2b = 6 - a \\ 3a - 2b = 6 \end{cases}$

16. $\begin{cases} f(x) = 2 \\ x = -3 \end{cases}$

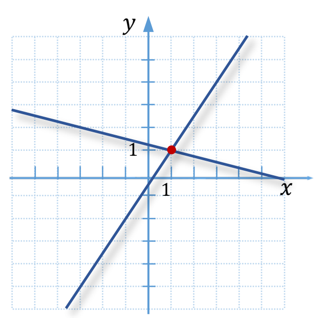
17. $\begin{cases} f(x) = x \\ g(x) = -1.5 \end{cases}$

18. Classify each system **I** to **VI** as *consistent* or *inconsistent* and the equations as *dependent* or *independent*. Then, match it with the corresponding system of equations **A** to **F**.

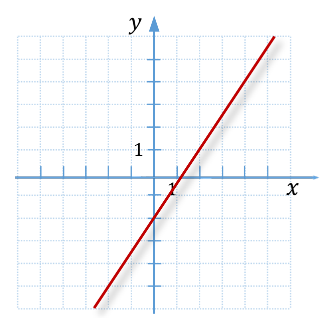
I



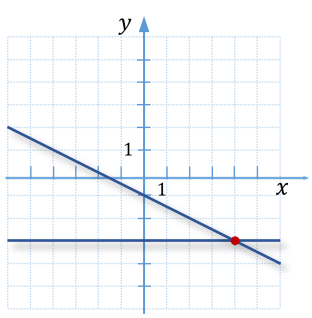
II



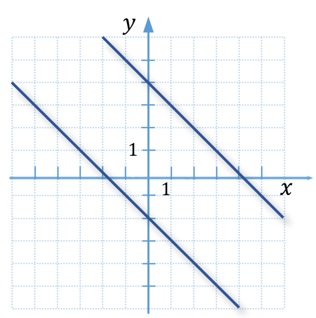
III



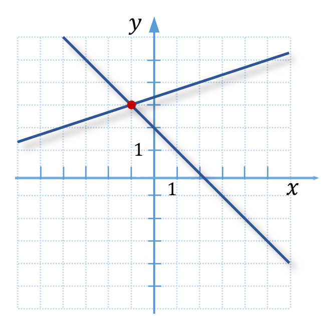
IV



V



VI



A $\begin{cases} 3y - x = 10 \\ x = -y + 2 \end{cases}$

B $\begin{cases} 9x - 6y = 12 \\ y = \frac{3}{2}x - 2 \end{cases}$

C $\begin{cases} 2y - 3x = -1 \\ x + 4y = 5 \end{cases}$

D $\begin{cases} x + y = 4 \\ y = -x - 2 \end{cases}$

E $\begin{cases} \frac{1}{2}x + y = -1 \\ y = -3 \end{cases}$

F $\begin{cases} x = 3 \\ y = 2 \end{cases}$

Solve each system by ***substitution****. If the system describes* ***parallel lines*** *or the* ***same line****, say so.*

19. $\begin{cases} y = 2x + 1 \\ 3x - 4y = 1 \end{cases}$

20. $\begin{cases} 5x - 6y = 23 \\ x = 6 - 3y \end{cases}$

21. $\begin{cases} x + 2y = 3 \\ 2x + y = 5 \end{cases}$

22. $\begin{cases} 2y = 1 - 4x \\ 2x + y = 0 \end{cases}$

23. $\begin{cases} y = 4 - 2x \\ y + 2x = 6 \end{cases}$

24. $\begin{cases} y - 2x = 3 \\ 4x - 2y = -6 \end{cases}$

25. $\begin{cases} 4s - 2t = 18 \\ 3s + 5t = 20 \end{cases}$

26. $\begin{cases} 4p + 2q = 8 \\ 5p - 7q = 1 \end{cases}$

27. $\begin{cases} \frac{x}{2} + \frac{y}{2} = 5 \\ \frac{3x}{2} - \frac{2y}{3} = 2 \end{cases}$

28. $\begin{cases} \frac{x}{4} + \frac{y}{3} = 0 \\ \frac{x}{8} - \frac{y}{6} = 2 \end{cases}$

29. $\begin{cases} 1.5a - 0.5b = 8.5 \\ 3a + 1.5b = 6 \end{cases}$

30. $\begin{cases} 0.3u - 2.4v = -2.1 \\ 0.04u + 0.03v = 0.7 \end{cases}$

Solve each system by ***elimination****. If the system describes* ***parallel lines*** *or the* ***same line****, say so.*

31. $\begin{cases} x + y = 20 \\ x - y = 4 \end{cases}$

32. $\begin{cases} 6x + 5y = -7 \\ -6x - 11y = 1 \end{cases}$

33. $\begin{cases} x - y = 5 \\ 3x + 2y = 10 \end{cases}$

34. $\begin{cases} x - 4y = -3 \\ -3x + 5y = 2 \end{cases}$

35. $\begin{cases} 2x + 3y = 1 \\ 3x - 5y = -8 \end{cases}$

36. $\begin{cases} -2x + 5y = 14 \\ 7x + 6y = -2 \end{cases}$

37. $\begin{cases} 2x + 3y = 1 \\ 4x + 6y = 2 \end{cases}$

38. $\begin{cases} 6x - 10y = -4 \\ 5y - 3x = 7 \end{cases}$

39. $\begin{cases} 0.3x - 0.2y = 4 \\ 0.2x + 0.3y = 1 \end{cases}$

40. $\begin{cases} \frac{2}{3}x + \frac{1}{7}y = -11 \\ \frac{1}{7}x - \frac{1}{3}y = -10 \end{cases}$

41. $\begin{cases} 3a + 2b = 3 \\ 9a - 8b = -2 \end{cases}$

42. $\begin{cases} 5m - 9n = 7 \\ 7n - 3m = -5 \end{cases}$

43. Can a linear system of two equations has exactly two solutions? Justify your answer.

Write each equation in ***slope-intercept form*** *and then tell how many solutions the system has. Do not actually solve.*

44. $\begin{cases} -x + 2y = 8 \\ 4x - 8y = 1 \end{cases}$

45. $\begin{cases} 6x = -9y + 3 \\ 2x = -3y + 1 \end{cases}$

46. $\begin{cases} y - x = 6 \\ x + y = 6 \end{cases}$

Solve each system by the method of your choice.

47. $\begin{cases} 3x + y = -7 \\ x - y = -5 \end{cases}$

48. $\begin{cases} 3x - 2y = 0 \\ 9x + 8y = 7 \end{cases}$

49. $\begin{cases} 3x - 5y = 7 \\ 2x + 3y = 30 \end{cases}$

50. $\begin{cases} 2x + 3y = 10 \\ -3x + y = 18 \end{cases}$

51. $\begin{cases} \frac{1}{6}x + \frac{1}{3}y = 8 \\ \frac{1}{4}x + \frac{1}{2}y = 30 \end{cases}$

52. $\begin{cases} \frac{1}{2}x - \frac{1}{8}y = -\frac{1}{2} \\ 4x - y = -2 \end{cases}$

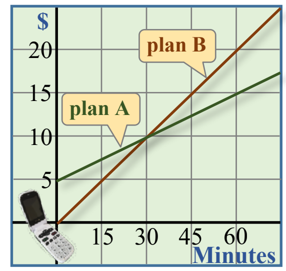
53. $\begin{cases} a + 4b = 2 \\ 5a - b = 3 \end{cases}$ **54.** $\begin{cases} 3a - b = 7 \\ 2a + 2b = 5 \end{cases}$ **55.** $\begin{cases} 6 \cdot f(x) = 2x \\ -7x + 15 \cdot g(x) = 10 \end{cases}$

Solve the system of linear equations. Assume that ***a*** *and* ***b*** *represent nonzero constants.*

56. $\begin{cases} x + ay = 1 \\ 2x + 2ay = 4 \end{cases}$ **57.** $\begin{cases} -ax + y = 4 \\ ax + y = 4 \end{cases}$ **58.** $\begin{cases} -ax + y = 2 \\ ax + y = 4 \end{cases}$

59. $\begin{cases} ax + by = 2 \\ -ax + 2by = 1 \end{cases}$ **60.** $\begin{cases} 2ax - y = 3 \\ y = 5ax \end{cases}$ **61.** $\begin{cases} 3ax + 2y = 1 \\ -ax + y = 2 \end{cases}$

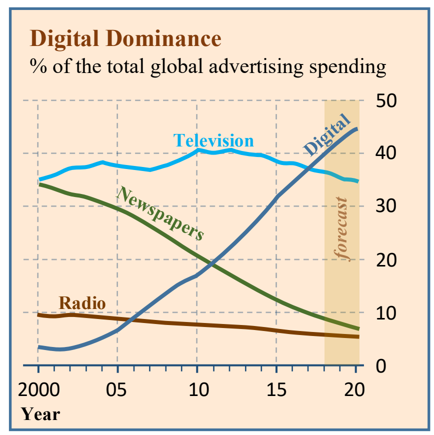
Refer to the accompanying graph to answer questions 72-73.



62. According to the graph, for how many long-distance minutes the charge would be the same in plan A as in plan B? Give an ordered pair of the form (minutes, dollars) to represent this situation.

63. For what range of long-distance minutes would plan B be cheaper?

Refer to the accompanying graph to answer questions 74-77.



64. According to the predictions shown in the graph, what percent of the global advertising spending will be allocated to digital advertising in 2020? Give an ordered pair of the form (year, percent) to represent this information.

65. Estimate the year in which spending on digital advertising matches spending on television advertising.

66. Since when did spending on digital advertising exceed spending on advertising in newspapers? What was this spending as a percentage of global advertising spending at that time?

67. Since when did spending on digital advertising exceed spending on radio advertising? What was this spending as a percentage of global advertising spending at that time?

E2 Applications of Systems of Linear Equations in Two Variables

Systems of equations are frequently used in solving applied problems. Although many problems with two unknowns can be solved with the use of a single equation with one variable, it is often easier to translate the information given in an application problem with two unknowns into two equations in two variables.

Here are some guidelines to follow when solving applied problems with two variables.

Solving Applied Problems with the Use of System of Equations

- **Read** the problem, several times if necessary. When reading, watch the given information and what the problem asks for. Recognize the type of problem, such as geometry, total value, motion, solution, percent, investment, etc.
- **Assign variables** for the unknown quantities. Use meaningful letters, if possible.
- **Organize** the given information. Draw appropriate tables or diagrams; list relevant formulas.
- **Write equations** by following a relevant formula(s) or a common sense pattern.
- **Solve** the system of equations.
- **Check** if the solution is reasonable in the context of the problem.
- **State the answer** to the problem.

Below we show examples of several types of applied problems that can be solved with the aid of systems of equations.

Number Relations Problems

Example 1 ► **Finding Unknown Numbers**

The difference between twice a number and a second number is 3. The sum of the two numbers is 18. Find the two numbers.

Solution ► Let a be the first number and b be the second number. The first sentence of the problem translates into the equation

$$2a - b = 3.$$

The second sentence translates to

$$a + b = 18.$$

Now, we can solve the system of the above equations, using the elimination method. Since

$$+\begin{cases} 2a - b = 3 \\ a + b = 18 \end{cases}$$
$$3a = 21$$
$$a = \mathbf{7},$$

then $b = 18 - a = 18 - 7 = \mathbf{11}$. Therefore, the two numbers are **7** and **11**.

Observation: *A single equation in two variables gives us infinitely many solutions. For example, some of the solutions of the equation* $a + b = 16$ *are* $(0, 16), (1, 15), (2, 14)$, *and so on. Generally, any ordered pair of the type* $(a, 16 - a)$ *is the solution to this equation. So, when working with two variables, to find a specific solution we are in need of a second equation (not equivalent to the first) that relates these variables. This is why problems with two unknowns are solved with the use of systems of two equations.*

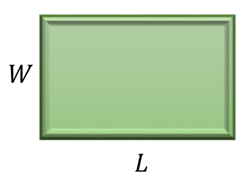
Geometry Problems

When working with geometry problems, we often use formulas for perimeter, area, or volume of basic figures. Sometimes, we rely on particular properties or theorems, such as *the sum of angles in a triangle is* $180°$ or *the ratios of corresponding sides of similar triangles are equal*.

Example 2 ► **Finding Dimensions of a Rectangle**

Pat plans a rectangular vegetable garden. The width of the rectangle is to be 5 meters shorter than the length. If the perimeter is planned to be 34 meters, what will the dimensions of the garden be?

Solution ►



The problem refers to the perimeter of a rectangular garden. Suppose L and W represent the length and width of the rectangle. Then the perimeter is represented by the expression $2L + 2W$. Since the perimeter of the garden should equal 34 meters, we set up the first equation

$$2L + 2W = 34 \qquad (1)$$

The second equation comes directly from translating the second sentence of the problem, which tells us that the width is to be 5 meters shorter than the length. So, we write

$$W = L - 5 \qquad (2)$$

Now, we can solve the system of the above equations, using the substitution method. After substituting equation (2) into equation (1), we obtain

$$2L + 2(L - 5) = 34$$
$$2L + 2L - 10 = 34$$
$$4L = 44$$
$$L = \mathbf{11}$$

So, $W = L - 5 = 11 - 5 = \mathbf{6}$.

Therefore, the garden is **11 meters** long and **6 meters** wide.

Number-Value Problems

Problems that refer to the number of different types of items and the value of these items are often solved by setting two equations. Either one equation compares the number of items, and the other compares the value of these items, like in coin types of problems, or both equations compare the values of different arrangements of these items.

Example 3 ► **Finding the Number of Each Type of Items**

A restoration company purchased 45 paintbrushes, some at \$7.99 each and some at \$9.49 each. If the total charge before tax was \$379.05, how many of each type were purchased?

Solution ► Let x represent the number of brushes at \$7.99 each, and let y represent the number of brushes at \$9.49 each. Then the value of x brushes at \$7.99 each is $7.99x$. Similarly, the value of y brushes at \$9.49 each is $9.49y$. To organize the given information, we suggest to create and complete the following table.

	brushes at \$7.99 each +	brushes at \$9.49 each	= Total
number of brushes	x	y	45
value of brushes (in \$)	$7.99x$	$9.49y$	379.05

Since we work with two variables, we need two different equations in these variables. The first equation comes from comparing the number of brushes, as in the middle row. The second equation comes from comparing the values of these brushes, as in the last row. So, we have the system

$$\begin{cases} x + y = 45 \\ 7.99x + 9.49y = 379.05 \end{cases}$$

to solve. This can be solved by substitution. From the first equation, we have $y = 45 - x$, which after substituting to the second equation gives us

$$7.99x + 9.49(45 - x) = 379.05$$
$$799x + 949(45 - x) = 37905$$
$$799x + 42705 - 949x = 37905$$
$$-150x = -4800$$
$$x = \mathbf{32}$$

Then, $y = 45 - x = 45 - 32 = \mathbf{13}$.

Therefore, the restoration company purchased **32** brushes at \$7.99 each and **13** brushes at \$9.49 each.

Example 4 ► **Finding the Unit Cost of Each Type of Items**

The cost of 48 ft of red oak and 72 ft of fibreboard is \$271.20. At the same prices, 32 ft of red oak and 60 ft of fibreboard cost \$200. Find the unit price of red oak and fibreboard.

Solution ► Let r be the unit price of red oak and let f be the unit price of fibreboard. Then the value of 48 ft of red oak is represented by $48r$, and the value of 72 ft of fibreboard is represented by $72f$. Using the total cost of \$271.20, we write the first equation

$$48r + 72f = 271.20$$

Similarly, using the total cost of \$200 for 32 ft of red oak and 60 ft of fibreboard, we write the second equation

$$32r + 60f = 200$$

We will solve the system of the above equations via the elimination method.

$$\begin{cases} 48r + 72f = 271.20 \\ 32r + 60f = 200 \end{cases}$$

$$+\begin{cases} -96r - 144f = -542.40 \\ 96r + 180f = 600 \end{cases} \quad 36f = 57.40$$

$$f = \mathbf{1.60}$$

After substituting $f = 1.60$ into the second equation, we obtain

$$32r + 60 \cdot 1.60 = 200$$
$$32r + 96 = 200$$
$$32r = 104$$
$$r = \mathbf{3.25}$$

So, red oak costs **3.25 \$/ft**, and fibreboard costs **1.60 \$/ft**.

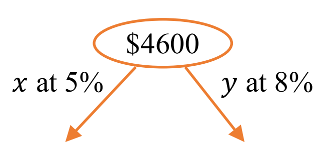
Investment Problems

Investment problems usually involve calculation of simple annual interest according to the formula $\boldsymbol{I = Prt}$, where I represents the amount of interest, P represents the principal which is the amount of investment, r represents the annual interest rate in decimal form, and t represents the time in years.

Example 5 ► **Finding the Amount of Each Loan**

Marven takes two student loans totaling \$4600 to cover the cost of his yearly tuition. One loan is at 5% simple interest, and the other is at 8% simple interest. If his total annual interest charge is \$278, find the amount of each loan.

Solution ► Suppose the amount of loan taken at 5% is x, and the amount of loan taken at 8% is y. The situation can be visualized by the diagram



one year interest: $0.05x$ + $0.08y$ = 278

So, the system of equations to solve is

$$\begin{cases} x + y = 4600 \\ 0.05x + 0.08y = 278 \end{cases}$$

Using the substitution method, we solve the first equation for x,

$$x = 4600 - y, \qquad (*)$$

substitute it into the second equation,

decimal elimination is optional

$$0.05(4600 - y) + 0.08y = 278,$$

and after elimination of decimals via multiplication by 100,

$$5(4600 - y) + 8y = 27800,$$

solve it for y:

$$23000 - 5y + 8y = 27800$$

$$3y = 4800$$

$$y = \mathbf{1600}$$

Then, after plugging in the y-value to the substitution equation $(*)$, we obtain

$$x = 4600 - 1600 = \mathbf{3000}$$

So, the amount of loan taken at 5% is $**3000**, and the amount of loan taken at 8% is $**1600**.

Mixture – Solution Problems

In mixture or solution problems, we typically mix two or more mixtures or solutions with different concentrations of a particular substance that we will refer to as the content. For example, if we are interested in the salt concentration in salty water, the salt is referred to as the content. When solving mixture problems, it is helpful to organize data in a table such as the one shown below.

	% ·	amount =	content
type I			
type II			
final mixture			

Example 6 ► **Solving a Mixture Problem**

Olivia wants to prepare 3 kg mixture of nuts and dried fruits that contains 30% of cranberries by mixing a blend that is 10% cranberry with a blend that is 40% cranberry. How much of each type of blend should she use to obtain the desired mixture?

Solution ► Suppose x is the amount of the 10% blend and y is the amount of the 40% blend. We complete the table

	% ·	amount (kg) =	cranberries (kg)

10% blend	0.1	x	$0.1x$
40% blend	0.4	y	$0.4y$
30% blend	0.3	3	0.9

The last column is completed by multiplying the "%" and "amount" columns.

The first equation comes from combining the weight of blends as shown in the "amount" column. The second equation comes from combining the weight of cranberries, as indicated in the last column. So, we solve

$$\begin{cases} x + y = 3 \\ 0.1x + 0.4y = 0.9 \end{cases}$$

Using the substitution method, we solve the first equation for x,

$$x = 3 - y, \qquad (*)$$

then substitute it into the second equation and solve for y:

$$\begin{aligned} 0.1(3 - y) + 0.4y &= 0.9 \\ 3 - y + 4y &= 9 \\ 3 + 3y &= 9 \\ 3y &= 6 \\ y &= \mathbf{2} \end{aligned}$$

Then, using the substitution equation $(*)$, we find the value of x:

$$x = 3 - 2 = \mathbf{1}$$

So, to obtain the desired blend, Olivia should mix **1 kg** of 10% and **2 kg** of 40% blend.

Motion Problems

In motion problems, we follow the formula $\boldsymbol{Rate} \cdot \boldsymbol{Time} = \boldsymbol{Distance}$. Drawing a diagram and completing a table based on the $\boldsymbol{R} \cdot \boldsymbol{T} = \boldsymbol{D}$ formula is usually helpful. In some motion problems, in addition to the rate of the moving object itself, we need to consider the rate of a moving medium such as water current or wind. The overall rate of a moving object is typically either the sum or the difference between the object's own rate and the rate of the moving medium.

Example 7 ► **Finding Rates in a Motion Problem**

A motorcycle travels 280 km in the same time that a car travels 245 km. If the motorcycle moves 14 kilometers per hour faster than the car, find the speed of each vehicle.

Solution ► Using meaningful letters, let m and c represent the speed of the motorcycle and the car, respectively. Since the speed of the motorcycle, m, is 14 km/h faster than the speed of the car, c, we can write the first equation:

$$m = c + 14$$

The second equation comes from comparing the travel time of each vehicle, as indicated in the table below.

	R ·	T	= D
motorcycle	m	$\frac{280}{m}$	280
car	c	$\frac{245}{c}$	245

To find the expression for time, we follow the formula $\boldsymbol{T} = \frac{\boldsymbol{D}}{\boldsymbol{R}}$, which comes from solving $R \cdot T = D$ for T.

So, we need to solve the system

$$\begin{cases} m = c + 14 \\ \frac{280}{m} = \frac{245}{c} \end{cases}$$

Cross-multiplication can **only** be applied to a **proportion** (an equation with a single fraction on each side.)

Notice that multiplication by the **LCD** would give the same result.

After substituting the first equation into the second, we obtain

$$\frac{280}{c + 14} = \frac{245}{c},$$

which can be solved by cross-multiplying

$$280c = 245(c + 14)$$

$$280c = 245c + 3430$$

$$35c = 3430$$

$$c = \mathbf{98}$$

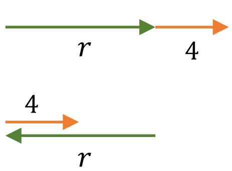
Then, we use this value to find $m = c + 14 = 98 + 14 = \mathbf{112}$.

So, the speed of the motorcycle is **112 km/h**, and the speed of the car is **98 km/h**.

Example 8 ► **Solving a Motion Problem with a Current**

A motorboat trip upstream a river takes 6 hours, while the return trip takes only 2 hours. Assuming the constant current of 4 mph, find the speed of the boat in still water.

Solution ►



Let r be the speed of the boat in still water.
Then the speed of the boat moving downstream is 4 mph faster because of the current going in the same direction as the boat. So, it is represented by $r + 4$.
The speed of the boat moving against the current is 4 mph slower. So, it is represented by $r - 4$.
Also, let d represent the distance covered by the boat going in one direction.

To organize the information, we can complete the table below.

	R ·	T	= D
downstream	$r + 4$	2	d
upstream	$r - 4$	6	d

The two equations come from following the formula $R \cdot T = D$, as indicated in each row.

$$\begin{cases}(r+4)\cdot 2 = d \\ (r-4)\cdot 6 = d\end{cases}$$

Since the left sides of both equations represent the same distance d, we can equal them and solve for r:

$$(r+4)\cdot 2 = (r-4)\cdot 6$$
$$2r + 8 = 6r - 24$$
$$32 = 4r$$
$$\mathbf{8} = r$$

So, the speed of the boat in still water was **8 mph**.

E.2 Exercises

1. If a barrel contains 50 liters of 16% alcohol wine, what is the volume of pure alcohol there?

2. If \$3000 is invested into bonds paying 4.5% simple annual interest, how much interest is expected in a year?

3. If a kilogram of baked chicken breast costs \$9.25, how much would n kilograms of this meat cost?

4. A ticket to a concert costs \$18. Write an expression representing the revenue from selling n such tickets.

5. A canoe that moves at r km/h in still water encounters c km/h current. Write an expression that would represent the speed of this canoe moving **a.** with the current (*downstream*), **b.** against the current (upstream).

6. A helicopter moving at r km/h in still air encounters a wind that blows at w km/h. Write an expression representing the rate of the helicopter travelling **a.** with the wind (*tailwind*), **b.** against the wind (*headwind*).

Solve each problem using two variables.

7. The larger out of two complementary angles is 6° more than three times the smaller one. Find the measure of each angle. (*Recall:* complementary angles add to 90°)

8. The larger out of two supplementary angles is 5° more than four times the smaller one. Find the measure of each angle. (*Recall:* supplementary angles add to 180°)

9. The sum of the height and the base of a triangle is 231 centimeters. The height is half the base. Find the base and height.

10. A tennis court is 13 meters longer than it is wide. What are the dimensions of the court if its perimeter is 72 meters.

11. A marathon is a run that covers about 42 km. In 2017, Mary Keitany won the "Women Only" marathon in London. During this marathon, at a particular moment, she was five times as far from the start of the course as she was from its end. At that time, how many kilometers had she already run?

12. Jane asked her students to find the two numbers that she had in mind. She told them that the smaller number is 2 more than one-third of the larger number and that three times the larger number is 1 less than eight times the smaller one. Find the numbers.

13. During the 2014 Winter Olympics in Sochi, Russia won a total of 33 medals. There were 4 more gold medals than bronze ones. If the number of silver medals was the average of the number of gold and the number of bronze medals, how many of each type of medal did Russia earn?

14. A 156 cm long piece of wire is cut into two pieces. Then, each piece is bent to make an equilateral triangle. If the side of one triangle is twice as long as the side of the other triangle, how should the wire be cut?

15. A bistro cafe sells espresso and cappuccino cups of coffee. One day, the cafe has sold 452 cups of coffee. How many cups of each type of coffee the shop has sold that day if the number of cappuccino cups was 3 times as large as the number of espresso cups of coffee sold?

16. During an outdoor festival, a retail booth was selling solid-colour scarfs for \$8.75 each and printed scarfs for \$11.49 each. If \$478.60 was collected for selling 50 scarfs, how many of each type were sold?

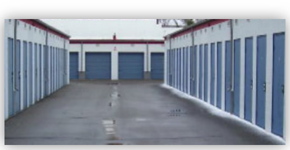
17. The Mission Folk Music Festival attracts local community every summer since 1988. In recent years, a one-day admission to this festival costs \$45 for an adult ticket and \$35 for a youth ticket. If the total of \$24,395 were collected from the sale of 605 tickets, how many of each type of tickets were sold?

18. Ellena's total GED score in Mathematics and Science was 328. If she scored 24 points higher in Mathematics than in Science, what were her GED scores in each subject?

19. One day, at his food stand, Tom sold 12 egg salad sandwiches and 18 meat sandwiches, totaling \$101.70. The next day he sold 23 egg salad sandwiches and 9 meat sandwiches, totaling \$93.18. How much did each type of sandwich cost?

20. At lunchtime, a group of conference members ordered three cappuccinos and four expressos for a total of \$20.07. Another group ordered two cappuccinos and three expressos for a total of \$14.43. How much did each type of coffee cost?

21. New York City and Paris are one of the most expensive cities to live in. Based on the average weekly cost of living in each city (not including accommodation), 2 weeks in New York and 3 weeks in Paris cost \$1852, while 4 weeks in New York and 2 weeks in Paris cost \$2344. Find the average weekly cost of living in each city?



22. One of the local storage facilities rents two types of storage lockers, a small one with 180 ft^2 in area, and a large one with the area of 600 ft^2. In total, the facility has 42 storage lockers that provide 15,120 ft^2 of the overall storage area. How many of each type of storage lockers does the facility have?

23. Ryan took two student loans for a total of \$4800. One of these loans was borrowed at 3.25% simple interest and the other one at 2.75%. If after one year Ryan's overall interest charge for both of the loans was \$143.50, what was the amount of each loan?

24. An investor made two investments totaling $36,000. In one year, these investments generated $1650 in simple interest. If the interest rate for the two investments were 5% and 3.75%, how much was invested at each rate?

25. A stockbroker invested some money in a low-risk fund and twice as much in a high-risk fund. In a year, the low-risk fund earned 3.7%, and the high-risk fund lost 8.2%. If the two investments resulted in the overall loss of $111.20, how much was invested in each fund?

26. Patricia's bank offers her two types of investments, one at 4.5% and the other one at 6.25% simple interest. Patricia invested $1500 more at 6.25% than at 4.5%. How much was invested at each rate if the total interest accumulated after one year was $462.25?

27. How many liters of 4% brine and 20% brine should be mixed to obtain 12 liters of 8% brine?

28. Sam has $11 in dimes and quarters. If he counted 71 coins in all, how many of each type of coin are there?

29. Cottage cheese contains 12% of protein and 6% of carbs while vanilla yogurt is 5% protein and 15% carbs. How many grams of each product should be used to serve a meal that contains 10 grams of protein and 10 grams of carbs?

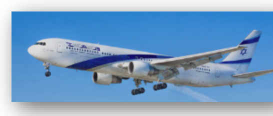
30. Kidney beans contain 24% protein while lima beans contain just 8% protein. How many dekagrams of each type of bean should be used to prepare 60 dekagrams of a bean-mix that is 12% protein?

31. Cezary purchased a shirt costing $42.75 with a $50 bill. The cashier gave him the change in quarters and loonies. If the change consisted of 14 coins, how many of each kind were there?

32. When travelling with the current, a speedboat covers 24 km in half an hour. It takes 40 minutes for the boat to cover the same distance against the current. Find the rate of the boat in still water and the rate of the current.

33. A houseboat travelling with the current went 45 km in 3 hours. It took 2 hours longer to travel the same distance against the current. Find the rate of the houseboat in still water and the rate of the current.

34. When flying with the wind, a passenger plane covers a distance of 1760 km in 2 hours. When flying against the same wind, the plane covers 2400 km in 3 hours. Determine the rate of the plane in still air and the rate of the wind.



35. Flying with a tailwind, a pilot of a small plane could cover the distance of 1500 km between two cities in 5 hours. Flying with the headwind of the same intensity, he would need 6 hours to cover the same distance. Find the rate of the plane in still air and the rate of the wind.

36. Robert kayaked 10 km downstream a river in 2 hours. When returning, Robert could kayak only 6 km in the same amount of time. What was his rate of kayaking in still water and what was the rate of the current?

37. A small private plane flying into a wind covered 1080 km in 4 hours. When flying back, with a tailwind of the same intensity, the plane needed only 3 hours to cover the same distance. What are the rate of the plane in still air and the rate of the wind?

38. Flight time against a headwind for a trip of 2300 kilometers is 4 hours. If the headwind were half as great, the same flight would take 10 minutes less time. Find the rate of the wind and the rate of the plane in still air.

39. Teresa was late for her conference presentation after driving at an average speed of 60 km/h. If she had driven 4 km/h faster, her travelling time would be half an hour shorter. How far was the conference?

40. Two vehicles leave a gas station at the same time and travel in the same direction. One travels at 96 km/h and the other at 108 km/h. The drivers of the two vehicles can communicate with each other with a short-distance radio device as long as they are within 10 km range. When will they lose this contact?

41. The windshield fluid tank in Izabella's car contains 5 L of 7% antifreeze. To the nearest tenths of a litre, how much of this mixture should be drained and replaced with pure antifreeze so that the mixture becomes 20% antifreeze?

42. Mike has two gallons of stain that is 10% brown and 90% neutral and a gallon of pure brown stain. To stain a deck, he needs 2 gallons of a stain that is 40% brown and 60% neutral. How much of each type of stain should he use to prepare the desired mix?

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Data:

p.**157** Global Add Spend Graph / Ooh Today