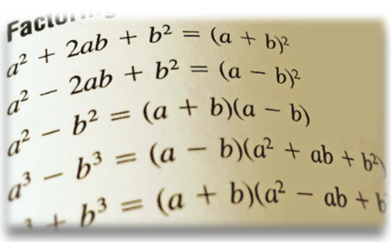
Factoring



Factoring is the reverse process of multiplication. Factoring polynomials in algebra has similar role as factoring numbers in arithmetic. Any number can be expressed as a product of prime numbers. For example, $6 = 2 \cdot 3$. Similarly, any polynomial can be expressed as a product of **prime** polynomials, which are polynomials that cannot be factored any further. For example, $x^2 + 5x + 6 = (x + 2)(x + 3)$. Just as factoring numbers helps in simplifying or adding fractions, factoring polynomials is very useful in simplifying or adding algebraic fractions. In addition, it helps identify zeros of polynomials, which in turn allows for solving higher degree polynomial equations.

In this chapter, we will examine the most commonly used factoring strategies with particular attention to special factoring. Then, we will apply these strategies in solving polynomial equations.

F1 Greatest Common Factor and Factoring by Grouping

Prime Factors

When working with integers, we are often interested in their factors, particularly prime factors. Likewise, we might be interested in factors of polynomials.

Definition 1.1 ► To **factor** a polynomial means to write the polynomial as a **product** of 'simpler' polynomials. For example,

$$5x + 10 = 5(x + 2), \quad \text{or} \quad x^2 - 9 = (x + 3)(x - 3).$$

In the above definition, 'simpler' means polynomials of **lower degrees** or polynomials with coefficients that **do not contain common factors** other than 1 or -1. If possible, we would like to see the polynomial factors, other than monomials, having **integral coefficients** and a **positive leading term**.

When is a polynomial factorization complete?

In the case of natural numbers, the complete factorization means a factorization into prime numbers, which are numbers divisible only by their own selves and 1. We would expect that similar situation is possible for polynomials. So, which polynomials should we consider as prime?

Observe that a polynomial such as $-4x + 12$ can be written as a product in many different ways, for instance

$$-(4x + 12), \quad 2(-2x + 6), \quad 4(-x + 3), \quad -4(x - 3), \quad -12\left(\tfrac{1}{3}x + 1\right), \ \text{etc.}$$

Since the terms of $4x + 12$ and $-2x + 6$ still contain common factors different than 1 or -1, these polynomials are not considered to be factored completely, which means that they should not be called prime. The next two factorizations, $4(-x + 3)$ and $-4(x - 3)$ are both complete, so both polynomials $-x + 3$ and $x - 3$ should be considered as prime. But what about the last factorization, $-12\left(\frac{1}{3}x + 1\right)$? Since the remaining binomial $\frac{1}{3}x + 1$ does not have integral coefficients, such a factorization is not always desirable.

Here are some examples of **prime polynomials**:

- any monomials such as $-2x^2$, πr^2, or $\frac{1}{3}xy$;
- any linear polynomials with integral coefficients that have no common factors other than 1 or -1, such as $x-1$ or $2x+5$;
- some quadratic polynomials with integral coefficients that cannot be factored into any lower degree polynomials with integral coefficients, such as x^2+1 or x^2+x+1.

For the purposes of this course, we will assume the following definition of a prime polynomial.

Definition 1.2 ► A polynomial with integral coefficients is called **prime** if one of the following conditions is true
- it is a **monomial**, or
- the only **common factors** of its terms are **1 or -1** and it **cannot be factored into any lower degree polynomials** with integral coefficients.

Definition 1.3 ► A **factorization** of a polynomial with integral coefficients is **complete** if all of its **factors** are **prime**.

Here is an example of a polynomial factored completely:

$$-6x^3-10x^2+4x=-2x(3x-1)(x+2)$$

In the next few sections, we will study several factoring strategies that will be helpful in finding complete factorizations of various polynomials.

Greatest Common Factor

The first strategy of factoring is to factor out the **greatest common factor** (**GCF**).

Definition 1.4 ► The **greatest common factor** (**GCF**) of two or more terms is the largest expression that is a factor of all these terms.

In the above definition, the "largest expression" refers to the expression with the most factors, disregarding their signs.

To find the greatest common factor, we take the product of the least powers of each type of common factor out of all the terms. For example, suppose we wish to find the GCF of the terms

$$6x^2y^3,\ -18x^5y, \text{ and } 24x^4y^2.$$

First, we look for the GCF of 6, 18, and 24, which is 6. Then, we take the lowest power out of x^2, x^5, and x^4, which is x^2. Finally, we take the lowest power out of y^3, y, and y^2, which is y. Therefore,

$$\mathrm{GCF}(6x^2y^3,\ -18x^5y,\ 24x^4y^2)=6x^2y$$

This GCF can be used to factor the polynomial $6x^2y^3-18x^5y+24x^4y^2$ by first seeing it as

$$6x^2y\cdot y^2-6x^2y\cdot 3x^3+6x^2y\cdot 4x^2y,$$

and then, using the **reverse distributing property**, 'pulling' the $6x^2y$ out of the bracket to obtain

$$6x^2y(y^2 - 3x^3 + 4x^2y).$$

Note 1: Notice that since 1 and -1 are factors of any expression, the GCF is defined up to the sign. Usually, we choose the positive GCF, but sometimes it may be convenient to choose the negative GCF. For example, we can claim that

$$\mathrm{GCF}(-2x, -4y) = 2 \quad \text{or} \quad \mathrm{GCF}(-2x, -4y) = -2,$$

depending on what expression we wish to leave after factoring the GCF out:

$$-2x - 4y = \underbrace{2}_{\substack{\text{positive}\\ \text{GCF}}} \underbrace{(-x - 2y)}_{\substack{\text{negative}\\ \text{leading}\\ \text{term}}} \quad \text{or} \quad -2x - 4y = \underbrace{-2}_{\substack{\text{negative}\\ \text{GCF}}} \underbrace{(x + 2y)}_{\substack{\text{positive}\\ \text{leading}\\ \text{term}}}$$

Note 2: If the GCF of the terms of a polynomial is equal to 1, we often say that these terms do not have any common factors. What we actually mean is that the terms do not have a common factor other than 1, as factoring 1 out does not help in breaking the original polynomial into a product of simpler polynomials. See *Definition 1.1*.

Example 1 ► **Finding the Greatest Common Factor**

Find the greatest common factor for the given expressions.

a. $6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2$ **b.** $4\pi(y-x),\ 8\pi(x-y)$

c. $ab^2,\ a^2b,\ b,\ a$ **d.** $3x^{-1}y^{-3},\ x^{-2}y^{-2}z$

Solution ►

a. Since $\mathrm{GCF}(6, 3, 9) = 3$, the lowest power out of x^4, x^3, and x is x, and the lowest power out of $(x+1)^3$, $(x+1)$, and $(x+1)^2$ is $(x+1)$, then

$$\mathrm{GCF}(6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2) = \mathbf{3x(x+1)}$$

b. Since $y - x$ is opposite to $x - y$, then $y - x$ can be written as $-(x - y)$. So 4, π, and $(x - y)$ is common for both expressions. Thus,

$$\mathrm{GCF}\big(4\pi(y-x),\ 8\pi(x-y)\big) = \mathbf{4\pi(x-y)}$$

Note: The greatest common factor is unique up to its sign. Notice that in the above example, we could write $x - y$ as $-(y - x)$ and choose the GCF to be $4\pi(y - x)$.

c. The terms ab^2, a^2b, b, and a have no common factor other than 1, so

$$\mathrm{GCF}(ab^2,\ a^2b,\ b,\ a) = \mathbf{1}$$

d. The lowest power out of x^{-1} and x^{-2} is x^{-2}, and the lowest power out of y^{-3} and y^{-2} is y^{-3}. Therefore,

$$\text{GCF}(3x^{-1}y^{-3},\ x^{-2}y^{-2}z) = \boldsymbol{x^{-2}y^{-3}}$$

Example 2 ► **Factoring out the Greatest Common Factor**

Factor each expression by taking the greatest common factor out. Simplify the factors, if possible.

a. $54x^2y^2 + 60xy^3$

b. $ab - a^2b(a-1)$

c. $-x(x-5) + x^2(5-x) - (x-5)^2$

d. $x^{-1} + 2x^{-2} - x^{-3}$

Solution ►

a. To find the greatest common factor of 54 and 60, we can use the method of dividing by any common factor, as presented below.

all common factors are listed in this column

$$\begin{array}{r|rr} 2 & 54, & 60 \\ 3 & 27, & 30 \\ \hline & 9, & 10 \end{array}$$

no more common factors for 9 and 10

So, $\text{GCF}(54, 60) = 2 \cdot 3 = 6$.

Since $\text{GCF}(54x^2y^2,\ 60xy^3) = 6xy^2$, we factor the $6xy^2$ out by dividing each term of the polynomial $54x^2y^2 + 60xy^3$ by $6xy^2$, as below.

$$54x^2y^2 + 60xy^3$$

$$= \boldsymbol{6xy^2}(\boldsymbol{9x} + \boldsymbol{10y})$$

$\frac{54x^2y^2}{6xy^2} = 9x$

$\frac{60xy^3}{6xy^2} = 10y$

Note: Since factoring is the reverse process of multiplication, it can be checked by finding the product of the factors. If the product gives us the original polynomial, the factorization is correct.

b. First, notice that the polynomial has two terms, ab and $-a^2b(a-1)$. The greatest common factor for these two terms is ab, so we have

$$ab - a^2b(a-1) = ab(\mathbf{1} - a(a-1))$$

remember to leave **1** for the first term

$$= ab(1 - a^2 + a)$$

$$= ab(-a^2 + a + 1)$$

simplify and arrange in decreasing powers

$$= \boldsymbol{-ab(a^2 - a - 1)}$$

take the "–" out

Note: Both factorizations, $ab(-a^2+a+1)$ and $-ab(a^2-a-1)$ are correct. However, we customarily leave the polynomial in the bracket with a positive leading coefficient.

c. Observe that if we write the middle term $x^2(5-x)$ as $-x^2(x-5)$ by factoring the negative out of the $(5-x)$, then $(5-x)$ is the common factor of all the terms of the equivalent polynomial

$$-x(x-5)-x^2(x-5)-(x-5)^2.$$

Then notice that if we take $-(x-5)$ as the GCF, then the leading term of the remaining polynomial will be positive. So, we factor

$$-x(x-5)+x^2(5-x)-(x-5)^2$$

$$=-x(x-5)-x^2(x-5)-(x-5)^2$$

$$=-(x-5)\left(x+x^2+(x-5)\right)$$ simplify and arrange in decreasing powers

$$=-(x-5)(\boldsymbol{x^2+2x-5})$$

d. The $\mathrm{GCF}(x^{-1},\ 2x^{-2},\ -x^{-3})=x^{-3}$, as -3 is the lowest exponent of the common factor x. So, we factor out x^{-3} as below.

$$x^{-1}+2x^{-2}-x^{-3}$$

$$=\boldsymbol{x^{-3}\left(x^2+2x-1\right)}$$

the exponent 2 is found by subtracting -3 from -1

the exponent 1 is found by subtracting -3 from -2

To check if the factorization is correct, we multiply

$$x^{-3}\left(x^2+2x-1\right)$$

add exponents

$$=x^{-3}x^2+2x^{-3}x-1x^{-3}$$

$$=x^{-1}+2x^{-2}-x^{-3}$$

Since the product gives us the original polynomial, the factorization is correct.

Factoring by Grouping

When referring to a common factor, we have in mind a common factor other than 1.

Consider the polynomial $x^2+x+xy+y$. It consists of four terms that do not have any common factors. Yet, it can still be factored if we group the first two and the last two terms. The first group of two terms contains the common factor of x and the second group of two terms contains the common factor of y. Observe what happens when we factor each group.

$$\underbrace{x^2+x}+\underbrace{xy+y}$$

$$=x(x+1)+y(x+1)$$

$$=(x+1)(x+y)$$

now $(x+1)$ is the common factor of the entire polynomial

This method is called **factoring by grouping**, in particular, two-by-two grouping.

Warning: After factoring each group, make sure to write the "+" or "−" between the terms. Failing to write these signs leads to the false impression that the polynomial is already factored. For example, if in the second line of the above calculations we would fail to write the middle "+", the expression would look like a product $x(x+1)\, y(x+1)$, which is not the case. Also, since the expression $x(x+1)+y(x+1)$ is a sum, not a product, we should not stop at this step. We need to factor out the common bracket $(x+1)$ to leave it as a product.

A two-by-two grouping leads to a factorization only if **the binomials**, after factoring out the common factors in each group, **are the same**. Sometimes a rearrangement of terms is necessary to achieve this goal.

For example, the attempt to factor $x^3-15+5x^2-3x$ by grouping the first and the last two terms,

$$\underbrace{x^3-15}+\underbrace{5x^2-3x}$$

$$=(x^3-15)+x(5x-3)$$

does not lead us to a common binomial that could be factored out.

However, rearranging terms allows us to factor the original polynomial in the following ways:

$$\begin{aligned}&x^3-15+5x^2-3x\\&=\underbrace{x^3+5x^2}+\underbrace{-3x-15}\\&=x^2(x+5)-3(x+5)\\&=(x+5)(x^2-3)\end{aligned}$$

or

$$\begin{aligned}&x^3-15+5x^2-3x\\&=\underbrace{x^3-3x}+\underbrace{5x^2-15}\\&=x(x^2-3)+5(x^2-3)\\&=(x^2-3)(x+5)\end{aligned}$$

Factoring by grouping applies to polynomials with more than three terms. However, not all such polynomials can be factored by grouping. For example, if we attempt to factor x^3+x^2+2x-2 by grouping, we obtain

$$\underbrace{x^3+x^2}+\underbrace{2x-2}$$

$$=x^2(x+1)+2(x-1).$$

Unfortunately, the expressions $x+1$ and $x-1$ are not the same, so there is no common factor to factor out. One can also check that no other rearrangments of terms allows us for factoring out a common binomial. So, this polynomial cannot be factored by grouping.

Example 3 ► **Factoring by Grouping**

Factor each polynomial by grouping, if possible. Remember to check for the GCF first.

a. $2x^3 - 6x^2 + x - 3$ **b.** $5x - 5y - ax + ay$

c. $2x^2y - 8 - 2x^2 + 8y$ **d.** $x^2 - x + y + 1$

Solution ► **a.** Since there is no common factor for all four terms, we will attempt the two-by-two grouping method.

$$\underbrace{2x^3 - 6x^2} + \underbrace{x - 3}$$

$$= 2x^2(x-3) + 1(x-3)$$

$$= \boldsymbol{(x-3)(2x^2+1)}$$

write the **1** for the second term

b. As before, there is no common factor for all four terms. The two-by-two grouping method works only if the remaining binomials after factoring each group are exactly the same. We can achieve this goal by factoring $-a$, rather than a, out of the last two terms. So,

$$\underbrace{5x - 5y}\ \underbrace{-ax + ay}$$

$$= 5(x-y) - a(x-y)$$

reverse signs when 'pulling' a "–" out

$$= \boldsymbol{(x-y)(5 - a)}$$

c. Notice that 2 is the GCF of all terms, so we factor it out first.

$$2x^2y - 8 - 2x^2 + 8y$$

$$= 2(x^2y - 4 - x^2 + 4y)$$

Then, observe that grouping the first and last two terms of the remaining polynomial does not help, as the two groups do not have any common factors. However, exchanging for example the second with the fourth term will help, as shown below.

$$= 2(\underbrace{x^2y + 4y}\ \underbrace{-x^2 - 4})$$

the square bracket is essential here because of the factor of 2

$$= 2[y(x^2+4) - (x^2+4)]$$

reverse signs when 'pulling' a "–" out

$$= \boldsymbol{2(x^2+4)(y-1)}$$

now, there is no need for the square bracket as multiplication is associative

d. The polynomial $x^2 - x + y + 1$ does not have any common factors for all four terms. Also, only the first two terms have a common factor. Unfortunately, when attempting to factor using the two-by-two grouping method, we obtain

$$x^2 - x + y + 1$$

$$= x(x-1) + (y+1),$$

which cannot be factored, as the expressions $x - 1$ and $y + 1$ are different.

One can also check that no other arrangement of terms allows for factoring of this polynomial by grouping. So, this polynomial cannot be factored by grouping.

Example 4 ► **Factoring in Solving Formulas**

Solve $ab = 3a + 5$ for a.

Solution ► First, we move the terms containing the variable a to one side of the equation,

$$ab = 3a + 5$$
$$ab - 3a = 5,$$

and then factor a out

$$a(b-3) = 5.$$

So, after dividing by $b-3$, we obtain $\boldsymbol{a = \frac{5}{b-3}}$.

F.1 Exercises

*In problems **1-2**, state whether the given sentence is **true** or **false**.*

1. The polynomial $6x + 8y$ is ***prime***.

2. The **GCF** of the terms of the polynomial $3(x-2) + x(2-x)$ is $(x-2)(2-x)$.

3. Observe the two factorizations of the polynomial $\frac{1}{2}x - \frac{3}{4}y$ performed by different students:

Student A: $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{2}\left(x - \frac{3}{2}y\right)$ *Student B:* $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{4}(2x - 3y)$

Are the two factorizations correct? Which one is preferable, and why?

Find the GCF with a positive coefficient for the given expressions.

4. $8xy,\ 10xz,\ -14xy$

5. $21a^3b^6,\ -35a^7b^5,\ 28a^5b^8$

6. $4x(x-1),\ 3x^2(x-1)$

7. $-x(x-3)^2,\ x^2(x-3)(x+2)$

8. $9(a-5),\ 12(5-a)$

9. $(x-2y)(x-1),\ (2y-x)(x+1)$

10. $-3x^{-2}y^{-3},\ 6x^{-3}y^{-5}$

11. $x^{-2}(x+2)^{-2},\ -x^{-4}(x+2)^{-1}$

Factor out the greatest common factor. Leave the remaining polynomial with a positive leading coeficient. Simplify the factors, if possible.

12. $9x^2 - 81x$

13. $8k^3 + 24k$

14. $6p^3 - 3p^2 - 9p^4$

15. $6a^3 - 36a^4 + 18a^2$

16. $-10r^2s^2 + 15r^4s^2$

17. $5x^2y^3 - 10x^3y^2$

18. $a(x-2) + b(x-2)$

19. $a(y^2-3) - 2(y^2-3)$

20. $(x-2)(x+3) + (x-2)(x+5)$

21. $(n-2)(n+3) + (n-2)(n-3)$

22. $y(x-1)+5(1-x)$

23. $(4x-y)-4x(y-4x)$

24. $4(3-x)^2-(3-x)^3+3(3-x)$

25. $2(p-3)+4(p-3)^2-(p-3)^3$

Factor out the least power of each variable.

26. $3x^{-3}+x^{-2}$

27. $k^{-2}+2k^{-4}$

28. $x^{-4}-2x^{-3}+7x^{-2}$

29. $3p^{-5}+p^{-3}-2p^{-2}$

30. $3x^{-3}y-x^{-2}y^2$

31. $-5x^{-2}y^{-3}+2x^{-1}y^{-2}$

Factor by grouping, if possible.

32. $20+5x+12y+3xy$

33. $2a^3+a^2-14a-7$

34. $ac-ad+bc-bd$

35. $2xy-x^2y+6-3x$

36. $3x^2+4xy-6xy-8y^2$

37. $x^3-xy+y^2-x^2y$

38. $3p^2+9pq-pq-3q^2$

39. $3x^2-x^2y-yz^2+3z^2$

40. $2x^3-x^2+4x-2$

41. $x^2y^2+ab-ay^2-bx^2$

42. $xy+ab+by+ax$

43. $x^2y-xy+x+y$

44. $xy-6y+3x-18$

45. x^ny-3x^n+y-5

46. $a^nx^n+2a^n+x^n+2$

Factor completely. Remember to check for the GCF first.

47. $5x-5ax+5abc-5bc$

48. $6rs-14s+6r-14$

49. $x^4(x-1)+x^3(x-1)-x^2+x$

50. $x^3(x-2)^2+2x^2(x-2)-(x+2)(x-2)$

51. One of possible factorizations of the polynomial $4x^2y^5-8xy^3$ is $2xy^3(2xy^2-4)$. Is this a complete factorization?

Use factoring the GCF strategy to ***solve*** *each formula* ***for the indicated variable****.*

52. $A=\boldsymbol{P}+\boldsymbol{P}r$, for $\boldsymbol{P}$

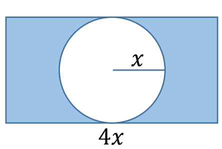
53. $M=\frac{1}{2}\boldsymbol{p}q+\frac{1}{2}\boldsymbol{p}r$, for $\boldsymbol{p}$

54. $2\boldsymbol{t}+c=k\boldsymbol{t}$, for $\boldsymbol{t}$

55. $w\boldsymbol{y}=3\boldsymbol{y}-x$, for $\boldsymbol{y}$

Write the area of each shaded region in factored form.

56.



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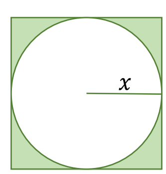

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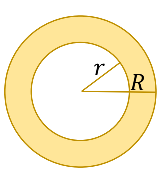

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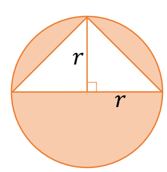
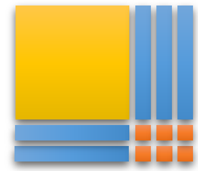


F2 Factoring Trinomials

In this section, we discuss factoring trinomials. We start with factoring quadratic trinomials of the form $x^2 + bx + c$, then quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, and finally trinomials reducible to quadratic by means of substitution.



Factorization of Quadratic Trinomials $x^2 + bx + c$

GUESSING METHOD

Factorization of a quadratic trinomial $x^2 + bx + c$ is the reverse process of the FOIL method of multiplying two linear binomials. Observe that

$$(x + p)(x + q) = x^2 + qx + px + pq = x^2 + (p + q)x + pq$$

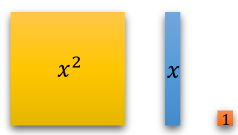
So, to reverse this multiplication, we look for two numbers p and q, such that the product pq equals to the free term c and the sum $p + q$ equals to the middle coefficient b of the trinomial.

$$x^2 + \underbrace{b}_{(p+q)} x + \underbrace{c}_{pq} = (x + p)(x + q)$$

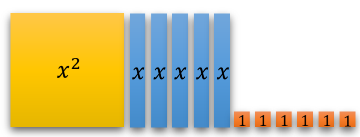
For example, to factor $x^2 + 5x + 6$, we think of two integers that multiply to 6 and add to 5. Such integers are 2 and 3, so $x^2 + 5x + 6 = (x + 2)(x + 3)$. Since multiplication is commutative, the order of these factors is not important.

VISUALIZATION OF FACTORING

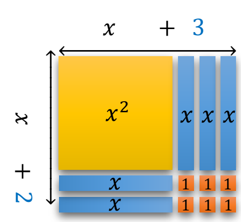
This could also be illustrated geometrically, using algebra tiles.



The area of a square with the side length x is equal to x^2. The area of a rectangle with the dimensions x by 1 is equal to x, and the area of a unit square is equal to 1. So, the trinomial $x^2 + 5x + 6$ can be represented as



To factor this trinomial, we would like to rearrange these tiles to fulfill a rectangle.



The area of such rectangle can be represented as the product of its length, $(x + 3)$, and width, $(x + 2)$ which becomes the factorization of the original trinomial.

In the trinomial examined above, the signs of the middle and the last terms are both positive. To analyse how different signs of these terms influence the signs used in the factors, observe the next three examples.

To factor $x^2 - 5x + 6$, we look for two integers that multiply to 6 and add to -5. Such integers are -2 and -3, so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

To factor $x^2 + x - 6$, we look for two integers that multiply to -6 and add to 1. Such integers are -2 and 3, so $x^2 + x - 6 = (x - 2)(x + 3)$.

To factor $x^2 - x - 6$, we look for two integers that multiply to -6 and add to -1. Such integers are 2 and -3, so $x^2 - x - 6 = (x + 2)(x - 3)$.

Observation: A **positive constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x + p)(x + q)$ are both of the **same sign** and their **sum** is the middle coefficient b. In addition, if b is positive, both p and q are positive, and if b is negative, both p and q are negative.

A **negative constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x + p)(x + q)$ are of **different signs** and the **difference** of their absolute values is the middle coefficient b. In addition, the integer whose absolute value is larger takes the sign of the middle coefficient b.

These observations are summarized in the following **Table of Signs**.

Assume that $|p| \geq |q|$.

sum $\boldsymbol{b}$	product $\boldsymbol{c}$	$\boldsymbol{p}$	$\boldsymbol{q}$	**comments**
+	+	+	+	b is the ***sum*** of p and q
−	+	−	−	b is the ***sum*** of p and q
+	−	+	−	b is the ***difference*** $|p| - |q|$
−	−	−	+	b is the ***difference*** $|q| - |p|$

Example 1 ► **Factoring Trinomials with the Leading Coefficient Equal to 1**

Factor each trinomial, if possible.

a. $x^2 - 10x + 24$
b. $x^2 + 9x - 36$
c. $x^2 - 39xy - 40y^2$
d. $x^2 + 7x + 9$

Solution ►

a. To factor the trinomial $x^2 - 10x + 24$, we look for two integers with a product of 24 and a sum of -10. The two integers are fairly easy to guess, -4 and -6. However, if one wishes to follow a more methodical way of finding these numbers, one can list the possible two-number factorizations of 24 and observe the sums of these numbers.

For simplicity, the table doesn't include signs of the integers. The signs are determined according to the **Table of Signs**.

product = 24 (pairs of factors of 24)	**sum = −10** (sum of factors)
1 · 24	25
2 · 12	14
3 · 8	11
4 · 6	10

Bingo!

Since the product is positive and the sum is negative, both integers must be negative. So, we take -4 and -6.

Thus, $x^2 - 10x + 24 = (\boldsymbol{x - 4})(\boldsymbol{x - 6})$. The reader is encouraged to check this factorization by multiplying the obtained binomials.

b. To factor the trinomial $x^2 + 9x - 36$, we look for two integers with a product of -36 and a sum of 9. So, let us list the possible factorizations of 36 into two numbers and observe the differences of these numbers.

product = −36 (pairs of factors of 36)	**sum = 9** (difference of factors)
1 · 36	35
2 · 18	16
3 · 12	9
4 · 9	5
6 · 6	0

This row contains the solution, so there is no need to list any of the subsequent rows.

Since the product is negative and the sum is positive, the integers are of different signs and the one with the larger absolute value assumes the sign of the sum, which is positive. So, we take 12 and -3.

Thus, $x^2 + 9x - 36 = (\boldsymbol{x + 12})(\boldsymbol{x - 3})$. Again, the reader is encouraged to check this factorization by multiplying the obtained binomials.

c. To factor the trinomial $x^2 - 39xy - 40y^2$, we look for two binomials of the form $(x + ?\,y)(x + ?\,y)$ where the question marks are two integers with a product of -40 and a sum of 39. Since the two integers are of different signs and the absolute values of these integers differ by 39, the two integers must be -40 and 1.

Therefore, $x^2 - 39xy - 40y^2 = (\boldsymbol{x - 40y})(\boldsymbol{x + y})$.

Suggestion: Create a table of pairs of factors <u>only</u> if guessing the two integers with the given product and sum becomes too difficult.

d. When attempting to factor the trinomial $x^2 + 7x + 9$, we look for a pair of integers that would multiply to 9 and add to 7. There are only two possible factorizations of 9: $9 \cdot 1$ and $3 \cdot 3$. However, neither of the sums, $9 + 1$ or $3 + 3$, are equal to 7. So, there is no possible way of factoring $x^2 + 7x + 9$ into two linear binomials with integral coefficients. Therefore, if we admit only integral coefficients, this polynomial is **not factorable**.

Factorization of Quadratic Trinomials $ax^2 + bx + c$ with $a \neq 0$

Before discussing factoring quadratic trinomials with a leading coefficient different than 1, let us observe the multiplication process of two linear binomials with integral coefficients.

$$(mx + p)(nx + q) = mnx^2 + mqx + npx + pq = \underbrace{a}_{mn}x^2 + \underbrace{b}_{(mq+np)}x + \underbrace{c}_{pq}$$

To reverse this process, notice that this time, we are looking for four integers m, n, p, and q that satisfy the conditions

$$mn = a, \quad pq = c, \quad mq + np = b,$$

where a, b, c are the coefficients of the quadratic trinomial that needs to be factored. This produces a lot more possibilities to consider than in the guessing method used in the case of the leading coefficient equal to 1. However, if at least one of the outside coefficients, a or c, are prime, the guessing method still works reasonably well.

GUESSING METHOD

For example, consider $2x^2 + x - 6$. Since the coefficient $a = 2 = mn$ is a prime number, there is only one factorization of a, which is $1 \cdot 2$. So, we can assume that $m = 2$ and $n = 1$. Therefore,

$$2x^2 + x - 6 = (2x \pm |p|)(x \mp |q|)$$

Since the constant term $c = -6 = pq$ is negative, the binomial factors have different signs in the middle. Also, since pq is negative, we search for such p and q that the inside and outside products **differ** by the middle term $b = x$, up to its sign. The only factorizations of 6 are $1 \cdot 6$ and $2 \cdot 3$. So we try

$$2x^2 + x - 6 = (2x \pm 1)(x \mp 6)$$

x

$12x$

differs by $11x$ → too much

Observe that these two trials can be disregarded at once as 2 is not a common factor of all the terms of the trinomial, while it is a common factor of the terms of one of the binomials.

$$2x^2 + x - 6 = (2x \pm 6)(x \mp 1)$$

$6x$

$2x$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (2x \pm 2)(x \mp 3)$$

$2x$

$6x$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (2x \pm 3)(x \mp 2)$$

$3x$

$4x$

differs by x → perfect!

Then, since the difference between the inner and outer products should be positive, the larger product must be positive and the smaller product must be negative. So, we distribute the signs as below.

$$2x^2 + x - 6 = (2x - 3)(x + 2)$$

$-3x$

$4x$

In the end, it is a good idea to multiply the product to check if it results in the original polynomial. We leave this task to the reader.

What if the outside coefficients of the quadratic trinomial are both composite? Checking all possible distributions of coefficients m, n, p, and q might be too cumbersome. Luckily, there is another method of factoring, called **decomposition**.

The decomposition method is based on the reverse FOIL process.
Suppose the polynomial $6x^2 + 19x + 15$ factors into $(mx + p)(nx + q)$. Observe that the FOIL multiplication of these two binomials results in the four term polynomial,

$$mnx^2 + mqx + npx + pq,$$

which after combining the two middle terms gives us the original trinomial. So, reversing these steps would lead us to the factored form of $6x^2 + 19x + 15$.

To reverse the FOIL process, we would like to:

This product is often referred to as the **master product** or the ***ac*-product**.

- Express the middle term, $19x$, as a sum of two terms, mqx and npx, such that the product of their coefficients, $mnpq$, is equal to the product of the outside coefficients $ac = 6 \cdot 15 = 90$.
- Then, factor the four-term polynomial by grouping.

Thus, we are looking for two integers with the product of 90 and the sum of 19. One can check that 9 and 10 satisfy these conditions. Therefore,

DECOMPOSITION METHOD

$$\begin{aligned} &6x^2 + 19x + 15 \\ &= 6x^2 + 9x + 10x + 15 \\ &= 3x(2x + 3) + 5(2x + 3) \\ &= (2x + 3)(3x + 5) \end{aligned}$$

Example 2 ► **Factoring Trinomials with the Leading Coefficient Different than 1**

Factor completely each trinomial.

a. $6x^3 + 14x^2 + 4x$ **b.** $-6y^2 - 10 + 19y$

c. $18a^2 - 19ab - 12b^2$ **d.** $2(x + 3)^2 + 5(x + 3) - 12$

Solution ► **a.** First, we factor out the GCF, which is $2x$. This gives us

$$6x^3 + 14x^2 + 4x = 2x(3x^2 + 7x + 2)$$

The outside coefficients of the remaining trinomial are prime, so we can apply the guessing method to factor it further. The first terms of the possible binomial factors must be $3x$ and x while the last terms must be 2 and 1. Since both signs in the trinomial are positive, the signs used in the binomial factors must be both positive as well. So, we are ready to give it a try:

$$2x(3x + 2)(x + 1) \quad \text{or} \quad 2x(3x + 1)(x + 2)$$

$2x$ x

$3x$ $6x$

The first distribution of coefficients does not work as it would give us $2x + 3x = 5x$ for the middle term. However, the second distribution works as $x + 6x = 7x$, which matches the middle term of the trinomial. So,

$$6x^3 + 14x^2 + 4x = \mathbf{2x(3x + 1)(x + 2)}$$

b. Notice that the trinomial is not arranged in decreasing order of powers of y. So, first, we rearrange the last two terms to achieve the decreasing order. Also, we factor out the -1, so that the leading term of the remaining trinomial is positive.

$$-6y^2 - 10 + 19y = -6y^2 + 19y - 10 = -(6y^2 - 19y + 10)$$

Then, since the outside coefficients are composite, we will use the decomposition method of factoring. The ac-product equals to 60 and the middle coefficient equals to -19. So, we are looking for two integers that multiply to 60 and add to -19. The integers that satisfy these conditions are -15 and -4. Hence, we factor

$$\begin{aligned} &-(6y^2 - 19y + 10) \\ &= -(6y^2 - 15y - 4y + 10) \\ &= -[3y(2y-5) - 2(2y-5)] \\ &= -\mathbf{(2y-5)(3y-2)} \end{aligned}$$

the **square bracket** is essential because of the negative sign outside

remember to **reverse the sign!**

c. There is no common factor to take out of the polynomial $18a^2 - 19ab - 12b^2$. So, we will attempt to factor it into two binomials of the type $(ma \pm pb)(na \mp qb)$, using the decomposition method. The ac-product equals $-12 \cdot 18 = -2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ and the middle coefficient equals -19. To find the two integers that multiply to the ac-product and add to -19, it is convenient to group the factors of the product

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

in such a way that the products of each group differ by 19. It turns out that grouping all the 2's and all the 3's satisfy this condition, as 8 and 27 differ by 19. Thus, the desired integers are -27 and 8, as the sum of them must be -19. So, we factor

$$\begin{aligned} &18a^2 - 19ab - 12b^2 \\ &= 18a^2 - 27ab + 8ab - 12b^2 \\ &= 9a(2a-3b) + 4b(2a-3b) \\ &= \mathbf{(2a-3b)(9a+4b)} \end{aligned}$$

d. To factor $2(x+3)^2 + 5(x+3) - 12$, first, we notice that treating the group $(x+3)$ as another variable, say a, simplifies the problem to factoring the quadratic trinomial

$$2a^2 + 5a - 12$$

This can be done by the guessing method. Since

$$2a^2 + 5a - 12 = (2a-3)(a+4),$$

$$-3a$$
$$8a$$

then

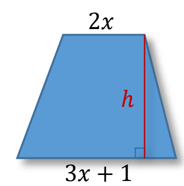
$$\begin{aligned} 2(x+3)^2 + 5(x+3) - 12 &= [2(x+3) - 3][(x+3) + 4] \\ &= (2x + 6 - 3)(x + 3 + 4) \\ &= \mathbf{(2x+3)(x+7)} \end{aligned}$$

Note 1: Polynomials that can be written in the form $\boldsymbol{a}(\quad)^2 + \boldsymbol{b}(\quad) + \boldsymbol{c}$, where $a \neq 0$ and $(\quad)$ represents any nonconstant polynomial expression, are referred to as **quadratic in form**. To factor such polynomials, it is convenient to **replace** the expression in the bracket by **a single variable**, different than the original one. This was illustrated in *Example 2d* by substituting a for $(x + 3)$. However, when using this **substitution method**, we must **remember to leave the final answer in terms of the original variable**. So, after factoring, we replace a back with $(x + 3)$, and then simplify each factor.

Note 2: Some students may feel comfortable factoring polynomials quadratic in form directly, without using substitution.

Example 3 ► **Application of Factoring in Geometry Problems**

Suppose that the area in square meters of a trapezoid is given by the polynomial $5x^2 - 9x - 2$. If the two bases are $2x$ and $(3x + 1)$ meters long, then what polynomial represents the height of the trapezoid?



Solution ► Using the formula for the area of a trapezoid, we write the equation

$$\frac{1}{2}h(a + b) = 5x^2 - 9x - 2$$

Since $a + b = 2x + (3x + 1) = 5x + 1$, then we have

$$\frac{1}{2}h(5x + 1) = 5x^2 - 9x - 2,$$

which after factoring the right-hand side gives us

$$\frac{1}{2}h(5x + 1) = (5x + 1)(x - 2).$$

To find h, it is enough to divide the above equation by the common factor $(5x + 1)$ and then multiply it by 2. So,

$$h = 2(x - 2) = \mathbf{2x - 4}.$$

F.2 Exercises

1. If $ax^2 + bx + c$ has no monomial factor, can either of the possible binomial factors have a monomial factor?
2. Is $(2x + 5)(2x - 4)$ a complete factorization of the polynomial $4x^2 + 2x - 20$?

3. When factoring the polynomial $-2x^2 - 7x + 15$, students obtained the following answers:

$$(-2x+3)(x+5),\ (2x-3)(-x-5),\ \text{or}\ -(2x-3)(x+5)$$

Which of the above factorizations are correct?

4. Is the polynomial $x^2 - x + 2$ factorable or is it prime?

Fill in the missing factor.

5. $x^2 - 4x + 3 = (\qquad)(x-1)$

6. $x^2 + 3x - 10 = (\qquad)(x-2)$

7. $x^2 - xy - 20y^2 = (x+4y)(\qquad)$

8. $x^2 + 12xy + 35y^2 = (x+5y)(\qquad)$

Factor, if possible.

9. $x^2 + 7x + 12$

10. $x^2 - 12x + 35$

11. $y^2 + 2y - 48$

12. $a^2 - a - 42$

13. $x^2 + 2x + 3$

14. $p^2 - 12p - 27$

15. $m^2 - 15m + 56$

16. $y^2 + 3y - 28$

17. $18 - 7n - n^2$

18. $20 + 8p - p^2$

19. $x^2 - 5xy + 6y^2$

20. $p^2 + 9pq + 20q^2$

Factor completely.

21. $-x^2 + 4x + 21$

22. $-y^2 + 14y + 32$

23. $n^4 - 13n^3 - 30n^2$

24. $y^3 - 15y^2 + 54y$

25. $-2x^2 + 28x - 80$

26. $-3x^2 - 33x - 72$

27. $x^4y + 7x^2y - 60y$

28. $24ab^2 + 6a^2b^2 - 3a^3b^2$

29. $40 - 35t^{15} - 5t^{30}$

30. $x^4y^2 + 11x^2y + 30$

31. $64n - 12n^5 - n^9$

32. $24 - 5x^a - x^{2a}$

33. If a polynomial $x^2 + \boxed{}x + 36$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of b?

Fill in the missing factor.

34. $2x^2 + 7x + 3 = (\qquad)(x+3)$

35. $3x^2 - 10x + 8 = (\qquad)(x-2)$

36. $4x^2 + 8x - 5 = (2x-1)(\qquad)$

37. $6x^2 - x - 15 = (2x+3)(\qquad)$

Factor completely.

38. $2x^2 - 5x - 3$

39. $6y^2 - y - 2$

40. $4m^2 + 17m + 4$

41. $6t^2 - 13t + 6$

42. $10x^2 + 23x - 5$

43. $42n^2 + 5n - 25$

44. $3p^2 - 27p + 24$

45. $-12x^2 - 2x + 30$

46. $6x^2 + 41xy - 7y^2$

47. $18x^2 + 27xy + 10y^2$

48. $8 - 13a + 6a^2$

49. $15 - 14n - 8n^2$

50. $30x^4 + 3x^3 - 9x^2$

51. $10x^3 - 6x^2 + 4x^4$

52. $2y^6 + 7xy^3 + 6x^2$

53. $9x^2y^2 - 4 + 5xy$

54. $16x^2y^3 + 3y - 16xy^2$

55. $4p^4 - 28p^2q + 49q^2$

56. $4(x-1)^2 - 12(x-1) + 9$

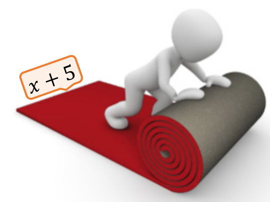
57. $2(a+2)^2 + 11(a+2) + 15$

58. $4x^{2a} - 4x^a - 3$

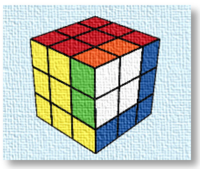
59. If a polynomial $3x^2 + \boxed{}x - 20$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of $\boxed{b}$?

60. The volume of a case of apples is $2x^3 - 3x^2 - 2x$ cubic feet, and the height of the case is $(x - 2)$ feet. Find a polynomial representing the area of the bottom of the case?

61. Suppose the width of a rectangular runner carpet is $(x + 5)$ feet. If the area of the carpet is $(3x^2 + 17x + 10)$ square feet, find the polynomial that represents the length of the carpet.



F3 Special Factoring and a General Strategy of Factoring



Recall that in *Section P2*, we considered formulas that provide a shortcut for finding special products, such as a product of two **conjugate** binomials,

$$(a+b)(a-b) = \boldsymbol{a^2 - b^2},$$

or the **perfect square** of a binomial,

$$(a \pm b)^2 = \boldsymbol{a^2 \pm 2ab + b^2}.$$

Since factoring reverses the multiplication process, these formulas can be used as shortcuts in factoring binomials of the form $\boldsymbol{a^2 - b^2}$ (**difference of squares)**, and trinomials of the form $\boldsymbol{a^2 \pm 2ab + b^2}$ **(perfect square)**. In this section, we will also introduce a formula for factoring binomials of the form $\boldsymbol{a^3 \pm b^3}$ (**sum or difference of cubes)**. These special product factoring techniques are very useful in simplifying expressions or solving equations, as they allow for more efficient algebraic manipulations.

At the end of this section, we give a summary of all the factoring strategies shown in this chapter.

Difference of Squares

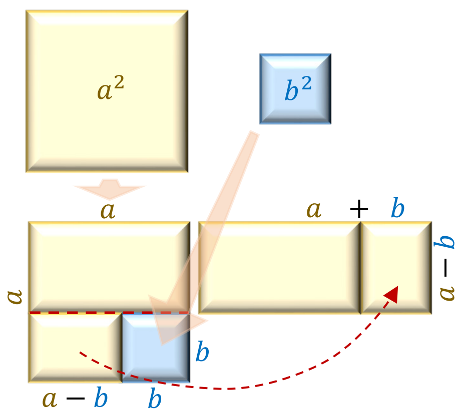


Figure 3.1

Out of the special factoring formulas, the easiest one to use is the difference of squares,

$$\boxed{\boldsymbol{a^2 - b^2 = (a+b)(a-b)}}$$

Figure 3.1 shows a geometric interpretation of this formula. The area of the yellow square, a^2, diminished by the area of the blue square, b^2, can be rearranged to a rectangle with the length of $(a+b)$ and the width of $(a-b)$.

To factor a difference of squares $\boldsymbol{a^2 - b^2}$, first, identify $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared, and then, form two factors, the sum $(\boldsymbol{a+b})$, and the difference $(\boldsymbol{a-b})$, as illustrated in the example below.

Example 1 ► **Factoring Differences of Squares**

Factor each polynomial completely.

a. $25x^2 - 1$ **b.** $3.6x^4 - 0.9y^6$

c. $x^4 - 81$ **d.** $16 - (a-2)^2$

Solution ► **a.** First, we rewrite each term of $25x^2 - 1$ as a perfect square of an expression.

$$25x^2 - 1 = (\underset{a}{5x})^2 - \underset{b}{1}^2$$

Then, treating $5x$ as the a and 1 as the b in the difference of squares formula $a^2 - b^2 = (a+b)(a-b)$, we factor:

$$a^2 - b^2 = (a + b)(a - b)$$

$$25x^2 - 1 = (5x)^2 - 1^2 = (\mathbf{5x + 1})(\mathbf{5x - 1})$$

b. First, we factor out 0.9 to leave the coefficients in a perfect square form. So,

$$3.6x^4 - 0.9y^6 = 0.9(4x^4 - y^6)$$

Then, after writing the terms of $4x^4 - y^6$ as perfect squares of expressions that correspond to a and b in the difference of squares formula $a^2 - b^2 = (a + b)(a - b)$, we factor

$$0.9(4x^4 - y^6) = 0.9[(\underbrace{2x^2}_{a})^2 - (\underbrace{y^3}_{b})^2] = \mathbf{0.9(2x^2 + y^3)(2x^2 - y^3)}$$

c. Similarly as in the previous two examples, $x^4 - 81$ can be factored by following the difference of squares pattern. So,

$$x^4 - 81 = (x^2)^2 - (9)^2 = (x^2 + 9)(x^2 - 9)$$

However, this factorization is not complete yet. Notice that $x^2 - 9$ is also a difference of squares, so the original polynomial can be factored further. Thus,

$$x^4 - 81 = (x^2 + 9)(x^2 - 9) = \mathbf{(x^2 + 9)(x + 3)(x - 3)}$$

Recall that $\boldsymbol{a^2 + b^2 \neq (a + b)^2}$

Attention: The sum of squares, $x^2 + 9$, cannot be factored using real coefficients.

Generally, except for a common factor, a quadratic binomial of the form $\boldsymbol{a^2 + b^2}$ is **not factorable** over the real numbers.

d. Following the difference of squares formula, we have

$$\begin{aligned} 16 - (a - 2)^2 &= 4^2 - (a - 2)^2 \\ &= [4 + (a - 2)][4 - (a - 2)] \\ &= (4 + a - 2)(4 - a + 2) && \text{work out the inner brackets} \\ &= \mathbf{(2 + a)(6 - a)} && \text{combine like terms} \end{aligned}$$

Remember to use brackets after the negative sign!

Perfect Squares

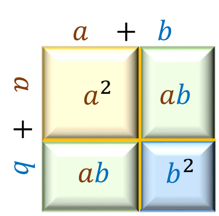


Figure 3.2

Another frequently used special factoring formula is the **perfect square** of a sum or a difference.

$$\boldsymbol{a^2 + 2ab + b^2 = (a + b)^2}$$

or

$$\boldsymbol{a^2 - 2ab + b^2 = (a - b)^2}$$

Figure 3.2 shows the geometric interpretation of the perfect square of a sum. We encourage the reader to come up with a similar interpretation of the perfect square of a difference.

To factor a perfect square trinomial $\boldsymbol{a^2 \pm 2ab + b^2}$, we find $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared. Then, depending on the middle sign, we use $\boldsymbol{a}$ and $\boldsymbol{b}$ to form the perfect square of the sum $(\boldsymbol{a} + \boldsymbol{b})^2$, or the perfect square of the difference $(\boldsymbol{a} - \boldsymbol{b})^2$.

Example 2 ► **Identifying Perfect Square Trinomials**

Decide whether the given polynomial is a perfect square.

a. $9x^2 + 6x + 4$ **b.** $9x^2 + 4y^2 - 12xy$

c. $25p^4 + 40p^2 - 16$ **d.** $49y^6 + 84xy^3 + 36x^2$

Solution ►

a. Observe that the outside terms of the trinomial $9x^2 + 6x + 4$ are perfect squares, as $9x^2 = (3x)^2$ and $4 = 2^2$. So, the trinomial would be a perfect square if the middle terms would equal $2 \cdot 3x \cdot 2 = 12x$. Since this is not the case, our trinomial is **not a perfect square**.

Attention: Except for a common factor, trinomials of the type $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable** over the real numbers!

b. First, we arrange the trinomial in decreasing order of the powers of x. So, we obtain $9x^2 - 12xy + 4y^2$. Then, since $9x^2 = (3x)^2$, $4y^2 = (2y)^2$, and the middle term (except for the sign) equals $2 \cdot 3x \cdot 2y = 12xy$, we claim that the trinomial **is a perfect square**. Since the middle term is negative, this is the perfect square of a difference. So, the trinomial $9x^2 - 12xy + 4y^2$ can be seen as

$$\begin{array}{c} a^2 - 2\,a\,b + b^2 = (a - b)^2 \\ \downarrow \\ (3x)^2 - 2 \cdot 3x \cdot 2y + (2y)^2 = (3x - 2y)^2 \end{array}$$

c. Even though the coefficients of the trinomial $25p^4 + 40p^2 - 16$ and the distribution of powers seem to follow the pattern of a perfect square, the last term is negative, which makes it **not a perfect square**.

d. Since $49y^6 = (7y^3)^2$, $36x^2 = (6x)^2$, and the middle term equals $2 \cdot 7y^3 \cdot 6x = 84xy^3$, we claim that the trinomial **is a perfect square**. Since the middle term is positive, this is the perfect square of a sum. So, the trinomial $49y^6 + 84xy^3 + 36x^2$ can be seen as

$$\begin{array}{c} a^2 + 2\,a\,b + b^2 = (a + b)^2 \\ \downarrow \\ (7y^3)^2 + 2 \cdot 7y^3 \cdot 6x + (6x)^2 = (7y^3 + 6x)^2 \end{array}$$

Example 3 ► **Factoring Perfect Square Trinomials**

Factor each polynomial completely.

a. $25x^2 + 10x + 1$ **b.** $a^2 - 12ab + 36b^2$

c. $m^2 - 8m + 16 - 49n^2$ **d.** $-4y^2 - 144y^8 + 48y^5$

Solution ►

a. The outside terms of the trinomial $25x^2 + 10x + 1$ are perfect squares of $5a$ and 1, and the middle term equals $2 \cdot 5x \cdot 1 = 10x$, so we can follow the perfect square formula. Therefore,

$$25x^2 + 10x + 1 = (\mathbf{5x} + \mathbf{1})^2$$

b. The outside terms of the trinomial $a^2 - 12ab + 36b^2$ are perfect squares of a and $6b$, and the middle term (disregarding the sign) equals $2 \cdot a \cdot 6b = 12ab$, so we can follow the perfect square formula. Therefore,

$$a^2 - 12ab + 36b^2 = (\mathbf{a} - \mathbf{6b})^2$$

c. Observe that the first three terms of the polynomial $m^2 - 8m + 16 - 49n^2$ form a perfect square of $m - 4$ and the last term is a perfect square of $7n$. So, we can write

$$m^2 - 8m + 16 - 49n^2 = (m - 4)^2 - (7n)^2$$

This is not in factored form yet!

Notice that this way we have formed a difference of squares. So we can factor it by following the difference of squares formula

$$(m - 4)^2 - (7n)^2 = (\mathbf{m - 4 - 7n})(\mathbf{m - 4 + 7n})$$

d. As in any factoring problem, first we check the polynomial $-4y^2 - 144y^8 + 48y^5$ for a common factor, which is $4y^2$. To leave the leading term of this polynomial positive, we factor out $-4y^2$. So, we obtain

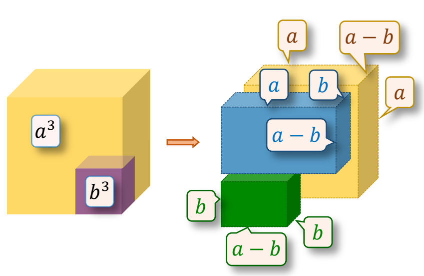
$$-4y^2 - 144y^8 + 48y^5$$
$$= -4y^2\,(1 + 36y^6 - 12y^3)$$
$$= -4y^2\,(36y^6 - 12y^3 + 1)$$

arrange the polynomial in decreasing powers

$$= \mathbf{-4y^2\,(6y^3 - 1)^2}$$

fold to the perfect square form

Sum or Difference of Cubes



$a^3 - b^3 = a^2(a-b) + ab(a-b) + b^2(a-b)$

$= (a-b)(a^2 + ab + b^2)$

Figure 3.3

The last special factoring formula to discuss in this section is the **sum or difference of cubes**.

$$\mathbf{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

or

$$\mathbf{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

The reader is encouraged to confirm these formulas by multiplying the factors in the right-hand side of each equation. In addition, we offer a geometric visualization of one of these formulas, the difference of cubes, as shown in *Figure 3.3*.

Hints for memorization of the sum or difference of cubes formulas:

- The binomial factor is a copy of the sum or difference of the terms that were originally cubed.
- The trinomial factor follows the pattern of a perfect square, except that the **middle term is single**, not doubled.
- The signs in the factored form follow the pattern ***S****ame*-***O****pposite*-***P****ositive* (**SOP**).

Example 4 ► **Factoring Sums or Differences of Cubes**

Factor each polynomial completely.

a. $8x^3 + 1$

b. $27x^7y - 125xy^4$

c. $2n^6 - 128$

d. $(p-2)^3 + q^3$

Solution ►

a. First, we rewrite each term of $8x^3 + 1$ as a perfect cube of an expression.

$$8x^3 + 1 = (\overset{a}{\overset{\downarrow}{2x}})^3 + \overset{b}{\overset{\downarrow}{1}}{}^3$$

Then, treating $2x$ as the a and 1 as the b in the sum of cubes formula $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, we factor:

$$\begin{aligned} \boldsymbol{a^3} + \boldsymbol{b^3} &= (\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a^2} - \boldsymbol{a}\ \boldsymbol{b} + \boldsymbol{b^2}) \\ 8x^3 + 1 = (2x)^3 + 1^3 &= (2x + 1)((2x)^2 - 2x \cdot 1 + 1^2) \\ &= \boldsymbol{(2x+1)(4x^2 - 2x + 1)} \end{aligned}$$

Quadratic trinomials of the form $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable**!

Notice that the trinomial $4x^2 - 2x + 1$ is not factorable anymore.

b. Since the two terms of the polynomial $27x^7y - 125xy^4$ contain the common factor xy, we factor it out and obtain

$$27x^7y - 125xy^4 = xy(27x^6 - 125y^3)$$

Observe that the remaining polynomial is a difference of cubes, $(3x^2)^3 - (5y)^3$. So, we factor,

$$\begin{aligned} 27x^7y - 125xy^4 &= xy[(3x^2)^3 - (5y)^3] \\ & \quad\ \ (\boldsymbol{a} - \boldsymbol{b})(\boldsymbol{a^2} + \boldsymbol{a}\ \boldsymbol{b} + \boldsymbol{b^2}) \\ &= xy(3x^2 - 5y)[(3x^2)^2 + 3x^2 \cdot 5y + (5y)^2] \\ &= \boldsymbol{xy(3x^2 - 5y)(9x^4 + 15x^2y + 25y^2)} \end{aligned}$$

c. After factoring out the common factor 2, we obtain

$$2n^6 - 128 = 2(n^6 - 64)$$

Difference of squares or **difference of cubes?**

Notice that n^6-64 can be seen either as a difference of squares, $(n^3)^2-8^2$, or as a difference of cubes, $(n^2)^3-4^3$. It turns out that applying the **difference of squares** formula first **leads us to a complete factorization** while starting with the difference of cubes does not work so well here. See the two approaches below.

$$(n^3)^2-8^2$$
$$=(n^3+8)(n^3-8)$$
$$=(n+2)(n^2-2n+4)(n-2)(n^2+2n+4)$$

4 prime factors, so the factorization is complete

$$(n^2)^3-4^3$$
$$=(n^2-4)(n^4+4n^2+16)$$
$$=(n+2)(n-2)(n^4+4n^2+16)$$

There is no easy way of factoring this trinomial!

Therefore, the original polynomial should be factored as follows:

$$2n^6-128=2(n^6-64)=2[(n^3)^2-8^2]=2(n^3+8)(n^3-8)$$
$$=\mathbf{2(n+2)(n^2-2n+4)(n-2)(n^2+2n+4)}$$

d. To factor $(p-2)^3+q^3$, we follow the sum of cubes formula $(a+b)(a^2-ab+b^2)$ by assuming $a=p-2$ and $b=q$. So, we have

$$(p-2)^3+q^3=(p-2+q)\,[(p-2)^2-(p-2)q+q^2]$$
$$=(p-2+q)\,[p^2-4p+4-pq+2q+q^2]$$
$$=\mathbf{(p+q-2)\,[p^2-pq+q^2-4p+2q+4]}$$

General Strategy of Factoring

Recall that a polynomial with integral coefficients is factored completely if all of its factors are prime over the integers.

How to Factorize Polynomials Completely?

1. Factor out all **common factors**. Leave the remaining polynomial with a positive leading term and integral coefficients, if possible.

2. Check the number of terms. If the polynomial has

- **more than three terms**, try to factor by **grouping**; a four term polynomial may require 2-2, 3-1, or 1-3 types of grouping.
- **three terms**, factor by **guessing**, **decomposition**, or follow the **perfect square** formula, if applicable.
- **two terms**, follow the **difference of squares**, or **sum or difference of cubes** formula, if applicable. Remember that sum of squares, $\boldsymbol{a^2+b^2}$, is **not factorable** over the real numbers, except for possibly a common factor.

3. Keep in mind the **special factoring formulas**:

Difference of Squares	$\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$
Perfect Square of a Sum	$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$
Perfect Square of a Difference	$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$
Sum of Cubes	$\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$
Difference of Cubes	$\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$

4. **Keep factoring** each of the obtained factors until all of them are **prime** over the integers.

Example 5 ► **Multiple-step Factorization**

Factor each polynomial completely.

a. $80x^5 - 5x$ **b.** $4a^2 - 4a + 1 - b^2$

c. $(5r+8)^2 - 6(5r+8) + 9$ **d.** $(p-2q)^3 + (p+2q)^3$

Solution ► **a.** First, we factor out the GCF of $80x^5$ and $-5x$, which equals to $5x$. So, we obtain

$$80x^5 - 5x = 5x(16x^4 - 1)$$

repeated difference of squares

Then, we notice that $16x^4 - 1$ can be seen as the difference of squares $(4x^2)^2 - 1^2$. So, we factor further

$$80x^5 - 5x = 5x(4x^2+1)(4x^2-1)$$

The first binomial factor, $4x^2 + 1$, cannot be factored any further using integral coefficients as it is the sum of squares, $(2x)^2 + 1^2$. However, the second binomial factor, $4x^2 - 1$, is still factorable as a difference of squares, $(2x)^2 - 1^2$. Therefore,

$$80x^5 - 5x = \boldsymbol{5x(4x^2+1)(2x+1)(2x-1)}$$

This is a complete factorization as all the factors are prime over the integers.

3-1 type of grouping

b. The polynomial $4a^2 - 4a + 1 - b^2$ consists of four terms, so we might be able to factor it by grouping. Observe that the 2-2 type of grouping has no chance to succeed, as the first two terms involve only the variable a while the second two terms involve only the variable b. This means that after factoring out the common factor in each group, the remaining binomials would not be the same. So, the 2-2 grouping would not lead us to a factorization. However, the 3-1 type of grouping should help. This is because the first three terms form the perfect square, $(2a-1)^2$, and there is a subtraction before the last term b^2, which is also a perfect square. So, in the end, we can follow the difference of squares formula to complete the factoring process.

$$\begin{aligned} \underbrace{4a^2 - 4a + 1} - \underbrace{b^2} &= (2a-1)^2 - b^2 \\ &= \boldsymbol{(2a-1-b)(2a-1+b)} \end{aligned}$$

factoring by substitution

c. To factor $(5r+8)^2 - 6(5r+8) + 9$, it is convenient to substitute a new variable, say $\boldsymbol{a}$, for the expression $5r+8$. Then,

$$(5r+8)^2 - 6(5r+8) + 9 = \boldsymbol{a}^2 - 6\boldsymbol{a} + 9$$

perfect square!

$$= (\boldsymbol{a} - 3)^2$$

Remember to represent the new variable by a **different letter** than the original variable!

$$= (5r + 8 - 3)^2$$

go back to the original variable

$$= (5r+5)^2$$

Notice that $5r+5$ can still be factored by taking the 5 out. So, for a complete factorization, we factor further

$$(5r+5)^2 = \left(5(r+1)\right)^2 = \mathbf{25}(\boldsymbol{r}+\mathbf{1})^{\mathbf{2}}$$

multiple special formulas and simplifying

d. To factor $(p-2q)^3 + (p+2q)^3$, we follow the sum of cubes formula $(a+b)(a^2 - ab + b^2)$ by assuming $a = p - 2q$ and $b = p + 2q$. So, we have

$$(p-2q)^3 + (p+2q)^3$$

$$= (p - \cancel{2q} + p + \cancel{2q})\,[(p-2q)^2 - (p-2q)(p+2q) + (p+2q)^2]$$

$$= 2p\,[p^2 - \cancel{4pq} + 4q^2 - (p^2 - 4q^2) + p^2 + \cancel{4pq} + 4q^2]$$

$$= 2p\,(2p^2 + 8q^2 - p^2 + 4q^2) = \mathbf{2}\boldsymbol{p}\left(\boldsymbol{p}^{\mathbf{2}} + \mathbf{12}\boldsymbol{q}^{\mathbf{2}}\right)$$

F.3 Exercises

Determine whether each polynomial in problems 7-18 is a perfect square, a difference of squares, a sum or difference of cubes, or neither.

1. $0.25x^2 - 0.16y^2$

2. $x^2 - 14x + 49$

3. $9x^4 + 4x^2 + 1$

4. $4x^2 - (x+4)^2$

5. $125x^3 - 64$

6. $y^{12} + 0.008x^3$

7. $-y^4 + 16x^4$

8. $64 + 48x^3 + 9x^6$

9. $25x^6 - 10x^3y^2 + y^4$

10. $-4x^6 - y^6$

11. $-8x^3 + 27y^6$

12. $81x^2 - 16x$

13. Generally, the sum of squares is not factorable. For example, $x^2 + 9$ cannot be factored in integral coefficients. However, some sums of squares can be factored. For example, the binomial $25x^2 + 100$ can be factored. Factor the above example and discuss what makes a sum of two squares factorable.

14. Insert the correct signs into the blanks.

a. $8 + a^3 = (2 \_\_ a)(4 \_\_ 2a \_\_ a^2)$ **b.** $b^3 - 1 = (b \_\_ 1)(b^2 \_\_ b \_\_ 1)$

Factor each polynomial completely, if possible.

15. $x^2 - y^2$
16. $x^2 + 2xy + y^2$
17. $x^3 - y^3$
18. $16x^2 - 100$
19. $4z^2 - 4z + 1$
20. $x^3 + 27$
21. $4z^2 + 25$
22. $y^2 + 18y + 81$
23. $125 - y^3$
24. $144x^2 - 64y^2$
25. $n^2 + 20nm + 100m^2$
26. $27a^3b^6 + 1$
27. $9a^4 - 25b^6$
28. $25 - 40x + 16x^2$
29. $p^6 - 64q^3$
30. $16x^2z^2 - 100y^2$
31. $4 + 49p^2 + 28p$
32. $x^{12} + 0.008y^3$
33. $r^4 - 9r^2$
34. $9a^2 - 12ab - 4b^2$
35. $\frac{1}{8} - a^3$
36. $0.04x^2 - 0.09y^2$
37. $x^4 + 8x^2 + 1$
38. $-\frac{1}{27} + t^3$
39. $16x^6 - 121x^2y^4$
40. $9 + 60pq + 100p^2q^2$
41. $-a^3b^3 - 125c^6$
42. $36n^{2t} - 1$
43. $9a^8 - 48a^4b + 64b^2$
44. $9x^3 + 8$
45. $(x+1)^2 - 49$
46. $\frac{1}{4}u^2 - uv + v^2$
47. $2t^4 - 128t$
48. $81 - (n+3)^2$
49. $x^{2n} + 6x^n + 9$
50. $8 - (a+2)^3$
51. $16z^4 - 1$
52. $5c^3 + 20c^2 + 20c$
53. $(x+5)^3 - x^3$
54. $a^4 - 81b^4$
55. $0.25z^2 - 0.7z + 0.49$
56. $(x-1)^3 + (x+1)^3$
57. $(x-2y)^2 - (x+y)^2$
58. $0.81p^8 + 9p^4 + 25$
59. $(x+2)^3 - (x-2)^3$

Factor each polynomial completely.

60. $3y^3 - 12x^2y$
61. $2x^2 + 50a^2 - 20ax$
62. $x^3 - xy^2 + x^2y - y^3$
63. $y^2 - 9a^2 + 12y + 36$
64. $64u^6 - 1$
65. $7m^3 + m^6 - 8$
66. $-7n^2 + 2n^3 + 4n - 14$
67. $a^8 - b^8$
68. $y^9 - y$
69. $(x^2-2)^2 - 4(x^2-2) - 21$
70. $8(p-3)^2 - 64(p-3) + 128$
71. $a^2 - b^2 - 6b - 9$
72. $25(2a-b)^2 - 9$
73. $3x^2y^2z + 25xyz^2 + 28z^3$
74. $x^{8a} - y^2$
75. $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$
76. $4x^2y^4 - 9y^4 - 4x^2z^4 + 9z^4$
77. $c^{2w+1} + 2c^{w+1} + c$

F4 Solving Polynomial Equations and Applications of Factoring

Many application problems involve solving polynomial equations. In Chapter L, we studied methods for solving linear, or first-degree, equations. Solving higher degree polynomial equations requires other methods, which often involve factoring. In this chapter, we study solving polynomial equations using the zero-product property, graphical connections between roots of an equation and zeros of the corresponding function, and some application problems involving polynomial equations or formulas that can be solved by factoring.

Zero-Product Property

Recall that to solve a linear equation, for example $2x + 1 = 0$, it is enough to isolate the variable on one side of the equation by applying reverse operations. Unfortunately, this method usually does not work when solving higher degree polynomial equations. For example, we would not be able to solve the equation $x^2 - x = 0$ through the reverse operation process, because the variable x appears in different powers.

So … how else can we solve it?

In this particular example, it is possible to guess the solutions. They are $x = 0$ and $x = 1$.

But how can we solve it algebraically?

It turns out that factoring the left-hand side of the equation $x^2 - x = 0$ helps. Indeed, $x(x - 1) = 0$ tells us that the product of x and $x - 1$ is 0. Since the product of two quantities is 0, at least one of them must be 0. So, either $x = 0$ or $x - 1 = 0$, which solves to $x = 1$.

The equation discussed above is an example of a second degree polynomial equation, more commonly known as a quadratic equation.

Definition 4.1 ► A **quadratic equation** is a second degree polynomial equation in one variable that can be written in the form,

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $a \neq 0$. This form is called **standard form**.

One of the methods of solving such equations involves factoring and the zero-product property that is stated below.

Zero-Product Property ► For any real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{ab = 0} \text{ if and only if } \boldsymbol{a = 0} \text{ or } \boldsymbol{b = 0}$$

This means that any product containing a factor of 0 is equal to 0, and conversely, if a product is equal to 0, then at least one of its factors is equal to 0.

Proof ► The implication "if $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$, then $\boldsymbol{ab = 0}$" is true by the multiplicative property of zero.

To prove the implication "if $\boldsymbol{ab = 0}$, then $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$", let us assume first that $a \neq 0$. (As, if $a = 0$, then the implication is already proven.)

Since $a \neq 0$, then $\frac{1}{a}$ exists. Therefore, both sides of $ab = 0$ can be multiplied by $\frac{1}{a}$ and we obtain

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$

$$b = 0,$$

which concludes the proof.

Attention: The zero-product property works only for a product equal to $\mathbf{0}$. For example, the fact that $\boldsymbol{ab} = \mathbf{1}$ does not mean that either a or b equals to 1.

Example 1 ► **Using the Zero-Product Property to Solve Polynomial Equations**

Solve each equation.

a. $(x-3)(2x+5) = 0$ **b.** $2x(x-5)^2 = 0$

Solution ► **a.** Since the product of $x - 3$ and $2x + 5$ is equal to zero, then by the zero-product property at least one of these expressions must equal to zero. So,

$$x - 3 = 0 \quad \text{or} \quad 2x + 5 = 0$$

which results in $\quad x = 3 \quad \text{or} \quad 2x = -5$

$$x = -\frac{5}{2}$$

Thus, $\left\{-\frac{5}{2}, 3\right\}$ is the solution set of the given equation.

b. Since the product $2x(x-5)^2$ is zero, then either $x = 0$ or $x - 5 = 0$, which solves to $x = 5$. Thus, the solution set is equal to $\{\mathbf{0}, \mathbf{5}\}$.

Note 1: The factor of 2 does not produce any solution, as 2 is never equal to 0.

Note 2: The perfect square $(x-5)^2$ equals to 0 if and only if the base $x - 5$ equals to 0.

Solving Polynomial Equations by Factoring

To solve polynomial equations of second or higher degree by factoring, we

- **arrange** the polynomial **in decreasing order** of powers **on one side** of the equation,
- keep the **other side** of the equation **equal to 0**,
- **factor** the polynomial **completely**,
- use the zero-product property to **form linear equations for each factor**,
- **solve** the linear equations to find the roots (solutions) to the original equation.

Example 2 ► **Solving Quadratic Equations by Factoring**

Solve each equation by factoring.

a. $x^2 + 9 = 6x$

b. $15x^2 - 12x = 0$

c. $(x + 2)(x - 1) = 4(3 - x) - 8$

d. $(x - 3)^2 = 36x^2$

Solution ►

a. To solve $x^2 + 9 = 6x$ by factoring we need one side of this equation equal to 0. So, first, we move the $6x$ term to the left side of the equation,

$$x^2 + 9 - 6x = 0,$$

and arrange the terms in decreasing order of powers of x,

$$x^2 - 6x + 9 = 0.$$

Then, by observing that the resulting trinomial forms a perfect square of $x - 3$, we factor

$$(x - 3)^2 = 0,$$

which is equivalent to

$$x - 3 = 0,$$

and finally

$$x = 3.$$

So, the solution is $x = \mathbf{3}$.

b. After factoring the left side of the equation $15x^2 - 12x = 0$,

$$3x(5x - 4) = 0,$$

we use the zero-product property. Since 3 is never zero, the solutions come from the equations

$$x = \mathbf{0} \quad \text{or} \quad 5x - 4 = 0.$$

Solving the second equation for x, we obtain

$$5x = 4,$$

and finally

$$x = \frac{\mathbf{4}}{\mathbf{5}}.$$

So, the solution set consists of **0** and $\frac{\mathbf{4}}{\mathbf{5}}$.

c. To solve $(x + 2)(x - 1) = 4(3 - x) - 8$ by factoring, first, we work out the brackets and arrange the polynomial in decreasing order of exponents on the left side of the equation. So, we obtain

$$x^2 + x - 2 = 12 - 4x - 8$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

Now, we can read the solutions from each bracket, that is, $x = \mathbf{-6}$ and $x = \mathbf{1}$.

Observation: In the process of solving a linear equation of the form $ax + b = 0$, first we subtract b and then we divide by a. So the solution, sometimes referred to as the root, is $x = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$. This allows us to read the solution directly from the equation. For example, the solution to $x - 1 = 0$ is $x = 1$ and the solution to $2x - 1 = 0$ is $x = \frac{1}{2}$.

d. To solve $(x-3)^2 = 36x^2$, we bring all the terms to one side and factor the obtained difference of squares, following the formula $a^2 - b^2 = (a+b)(a-b)$. So, we have

$$(x-3)^2 - 36x^2 = 0$$
$$(x-3+6x)(x-3-6x) = 0$$
$$(7x-3)(-5x-3) = 0$$

Then, by the zero-product property,

$$7x - 3 = 0 \text{ or } -5x - 3 = 0,$$

which results in

$$x = \frac{\mathbf{3}}{\mathbf{7}} \text{ or } x = -\frac{\mathbf{3}}{\mathbf{5}}.$$

Example 3 ► **Solving Polynomial Equations by Factoring**

Solve each equation by factoring.

a. $2x^3 - 2x^2 = 12x$

b. $x^4 + 36 = 13x^2$

Solution ► **a.** First, we bring all the terms to one side of the equation and then factor the resulting polynomial.

$$2x^3 - 2x^2 = 12x$$
$$2x^3 - 2x^2 - 12x = 0$$
$$2x(x^2 - x - 6) = 0$$
$$2x(x-3)(x+2) = 0$$

By the zero-product property, the factors $x, (x-3)$ and $(x+2)$, give us the corresponding solutions, $0, 3,$ and -2. So, the solution set of the given equation is $\{\mathbf{0}, \mathbf{3}, \mathbf{-2}\}$.

b. Similarly as in the previous examples, we solve $x^4 + 36 = 13x^2$ by factoring and using the zero-product property. Since

$$x^4 - 13x^2 + 36 = 0$$

$$(x^2-4)(x^2-9)=0$$
$$(x+2)(x-2)(x+3)(x-3)=0,$$

then, the solution set of the original equation is $\{\boldsymbol{-2,2,-3,3}\}$

Observation: n-th degree polynomial equations may have up to n roots (solutions).

Factoring in Applied Problems

Factoring is a useful strategy when solving applied problems. For example, factoring is often used in **solving formulas** for a variable, in **finding roots** of a polynomial function, and generally, in any problem involving **polynomial equations** that can be solved by factoring.

Example 4 ► **Solving Formulas with the Use of Factoring**

Solve each formula for the specified variable.

a. $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$, for $\boldsymbol{h}$ **b.** $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solution ►

a. To solve $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$ for $\boldsymbol{h}$, we want to keep both terms containing $\boldsymbol{h}$ on the same side of the equation and bring the remaining terms to the other side. Here is an equivalent equation,

$$A-2wl=2\boldsymbol{h}w+2l\boldsymbol{h},$$

which, for convenience, could be written starting with h-terms:

$$2\boldsymbol{h}w+2l\boldsymbol{h}=A-2wl$$

Now, factoring $\boldsymbol{h}$ out causes $\boldsymbol{h}$ to appear in only one place, which is what we need to isolate it. So,

$$(2w+2l)\boldsymbol{h}=A-2wl$$
$$\boldsymbol{h}=\boldsymbol{\frac{A-2wl}{2w+2l}}$$

Notice: In the above formula, there is nothing that can be simplified. Trying to reduce 2 or $2w$ or l would be an error, as there is no essential common factor that can be carried out of the numerator.

b. When solving $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$ for $\boldsymbol{t}$, our goal is to, firstly, keep the variable $\boldsymbol{t}$ in the numerator and secondly, to keep it in a single place. So, we have

$$s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$$
$$s\boldsymbol{t}=2\boldsymbol{t}+3$$

factor t

$$st - 2t = 3$$
$$t(s-2) = 3$$
$$t = \frac{3}{s-2}.$$

Example 5 ► **Finding Roots of a Polynomial Function**

A toy-rocket is launched vertically with an initial velocity of 40 meters per second. If its height in meters after t seconds is given by the function

$$h(t) = -5t^2 + 40t,$$

in how many seconds will the rocket hit the ground?

Solution ► The rocket hits the ground when its height is 0. So, we need to find the time t for which $h(t) = 0$. Therefore, we solve the equation

$$-5t^2 + 40t = 0$$

for t. From the factored form

$$-5t(t-8) = 0$$

we conclude that the rocket is on the ground at times 0 and 8 seconds. So, the rocket hits the ground **8 seconds** after it was launched.

Example 6 ► **Solving an Application Problem with the Use of Factoring**

The height of a triangle is 1 meter less than twice the length of the base. If the area of the triangle is 14 m^2, how long are the base and the height?

Solution ► Let b and h represent the base and the height of the triangle, correspondingly. The first sentence states that h is 1 less than 2 times b. So, we record

$$h = 2b - 1.$$

Using the formula for area of a triangle, $A = \frac{1}{2}bh$, and the fact that $A = 14$, we obtain

$$14 = \frac{1}{2}b(2b-1).$$

Since this is a one-variable quadratic equation, we will attempt to solve it by factoring, after bringing all the terms to one side of the equation. So, we have

to clear the fraction, multiply each term by 2 before working out the bracket

$$0 = \frac{1}{2}b(2b-1) - 14$$
$$0 = b(2b-1) - 28$$
$$0 = 2b^2 - b - 28$$

$$0 = (2b + 7)(b - 4),$$

which by the zero-product property gives us $b = -\frac{7}{2}$ or $b = 4$. Since b represents the length of the base, it must be positive. So, the base is **4** meters long and the height is $h = 2b - 1 = 2 \cdot 4 - 1 = \mathbf{7}$ meters long.

F.4 Exercises

True or false.

1. If $xy = 0$ then $x = 0$ or $y = 0$.

2. If $ab = 1$ then $a = 1$ or $b = 1$.

3. If $x + y = 0$ then $x = 0$ or $y = 0$.

4. If $a^2 = 0$ then $a = 0$.

5. If $x^2 = 1$ then $x = 1$.

6. Which of the following equations is **<u>not</u>** in proper form for using the zero-product property.

a. $x(x-1) + 3(x-1) = 0$

b. $(x+3)(x-1) = 0$

c. $x(x-1) = 3(x-1)$

d. $(x+3)(x-1) = -3$

Solve each equation.

7. $3(x-1)(x+4) = 0$

8. $2(x+5)(x-7) = 0$

9. $(3x+1)(5x+4) = 0$

10. $(2x-3)(4x-1) = 0$

11. $x^2 + 9x + 18 = 0$

12. $x^2 - 18x + 80 = 0$

13. $2x^2 = 7 - 5x$

14. $3k^2 = 14k - 8$

15. $x^2 + 6x = 0$

16. $6y^2 - 3y = 0$

17. $(4-a)^2 = 0$

18. $(2b+5)^2 = 0$

19. $0 = 4n^2 - 20n + 25$

20. $0 = 16x^2 + 8x + 1$

21. $p^2 - 32 = -4p$

22. $19a + 36 = 6a^2$

23. $x^2 + 3 = 10x - 2x^2$

24. $3x^2 + 9x + 30 = 2x^2 - 2x$

25. $(3x+4)(3x-4) = -10x$

26. $(5x+1)(x+3) = -2(5x+1)$

27. $4(y-3)^2 - 36 = 0$

28. $3(a+5)^2 - 27 = 0$

29. $(x-3)(x+5)=-7$

30. $(x+8)(x-2)=-21$

31. $(2x-1)(x-3)=x^2-x-2$

32. $4x^2+x-10=(x-2)(x+1)$

33. $4(2x+3)^2-(2x+3)-3=0$

34. $5(3x-1)^2+3=-16(3x-1)$

35. $x^3+2x^2-15x=0$

36. $6x^3-13x^2-5x=0$

37. $25x^3=64x$

38. $9x^3=49x$

39. $y^4-26y^2+25=0$

40. $n^4-50n^2+49=0$

41. $x^3-6x^2=-8x$

42. $x^3-2x^2=3x$

43. $a^3+a^2-9a-9=0$

44. $2x^3-x^2-2x+1=0$

45. $5x^3+2x^2-20x-8=0$

46. $2x^3+3x^2-18x-27=0$

47. Discuss the validity of the following solution:

$$x^3=9x$$
$$x^2=9$$
$$x=3$$

How many solutions should we expect? What is the solution set of the original equation? What went wrong in the above procedure?

48. Given that $f(x)=x^2+14x+50$, find all values of x such that $f(x)=5$.

49. Given that $g(x)=2x^2-15x$, find all values of x such that $g(x)=-7$.

50. Given that $f(x)=2x^2+3x$ and $g(x)=-6x+5$, find all values of x such that $f(x)=g(x)$.

51. Given that $g(x)=2x^2+11x-16$ and $h(x)=5+9x-x^2$, find all values of x such that $g(x)=h(x)$.

Solve each equation for the specified variable.

52. $\boldsymbol{P}rt=A-\boldsymbol{P}$, for $\boldsymbol{P}$

53. $3\boldsymbol{s}+2p=5-r\boldsymbol{s}$, for $\boldsymbol{s}$

54. $5a+b\boldsymbol{r}=\boldsymbol{r}-2c$, for $\boldsymbol{r}$

55. $E=\frac{R+\boldsymbol{r}}{\boldsymbol{r}}$, for $\boldsymbol{r}$

56. $z=\frac{x+2\boldsymbol{y}}{\boldsymbol{y}}$, for $\boldsymbol{y}$

57. $c=\frac{-2\boldsymbol{t}+4}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solve each problem.

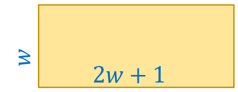
58. Bartek threw down a small rock from the top of a 120 m high observation tower. Suppose the distance travelled by the rock, in meters, is modelled by the function $d(t)=vt+4t^2$, where v is the initial velocity in m/s, and t is the time in seconds. In how many seconds will the rock hit the ground if it was thrown with the initial velocity of 4 m/s?

59. A camera is dropped from a hot-air balloon 320 meters above the ground. Suppose the height of the camera above the ground, in meters, is given by the function $h(t)=320-5t^2$, where t is the time in seconds. How long will it take for the camera to hit the ground?

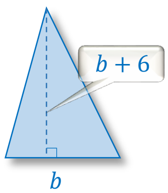
60. The sum of squares of two consecutive numbers is 85. Find the smaller number.

61. The difference between a number and its square is -156. Find the number.

62. The length of a rectangle is 1 centimeter more than twice the width. If the area of this rectangle is 105 cm^2, find its width and length.



63. A postcard is 7 cm longer than it is wide. The area of this postcard is 144 cm^2. Find its length and width.



64. A triangle with the area of 80 cm^2 is 6 cm taller than the length of its base. Find the dimensions of the triangle.

65. A triangular house is 3 m taller than it is wide. If the cross-sectional area (see the accompanying picture) of the house is 35 m^2, what are the width and the height of this house?

66. Amira designs a rectangular flower bed with a pathway of uniform width around it. She has 42 square meters of ground available for the whole project (including the path). If the flower bed is planned to be 3 meters by 4 meters, how wide would be the pathway around it?

67. Suppose a rectangular flower bed is 5 m longer than it is wide. What are the dimensions of the flower bed if its area is 84 m^2 ?

68. Suppose a picture frame measures 10 cm by 18 cm, and it frames a picture with 48 cm^2 of area. How wide is the frame?

69. When 187 cm^2 picture is framed, its outside dimensions become 15 cm by 21 cm. How wide is the frame?

70. After lengthening each side of a square by 4 cm, the area of the enlarged square turns out to be 225 cm^2. How long is the side of the original square?

71. A square piece of drywall was used to fix a hole in a wall. The sides of the piece of drywall had to be shortened by 2 inches in order to cover the required area of 49 in^2. What were the dimensions of the original piece of drywall?

Attributions