Radicals and Radical Functions

So far we have discussed polynomial and rational expressions and functions. In this chapter, we study algebraic expressions that contain radicals. For example, $3+\sqrt{2}$, $\sqrt[3]{x}-1$, or $\frac{1}{\sqrt{5x-1}}$. Such expressions are called ***radical expressions***. Familiarity with radical expressions is essential when solving a wide variety of problems. For instance, in algebra, some polynomial or rational equations have radical solutions that need to be simplified. In geometry, due to the frequent use of the Pythagorean equation, $a^2+b^2=c^2$, the exact distances are often radical expressions. In sciences, many formulas involve radicals.

We begin the study of radical expressions with defining radicals of various degrees and discussing their properties. Then, we show how to simplify radicals and radical expressions, and introduce operations on radical expressions. Finally, we study the methods of solving radical equations. In addition, similarly as in earlier chapters where we looked at the related polynomial and rational functions, we will also define and look at properties of radical functions.

RD1 Radical Expressions, Functions, and Graphs

Roots and Radicals

The operation of taking a **square root** of a number is the **reverse operation of squaring** a number. For example, a square root of 25 is 5 because raising 5 to the second power gives us 25.

Note: Observe that raising -5 to the second power also gives us 25. So, the square root of 25 could have two answers, 5 or -5. To avoid this duality, we choose the **nonnegative value**, called the **principal square root**, for the value of a square root of a number.

The operation of taking a square root is denoted by the symbol $\sqrt{}$. So, we have

$$\sqrt{25}=5,\quad \sqrt{0}=0,\quad \sqrt{1}=1,\quad \sqrt{9}=3, \text{ etc.}$$

What about $\sqrt{-4}=$? Is there a number such that when it is squared, it gives us -4 ?

Since the square of any real number is nonnegative, the square root of a negative number is not a real number. So, when working in the set of real numbers, we can conclude that

$$\sqrt{\boldsymbol{positive}}=\boldsymbol{positive},\quad \sqrt{0}=0,\quad \text{and}\quad \sqrt{\boldsymbol{negative}}=\boldsymbol{DNE}$$

does not exist

The operation of taking a **cube root** of a number is the **reverse operation of cubing** a number. For example, a cube root of 8 is 2 because raising 2 to the third power gives us 8.

This operation is denoted by the symbol $\sqrt[3]{}$. So, we have

$$\sqrt[3]{8}=2,\quad \sqrt[3]{0}=0,\quad \sqrt[3]{1}=1,\quad \sqrt[3]{27}=3, \text{ etc.}$$

Note: Observe that $\sqrt[3]{-8}$ exists and is equal to -2. This is because $(-2)^3 = -8$. Generally, a cube root can be applied to any real number and the **sign** of the resulting value **is the same** as the sign of the original number.

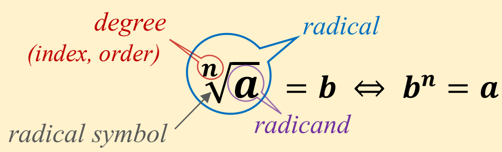
Thus, we have

$$\sqrt[3]{\textit{positive}} = \textit{positive}, \quad \sqrt[3]{0} = 0, \quad \text{and} \quad \sqrt[3]{\textit{negative}} = \textit{negative}$$

The square or cube roots are special cases of n-th degree radicals.

Definition 1.1 ► The ***n*-th degree radical** of a number a is a number b such that $\boldsymbol{b^n = a}$.

Notation:



$$\sqrt[n]{a} = b \Leftrightarrow b^n = a$$

For example, $\sqrt[4]{16} = 2$ because $2^4 = 16$,
$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$,
$\sqrt[3]{0.027} = 0.3$ because $(0.3)^3 = 0.027$.

Notice: A square root is a second degree radical, customarily denoted by $\sqrt{\ }$ rather than $\sqrt[2]{\ }$.

Example 1 ► Evaluating Radicals

Evaluate each radical, if possible.

a. $\sqrt{0.64}$ **b.** $\sqrt[3]{125}$ **c.** $\sqrt[4]{-16}$ **d.** $\sqrt[5]{-\frac{1}{32}}$

Solution ► **a.** Since $0.64 = (0.8)^2$, then $\sqrt{0.64} = 0.8$.

take half of the decimal places

Advice: To become fluent in evaluating square roots, it is helpful to be familiar with the following perfect square numbers:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, …, 400, …, 625, …

b. $\sqrt[3]{125} = 5$ as $5^3 = 125$

Advice: To become fluent in evaluating cube roots, it is helpful to be familiar with the following cubic numbers:
1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …

c. $\sqrt[4]{-16}$ is not a real number as there is no real number which raised to the 4-th power becomes negative.

d. $\sqrt[5]{-\frac{1}{32}} = -\frac{1}{2}$ as $\left(-\frac{1}{2}\right)^5 = -\frac{1^5}{2^5} = -\frac{1}{32}$

Note: Observe that $\frac{\sqrt[5]{-1}}{\sqrt[5]{32}} = \frac{-1}{2}$, so $\sqrt[5]{-\frac{1}{32}} = \frac{\sqrt[5]{-1}}{\sqrt[5]{32}}$.

Generally, to take a radical of a quotient, $\sqrt[n]{\frac{a}{b}}$, it is the same as to take the quotient of radicals, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Example 2 ► **Evaluating Radical Expressions**

Evaluate each radical expression.

a. $-\sqrt{121}$ **b.** $-\sqrt[3]{-64}$ **c.** $\sqrt[4]{(-3)^4}$ **d.** $\sqrt[3]{(-6)^3}$

Solution ►

a. $-\sqrt{121} = -11$

b. $-\sqrt[3]{-64} = -(-4) = 4$

c. $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$

the result is positive

Note: If n is **even**, then $\sqrt[n]{a^n} = \begin{cases} a, & if\ a \ge 0 \\ -a, & if\ a < 0 \end{cases} = |a|$.

For example, $\sqrt{7^2} = 7$ and $\sqrt{(-7)^2} = 7$.

d. $\sqrt[3]{(-6)^3} = \sqrt[3]{-216} = -6$

the result has the same sign

Note: If n is **odd**, then $\sqrt[n]{a^n} = a$. For example, $\sqrt[3]{5^3} = 5$ but $\sqrt[3]{(-5)^3} = -5$.

Summary of Properties of n-th Degree Radicals

- If n is **EVEN**, then

$$\sqrt[n]{\text{positive}} = \text{positive}, \quad \sqrt[n]{\text{negative}} = DNE, \quad \text{and} \quad \sqrt[n]{a^n} = |a|$$

- If n is **ODD**, then

$$\sqrt[n]{\text{positive}} = \text{positive}, \quad \sqrt[n]{\text{negative}} = \text{negative}, \quad \text{and} \quad \sqrt[n]{a^n} = a$$

- For any natural $n \geq 0$, $\sqrt[n]{0} = \mathbf{0}$ and $\sqrt[n]{1} = \mathbf{1}$.

Example 3 ► **Simplifying Radical Expressions Using Absolute Value Where Appropriate**

Simplify each radical, assuming that all variables represent any real number.

a. $\sqrt{9x^2y^4}$ **b.** $\sqrt[3]{-27y^3}$ **c.** $\sqrt[4]{a^{20}}$ **d.** $-\sqrt[4]{(k-1)^4}$

Solution ►

a. $\sqrt{9x^2y^4} = \sqrt{(3xy^2)^2} = |3xy^2| = \mathbf{3|x|y^2}$

An **even** degree radical is **nonnegative**, so we must use the **absolute value** operator.

Recall: As discussed in *Section L6*, the absolute value operator has the following properties:

$$|xy| = |x||y|$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

Note: $|y^2| = y^2$ as y^2 is already nonnegative.

b. $\sqrt[3]{-27y^3} = \sqrt[3]{(-3y)^3} = \mathbf{-3y}$

An **odd** degree radical assumes the sign of the radicand, so we **do not apply the absolute value** operator.

c. $\sqrt[4]{a^{20}} = \sqrt[4]{(a^5)^4} = |a^5| = \mathbf{|a|^5}$

Note: To simplify an expression with an absolute value, we keep the absolute value operator as close as possible to the variable(s).

d. $-\sqrt[4]{(k-1)^4} = \mathbf{-|k-1|}$

Radical Functions

Since each nonnegative real number x has exactly one principal square root, we can define the **square root** function, $\boldsymbol{f(x) = \sqrt{x}}$. The **domain** D_f of this function is the set of nonnegative real numbers, $[\mathbf{0}, \infty)$, and so is its **range** (as indicated in *Figure 1*).

To graph the square root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect square numbers. However, sometimes we want to use additional x-values that are not perfect squares. Since a square root of such a number, for example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$, etc., is an irrational number, we approximate these values using a calculator.

x	y
0	0
$\frac{1}{4}$	$\frac{1}{2}$
1	1
4	2
6	$\sqrt{6} \approx 2.4$

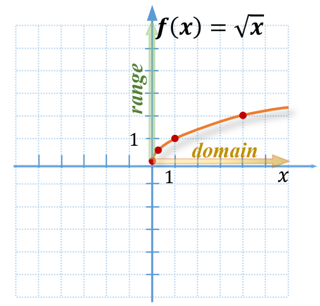


Figure 1

For example, to approximate $\sqrt{6}$, we use the sequence of keying: [√‾] [6] [ENTER] or [6] [^] [(] [1] [/] [2] [)] [ENTER]. This is because a square root operator works the same way as the exponent of $\frac{1}{2}$.

Note: When graphing an even degree radical function, it is essential that we find its domain first. The end-point of the domain indicates the starting point of the graph, often called the vertex.
For example, since the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$, the graph starts from the point $(0, f(0)) = (0,0)$, as in *Figure 1*.

Since the cube root can be evaluated for any real number, the **domain** D_f of the related **cube root** function, $\boldsymbol{f(x) = \sqrt[3]{x}}$, is the set of **all real numbers**, $\mathbb{R}$. The **range** can be observed in the graph (see *Figure 2*) or by inspecting the expression $\sqrt[3]{x}$. It is also $\mathbb{R}$.

To graph the cube root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect cube numbers. As before, sometimes we might need to estimate additional x-values. For example, to approximate $\sqrt[3]{6}$, we use the sequence of keying:
[∛‾] [6] [ENTER] or
[6] [^] [(] [1] [/] [3] [)] [ENTER].

x	y
−8	−2
−6	$-\sqrt[3]{6} \approx -1.8$
−1	−1
$-\frac{1}{8}$	$-\frac{1}{2}$
0	0
$\frac{1}{8}$	$\frac{1}{2}$
1	1
6	$\sqrt[3]{6} \approx 1.8$
8	2

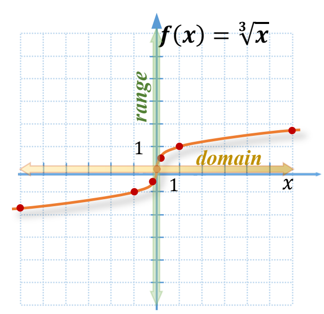


Figure 2

Example 4 ► **Finding a Calculator Approximations of Roots**

Use a calculator to approximate the given root up to three decimal places.

a. $\sqrt{3}$ **b.** $\sqrt[3]{5}$ **c.** $\sqrt[5]{100}$

Solution ►

a. $\sqrt{3} \approx 1.732$

b. $\sqrt[3]{5} \approx 1.710$

5 ^ (1 / 3) ENTER or MATH 4 5 ENTER on a graphing calculator

c. $\sqrt[5]{100} \approx 2.512$

100 ^ (1 / 5) ENTER or 5 MATH 5 100 ENTER on a graphing calculator

Example 5 ► **Finding the Best Integer Approximation of a Square Root**

Without the use of a calculator, determine the best integer approximation of the given root.

a. $\sqrt{68}$ **b.** $\sqrt{140}$

Solution ►

a. Observe that 68 lies between the following two consecutive perfect square numbers, 64 and 81. Also, 68 lies closer to 64 than to 81. Therefore, $\sqrt{68} \approx \sqrt{64} = 8$.

b. 140 lies between the following two consecutive perfect square numbers, 121 and 144. In addition, 140 is closer to 144 than to 121. Therefore, $\sqrt{140} \approx \sqrt{144} = 12$.

Example 6 ► **Finding the Domain of a Radical Function**

Find the domain of each of the following functions.

a. $f(x) = \sqrt{2x+3}$ **b.** $g(x) = 2 - \sqrt{1-x}$

Solution ►

The **domain** of an **even degree** radical is the solution set of the inequality ***radicand*** $\boldsymbol{\geq 0}$

The **domain** of an **odd degree** radical is $\mathbb{R}$.

a. When finding domain D_f of function $f(x) = \sqrt{2x+3}$, we need to protect the radicand $2x+3$ from becoming negative. So, an x-value belongs to the domain D_f if it satisfies the condition

$$2x + 3 \geq 0.$$

This happens for $x \geq -\frac{3}{2}$. Therefore, $\boldsymbol{D_f} = \left[-\frac{3}{2}, \infty\right)$.

b. To find the domain D_g of function $g(x) = 2 - \sqrt{1-x}$, we solve the condition

$$1 - x \geq 0$$
$$1 \geq x$$

Thus, $\boldsymbol{D_g} = (-\infty, \mathbf{1}]$.

Example 7 ► **Graphing Radical Functions**

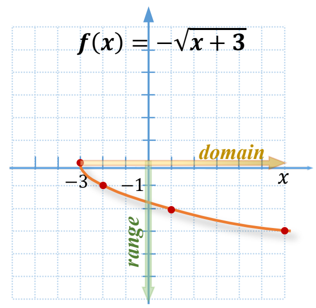
For each function, find its domain, graph it, and find its range. Then, observe what transformation(s) of a basic root function result(s) in the obtained graph.

a. $f(x) = -\sqrt{x+3}$ **b.** $g(x) = \sqrt[3]{x} - 2$

Solution ► **a.** The **domain** D_f is the solution set of the inequality $x + 3 \geq 0$, which is equivalent to $x \geq -3$. Hence, $\boldsymbol{D_f = [-3, \infty)}$.

start from the endpoint of the domain

x	y
−3	0
−2	−1
1	−2
6	−3



The projection of the graph onto the y-axis indicates the **range** of this function, which is $(-\infty, \mathbf{0}]$.

The graph of $f(x) = -\sqrt{x+3}$ has the same shape as the graph of the basic square root function $f(x) = \sqrt{x}$, except that it is flipped over the x-axis and moved to the left by three units. These transformations are illustrated in *Figure 3*.

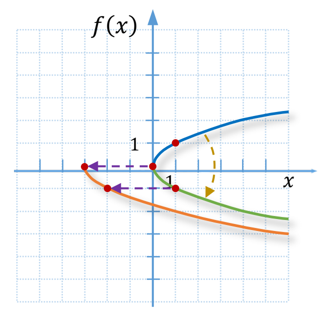
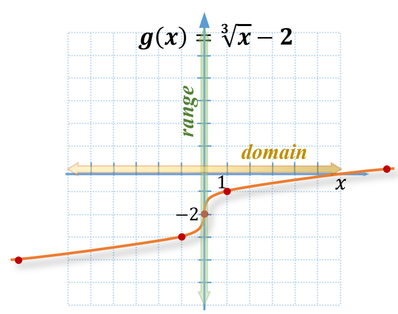


Figure 3

b. The **domain** and **range** of any odd degree radical are both the set of all real numbers. So, $\boldsymbol{D_g} = \mathbb{R}$ and $\boldsymbol{range_g} = \mathbb{R}$.

x	y
−8	−4
−1	−3
0	−2
1	−1
8	0



The graph of $g(x) = \sqrt[3]{x} - 2$ has the same shape as the graph of the basic cube root function $f(x) = \sqrt[3]{x}$, except that it is moved down by two units. This transformation is illustrated in *Figure 4*.

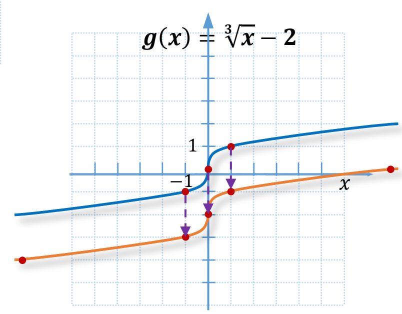


Figure 4

Radicals in Application Problems

Some application problems require evaluation of formulas that involve radicals. For example, the formula $c = \sqrt{a^2 + b^2}$ allows for finding the hypotenuse in a right angle triangle (see *Section RD3*), **Heron's** formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ allows for finding the area of any triangle given the lengths of its sides (see *Section T5*), the formula $T = 2\pi\sqrt{\frac{d^3}{Gm}}$ allows for finding the time needed for a planet to make a complete orbit around the Sun, and so on.

Example 8 ► **Using a Radical Formula in an Application Problem**

The time T, in seconds, needed for a pendulum to complete a full swing can be calculated using the formula

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where L denotes the length of the pendulum in feet, and g is the acceleration due to gravity, which is about 32 ft/sec^2. To the nearest hundredths of a second, find the time of a complete swing of an 18-inch long pendulum.

Solution ► Since $L = 18$ in $= \frac{18}{12}$ ft $= \frac{3}{2}$ ft and $g = 32$ ft/sec^2, then

$$T = 2\pi\sqrt{\frac{\frac{3}{2}}{32}} = 2\pi\sqrt{\frac{3}{2\cdot 32}} = 2\pi\sqrt{\frac{3}{64}} = 2\pi\cdot\frac{\sqrt{3}}{8} = \frac{\pi\sqrt{3}}{4} \approx 1.36$$

So, the approximate time of a complete swing of an 18-in pendulum is **1.36 seconds**.

RD.1 Exercises

Evaluate each radical, if possible.

1. $\sqrt{49}$

2. $-\sqrt{81}$

3. $\sqrt{-400}$

4. $\sqrt{0.09}$

5. $\sqrt{0.0016}$

6. $\sqrt{\frac{64}{225}}$

7. $\sqrt[3]{64}$

8. $\sqrt[3]{-125}$

9. $\sqrt[3]{0.008}$

10. $-\sqrt[3]{-1000}$

11. $\sqrt[3]{\frac{1}{0.000027}}$

12. $\sqrt[4]{16}$

13. $\sqrt[5]{0.00032}$

14. $\sqrt[7]{-1}$

15. $\sqrt[8]{-256}$

16. $-\sqrt[6]{\frac{1}{64}}$

17. Decide whether the expression $-\sqrt{-a}$ is positive, negative, 0, or not a real number, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

18. Assuming that n is odd, decide whether the expression $-\sqrt[n]{a}$ is positive, negative, or 0, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

Simplify each radical. Assume that letters can represent *<u>any real number</u>*.

19. $\sqrt{15^2}$ **20.** $\sqrt{(-15)^2}$ **21.** $\sqrt{x^2}$ **22.** $\sqrt{(-x)^2}$

23. $\sqrt{81x^2}$ **24.** $\sqrt{(-12y)^2}$ **25.** $\sqrt{(a+3)^2}$ **26.** $\sqrt{(2-x)^2y^4}$

27. $\sqrt{x^2-4x+4}$ **28.** $\sqrt{9y^2+30y+25}$ **29.** $\sqrt[3]{(-5)^3}$ **30.** $\sqrt[3]{x^3}$

31. $\sqrt[3]{-125a^3}$ **32.** $-\sqrt[3]{0.008(x-1)^3}$ **33.** $\sqrt[4]{(5x)^4}$ **34.** $\sqrt[8]{(-10)^8}$

35. $\sqrt[5]{(y-3)^5}$ **36.** $\sqrt[2017]{(a+b)^{2017}}$ **37.** $\sqrt[2018]{(2a-b)^{2018}}$ **38.** $\sqrt[6]{x^{18}}$

39. $\sqrt[4]{(a+1)^{12}}$ **40.** $\sqrt[5]{(-a)^{20}}$ **41.** $\sqrt[7]{(-k)^{35}}$ **42.** $\sqrt[4]{x^4(-y)^8}$

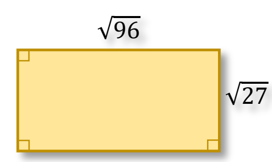
Find a decimal approximation for each radical. Round the answer to three decimal places.

43. $\sqrt{350}$ **44.** $-\sqrt{0.859}$ **45.** $\sqrt[3]{5}$ **46.** $\sqrt[5]{3}$

Without the use of a calculator, give the best integer approximation of each square root.

47. $\sqrt{67}$ **48.** $\sqrt{95}$ **49.** $\sqrt{115}$ **50.** $\sqrt{87}$

Questions in Exercises 60 and 61 refer to the accompanying rectangle. Answer these questions without the use of a calculator.



51. Give the best integer estimation of the area of the rectangle.

52. Give the best integer estimation of the perimeter of the rectangle.

Solve each problem. Do not use any calculator.

53. The length of a rectangular garden is $\sqrt{189}$ m and the width is $\sqrt{50}$ m. Give the best integral estimate for the dimentions of the garden and then estimate the perimeter of the garden to the nearest meter.

54. The sides of a triangle parking lot are $\sqrt{65}$ m, $\sqrt{78}$ m, and $\sqrt{220}$ m. Give the best integral estimate for each side and then estimate the perimeter of this parking lot to the nearest meter.

Graph *each function and give its* ***domain*** *and* ***range****. Then, discuss the* ***transformations*** *of a basic root function needed to obtain the graph of the given function.*

55. $f(x) = \sqrt{x+1}$ **56.** $g(x) = \sqrt{x} + 1$ **57.** $h(x) = -\sqrt{x}$

58. $f(x) = \sqrt{x-3}$

59. $g(x) = \sqrt{x} - 3$

60. $h(x) = 2 - \sqrt{x}$

61. $f(x) = \sqrt[3]{x-2}$

62. $g(x) = \sqrt[3]{x} + 2$

63. $h(x) = -\sqrt[3]{x} + 2$

***Graph** each function and give its **domain** and **range**.*

64. $f(x) = 2 + \sqrt{x-1}$

65. $g(x) = 2\sqrt{x}$

66. $h(x) = -\sqrt{x+3}$

67. $f(x) = \sqrt{3x+9}$

68. $g(x) = \sqrt{3x-6}$

69. $h(x) = -\sqrt{2x-4}$

70. $f(x) = \sqrt{12-3x}$

71. $g(x) = \sqrt{8-4x}$

72. $h(x) = -2\sqrt{-x}$

Graph the three given functions on the same grid and discuss the relationship between them.

73. $f(x) = 2x + 1;\; g(x) = \sqrt{2x+1};\; h(x) = \sqrt[3]{2x+1}$

74. $f(x) = -x + 2;\; g(x) = \sqrt{-x+2};\; h(x) = \sqrt[3]{-x+2}$

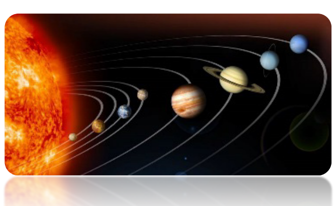
75. $f(x) = \frac{1}{2}x + 1;\; g(x) = \sqrt{\frac{1}{2}x+1};\; h(x) = \sqrt[3]{\frac{1}{2}x+1}$

Solve each problem.

76. The distance D, in kilometers, from the point of sight to the horizon is given by the formula $D = 4\sqrt{H}$, where H denotes the height of the point of sight above the sea level, in meters. To the nearest tenth of a kilometer, how far away is the horizon for a 180 cm tall man standing on a 40-m high cliff?

77. Let T represents the threshold body weight, in kilograms, above which the risk of death of a person increases significantly. Suppose the formula $h = 40\sqrt[3]{T}$ can be used to calculate the height h, in centimeters, of a middle age man with the threshold body weight T. To the nearest centimeter, find the height corresponding to a threshold weight of a 100 kg man at his forties.

78. The orbital period (time needed for a planet to make a complete rotation around the Sun) is given by the formula $T = 2\pi\sqrt{\frac{r^3}{GM}}$, where r is the average distance of the planet from the Sun, G is the universal gravitational constant, and M is the mass of the Sun. To the nearest day, find the orbital period of Mercury, knowing that its average distance from the Sun is $5.791 \cdot 10^7$ km, the mass of the Sun is $1.989 \cdot 10^{30}$ kg, and $G = 6.67408 \cdot 10^{-11}$ $m^3/(kg \cdot s^2)$. (Attention: *Watch the units!*)



79. Suppose that the time t, in seconds, needed for an object to fall a certain distance can be found by using the formula $t = \sqrt{\frac{2d}{g}}$, where d is the distance in meters, and g is the acceleration due to gravity. An astronaut standing on a platform above the moon's surface drops an object, which hits the ground 2 seconds after it was dropped. Assume that the acceleration due to gravity on the moon is 1.625 m/s^2. How high above the surface was the object at the time it was dropped?

*Half of the perimeter (**semiperimeter**) of a triangle with sides a, b, and c is* $\boldsymbol{s=\frac{1}{2}(a+b+c)}$. *The area of such a triangle is given by the **Heron's Formula**:* $\boldsymbol{A=\sqrt{s(s-a)(s-b)(s-c)}}$.

*In problems **89-90**, find the area of a triangular piece of land with the given sides.*

80. $a=3\text{ m}, b=4\text{ m}, c=5\text{ m}$

81. $a=80\text{ m}, b=80\text{ m}, c=140\text{ m}$

RD2 Rational Exponents

In *Sections P2* and *RT1*, we reviewed the properties of powers with natural and integral exponents. All of these properties hold for real exponents as well. In this section, we give meaning to expressions with rational exponents, such as $a^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, or $(2x)^{0.54}$, and use the rational exponent notation as an alternative way to write and simplify radical expressions.

Rational Exponents

Observe that $\sqrt{9} = 3 = 3^{2 \cdot \frac{1}{2}} = 9^{\frac{1}{2}}$. Similarly, $\sqrt[3]{8} = 2 = 2^{3 \cdot \frac{1}{3}} = 8^{\frac{1}{3}}$. This suggests the following generalization:

For any real number a and a natural number $n > 1$, we have

$$\sqrt[n]{\boldsymbol{a}} = \boldsymbol{a}^{\frac{1}{n}}.$$

Notice: The **denominator** of the rational exponent is the **index** of the radical.

Caution! If $a < 0$ and n is an even natural number, then $a^{\frac{1}{n}}$ is not a real number.

Example 1 ► **Converting Radical Notation to Rational Exponent Notation**

Convert each radical to a power with a rational exponent and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\sqrt[6]{16}$ **b.** $\sqrt[3]{27x^3}$ **c.** $\sqrt{\frac{4}{b^6}}$

Solution ►

a. $\sqrt[6]{16} = 16^{\frac{1}{6}} = (2^4)^{\frac{1}{6}} = 2^{\frac{4}{6}} = \mathbf{2^{\frac{2}{3}}}$

Observation: Expressing numbers as **powers of prime numbers** often allows for further simplification.

b. $\sqrt[3]{27x^3} = (27x^3)^{\frac{1}{3}} = 27^{\frac{1}{3}} \cdot (x^3)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} \cdot x = \mathbf{3x}$

distribution of exponents change into a power of a prime number

Note: The above example can also be done as follows:

$$\sqrt[3]{27x^3} = \sqrt[3]{3^3x^3} = (3^3x^3)^{\frac{1}{3}} = \mathbf{3x}$$

c. $\sqrt{\frac{9}{b^6}} = \left(\frac{9}{b^6}\right)^{\frac{1}{2}} = \frac{(3^2)^{\frac{1}{2}}}{(b^6)^{\frac{1}{2}}} = \frac{\mathbf{3}}{\mathbf{b^3}}$, as $b > 0$.

Observation: $\sqrt{a^4} = a^{\frac{4}{2}} = a^2$.

Generally, for any real number $a \neq 0$, natural number $n > 1$, and integral number m, we have

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

Rational exponents are introduced in such a way that they automatically agree with the rules of exponents, as listed in *Section RT1*.
Furthermore, the rules of exponents hold not only for rational but also for **real exponents**.

Observe that following the rules of exponents and the commutativity of multiplication, we have

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m,$$

provided that $\sqrt[n]{a}$ exists.

Example 2 ► **Converting Rational Exponent Notation to the Radical Notation**

Convert each power with a rational exponent to a radical and simplify, if possible.

a. $5^{\frac{3}{4}}$ **b.** $(-27)^{\frac{1}{3}}$ **c.** $3x^{-\frac{2}{5}}$

Solution ►

a. $5^{\frac{3}{4}} = \sqrt[4]{5^3} = \boldsymbol{\sqrt[4]{125}}$

b. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = \boldsymbol{-3}$

Notice that $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$, so $(-27)^{\frac{1}{3}} = -27^{\frac{1}{3}}$.

However, $(-9)^{\frac{1}{2}} \neq -9^{\frac{1}{2}}$, as $(-9)^{\frac{1}{2}}$ is not a real number while $-9^{\frac{1}{2}} = -\sqrt{9} = -3$.

c. $3x^{-\frac{2}{5}} = \frac{3}{x^{\frac{2}{5}}} = \boldsymbol{\frac{3}{\sqrt[5]{x^2}}}$

Caution: A **negative exponent** indicates a **reciprocal** not a negative number!

Also, the exponent refers to x only, so 3 remains in the numerator.

Observation: If $a^{\frac{m}{n}}$ is a real number, then

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}},$$

provided that $a \neq 0$.

Caution! Make sure to distinguish between a negative exponent and a negative result. A negative exponent leads to a reciprocal of the base. The result can be either positive or negative, depending on the sign of the base. For example,

$$8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{2}, \text{ but } (-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{-2} = -\frac{1}{2} \text{ and } -8^{-\frac{1}{3}} = -\frac{1}{8^{\frac{1}{3}}} = -\frac{1}{2}.$$

Example 3 ► **Applying Rules of Exponents When Working with Rational Exponents**

Simplify each expression. Write your answer with only positive exponents. Assume that all variables represent positive real numbers.

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}}$ **b.** $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}}$ **c.** $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$

Solution ►

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}} = 2a^{\frac{3}{4}+\left(-\frac{2}{3}\right)} = 2a^{\frac{9}{12}-\frac{8}{12}} = \mathbf{2a^{\frac{1}{12}}}$

b. $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}} = 4^{\frac{1}{3}-\frac{5}{3}} = 4^{-\frac{4}{3}} = \mathbf{\frac{1}{4^{\frac{4}{3}}}}$

c. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}} = x^{\frac{3}{8}\cdot\frac{4}{3}} \cdot y^{\frac{5}{2}\cdot\frac{4}{3}} = \mathbf{x^{\frac{1}{2}}y^{\frac{10}{3}}}$

Example 4 ► **Evaluating Powers with Rational Exponents**

Evaluate each power.

a. $64^{-\frac{1}{3}}$ **b.** $\left(-\frac{8}{125}\right)^{\frac{2}{3}}$

Solution ►

a. $64^{-\frac{1}{3}} = (2^6)^{-\frac{1}{3}} = 2^{-2} = \frac{1}{2^2} = \mathbf{\frac{1}{4}}$

It is helpful to change the base into a **power of a prime number**, if possible.

b. $\left(-\frac{8}{125}\right)^{\frac{2}{3}} = \left(\left(-\frac{2}{5}\right)^3\right)^{\frac{2}{3}} = \left(-\frac{2}{5}\right)^2 = \mathbf{\frac{4}{25}}$

Observe that if m in $\sqrt[n]{a^m}$ is a multiple of n, that is if $m = kn$ for some integer k, then

$$\sqrt[n]{a^{kn}} = a^{\frac{kn}{n}} = a^k$$

Example 5 ► **Simplifying Radical Expressions by Converting to Rational Exponents**

Simplify. Assume that all variables represent positive real numbers. Leave your answer in simplified single radical form.

a. $\sqrt[5]{3^{20}}$ **b.** $\sqrt{x}\cdot\sqrt[4]{x^3}$ **c.** $\sqrt[3]{2\sqrt{2}}$

Solution ► **a.** $\sqrt[5]{3^{20}} = (3^{20})^{\frac{1}{5}} = 3^4 = \mathbf{81}$

divide at the exponential level

b. $\sqrt{x}\cdot\sqrt[4]{x^3} = x^{\frac{1}{2}}\cdot x^{\frac{3}{4}} = x^{\frac{1\cdot 2}{2\cdot 2}+\frac{3}{4}} = x^{\frac{5}{4}} = x^1\cdot x^{\frac{1}{4}} = \boldsymbol{x\sqrt[4]{x}}$

add exponents as $\frac{5}{4}=1+\frac{1}{4}$

c. $\sqrt[3]{2\sqrt{2}} = \left(2\cdot 2^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{1+\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{2}} = \boldsymbol{\sqrt{2}}$

This bracket is essential!

Another solution:

$\sqrt[3]{2\sqrt{2}} = 2^{\frac{1}{3}}\cdot\left(2^{\frac{1}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}}\cdot 2^{\frac{1}{6}} = 2^{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}} = 2^{\frac{1}{2}} = \boldsymbol{\sqrt{2}}$

RD.2 Exercises

Match each expression from Column I with the equivalent expression from Column II.

1.

Column I	Column II
a. $9^{\frac{1}{2}}$	**A.** $\frac{1}{3}$
b. $9^{-\frac{1}{2}}$	**B.** 3
c. $-9^{\frac{3}{2}}$	**C.** -27
d. $-9^{-\frac{1}{2}}$	**D.** not a real number
e. $(-9)^{\frac{1}{2}}$	**E.** $\frac{1}{27}$
f. $9^{-\frac{3}{2}}$	**F.** $-\frac{1}{3}$

2.

Column I	Column II
a. $(-32)^{\frac{2}{5}}$	**A.** 2
b. $-27^{\frac{2}{3}}$	**B.** $\frac{1}{4}$
c. $32^{\frac{1}{5}}$	**C.** -8
d. $32^{-\frac{2}{5}}$	**D.** -9
e. $-4^{\frac{3}{2}}$	**E.** not a real number
f. $(-4)^{\frac{3}{2}}$	**F.** 4

Write the base as a ***power of a prime number*** *to evaluate each expression, if possible.*

3. $32^{\frac{1}{5}}$ **4.** $27^{\frac{4}{3}}$ **5.** $-49^{\frac{3}{2}}$ **6.** $16^{\frac{3}{4}}$

7. $-100^{-\frac{1}{2}}$ **8.** $125^{-\frac{1}{3}}$ **9.** $\left(\frac{64}{81}\right)^{\frac{3}{4}}$ **10.** $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

11. $(-36)^{\frac{1}{2}}$ **12.** $(-64)^{\frac{1}{3}}$ **13.** $\left(-\frac{1}{8}\right)^{-\frac{1}{3}}$ **14.** $(-625)^{-\frac{1}{4}}$

*Rewrite **with** rational exponents and simplify, if possible. Assume that all variables represent positive real numbers.*

15. $\sqrt{5}$ **16.** $\sqrt[3]{6}$ **17.** $\sqrt{x^6}$ **18.** $\sqrt[5]{y^2}$

19. $\sqrt[3]{64x^6}$ **20.** $\sqrt[3]{16x^2y^3}$ **21.** $\sqrt{\frac{25}{x^5}}$ **22.** $\sqrt[4]{\frac{16}{a^6}}$

*Rewrite **without** rational exponents, and simplify, if possible. Assume that all variables represent positive real numbers.*

23. $4^{\frac{5}{2}}$ **24.** $8^{\frac{3}{4}}$ **25.** $x^{\frac{3}{5}}$ **26.** $a^{\frac{7}{3}}$

27. $(-3)^{\frac{2}{3}}$ **28.** $(-2)^{\frac{3}{5}}$ **29.** $2x^{-\frac{1}{2}}$ **30.** $x^{\frac{1}{3}}y^{-\frac{1}{2}}$

*Use the **laws of exponents** to simplify. Write the answers with positive exponents. Assume that all variables represent positive real numbers.*

31. $3^{\frac{3}{4}} \cdot 3^{\frac{1}{8}}$ **32.** $x^{\frac{2}{3}} \cdot x^{-\frac{1}{4}}$ **33.** $\frac{2^{\frac{5}{8}}}{2^{-\frac{1}{8}}}$ **34.** $\frac{a^{\frac{1}{3}}}{a^{\frac{2}{3}}}$

35. $\left(5^{\frac{15}{8}}\right)^{\frac{2}{3}}$ **36.** $\left(y^{\frac{2}{3}}\right)^{-\frac{3}{7}}$ **37.** $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$ **38.** $\left(a^{-\frac{2}{3}} \cdot b^{\frac{5}{8}}\right)^{-4}$

39. $\left(\frac{y^{-\frac{3}{2}}}{x^{-\frac{5}{3}}}\right)^{\frac{1}{3}}$ **40.** $\left(\frac{a^{-\frac{2}{3}}}{b^{-\frac{5}{6}}}\right)^{\frac{3}{4}}$ **41.** $x^{\frac{2}{3}} \cdot 5x^{-\frac{2}{5}}$ **42.** $x^{\frac{2}{5}} \cdot \left(4x^{-\frac{4}{5}}\right)^{-\frac{1}{4}}$

*Use rational exponents to **simplify**. Write the answer **in radical notation** if appropriate. Assume that all variables represent positive real numbers.*

43. $\sqrt[6]{x^2}$ **44.** $\left(\sqrt[3]{ab}\right)^{15}$ **45.** $\sqrt[6]{y^{-18}}$ **46.** $\sqrt{x^4y^{-6}}$

47. $\sqrt[6]{81}$ **48.** $\sqrt[14]{128}$ **49.** $\sqrt[3]{8y^6}$ **50.** $\sqrt[4]{81p^6}$

51. $\sqrt[3]{(4x^3y)^2}$ **52.** $\sqrt[5]{64(x+1)^{10}}$ **53.** $\sqrt[4]{16x^4y^2}$ **54.** $\sqrt[5]{32a^{10}d^{15}}$

*Use rational exponents to rewrite in a **single radical expression** in a simplified form. Assume that all variables represent positive real numbers.*

55. $\sqrt[3]{5} \cdot \sqrt{5}$ **56.** $\sqrt[3]{2} \cdot \sqrt[4]{3}$ **57.** $\sqrt{a} \cdot \sqrt[3]{3a}$ **58.** $\sqrt[3]{x} \cdot \sqrt[5]{2x}$

59. $\sqrt[6]{x^5} \cdot \sqrt[3]{x^2}$ **60.** $\sqrt[3]{xz} \cdot \sqrt{z}$ **61.** $\frac{\sqrt{x^5}}{\sqrt{x^8}}$ **62.** $\frac{\sqrt[3]{a^5}}{\sqrt{a^3}}$

63. $\frac{\sqrt[3]{8x}}{\sqrt[4]{x^3}}$ **64.** $\sqrt[3]{\sqrt{a}}$ **65.** $\sqrt[4]{\sqrt[3]{xy}}$ **66.** $\sqrt{\sqrt[3]{(3x)^2}}$

67. $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$ **68.** $\sqrt[3]{3\sqrt{3}}$ **69.** $\sqrt[4]{x\sqrt{x}}$ **70.** $\sqrt[3]{2\sqrt{x}}$

71. Consider two expressions: $\sqrt[n]{x^n + y^n}$ and $x + y$. Observe that for $x = 1$ and $y = 0$ both expressions are equal: $\sqrt[n]{x^n + y^n} = \sqrt[n]{1^n + 0^n} = 1 = 1 + 9 = x + y$. Does this mean that $\sqrt[n]{x^n + y^n} = x + y$? Justify your answer.

Solve each problem.

72. When counting both the black and white keys on a piano, an octave contains 12 keys. The frequencies of consecutive keys increase by a factor of $2^{\frac{1}{12}}$. For example, the frequency of the tone D that is two keys above middle C is

$$2^{\frac{1}{12}} \cdot 2^{\frac{1}{12}} = \left(2^{\frac{1}{12}}\right)^2 = 2^{\frac{1}{6}} \approx 1.12$$

times the frequency of the middle C.

a. If tone G, which is five keys below the middle C, has a frequency of about 196 cycles per second, estimate the frequency of the middle C to the nearest tenths of a cycle.

b. Find the relation between frequencies of two tones that are one octave apart.

73. An animal's heart rate is related to the animal's weight. Suppose that the average heart rate R, in beats per minute, for an animal that weighs k kilograms can be estimated by using the function $R(w) = 600w^{-\frac{1}{2}}$. What is the expected average heart rate of a horse that weighs 400 kilograms?

74. Suppose that the duration of a storm T, in hours, can be determined by using the function $T(D) = 0.03D^{\frac{3}{2}}$, where D denotes the diameter of a storm in kilometers. To the nearest minute, what is the duration of a storm with a diameter of 20 kilometers?

RD3 Simplifying Radical Expressions and the Distance Formula

In the previous section, we simplified some radical expressions by replacing radical signs with rational exponents, applying the rules of exponents, and then converting the resulting expressions back into radical notation. In this section, we broaden the above method of simplifying radicals by examining products and quotients of radicals with the same indexes, as well as explore the possibilities of decreasing the index of a radical.

In the second part of this section, we will apply the skills of simplifying radicals in problems involving the Pythagorean Theorem. In particular, we will develop the distance formula and apply it to calculate distances between two given points in a plane.

Multiplication, Division, and Simplification of Radicals

Suppose we wish to multiply radicals with the same indexes. This can be done by converting each radical to a rational exponent and then using properties of exponents as follows:

PRODUCT RULE

$$\sqrt[n]{\boldsymbol{a}} \cdot \sqrt[n]{\boldsymbol{b}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{\boldsymbol{ab}}$$

This shows that the **product of same index radicals** is the **radical of the product** of their radicands.

Similarly, the **quotient of same index radicals** is the **radical of the quotient** of their radicands, as we have

QUOTIENT RULE

$$\frac{\sqrt[n]{\boldsymbol{a}}}{\sqrt[n]{\boldsymbol{b}}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{\boldsymbol{a}}{\boldsymbol{b}}}$$

So, $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = \mathbf{4}$. Similarly, $\frac{\sqrt[3]{16}}{\sqrt[3]{2}} = \sqrt[3]{\frac{16}{2}} = \sqrt[3]{8} = \mathbf{2}$.

Attention! There is no such rule for addition or subtraction of terms. For instance,

$$\sqrt{\boldsymbol{a}+\boldsymbol{b}} \neq \sqrt{\boldsymbol{a}} \pm \sqrt{\boldsymbol{b}},$$

and generally

$$\sqrt[n]{\boldsymbol{a} \pm \boldsymbol{b}} \neq \sqrt[n]{\boldsymbol{a}} \pm \sqrt[n]{\boldsymbol{b}}.$$

Here is a counterexample: $\sqrt[n]{2} = \sqrt[n]{1+1} \neq \sqrt[n]{1} + \sqrt[n]{1} = 1 + 1 = 2$

Example 1 ► **Multiplying and Dividing Radicals of the Same Indexes**

Perform the indicated operations and simplify, if possible. Assume that all variables are positive.

a. $\sqrt{10} \cdot \sqrt{15}$

b. $\sqrt{2x^3}\sqrt{6xy}$

c. $\frac{\sqrt{10x}}{\sqrt{5}}$

d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}}$

Solution ► **a.** $\sqrt{10}\cdot\sqrt{15} = \sqrt{10\cdot 15} = \sqrt{2\cdot 5\cdot 5\cdot 3} = \sqrt{5\cdot 5\cdot 2\cdot 3} = \sqrt{25}\cdot\sqrt{6} = \mathbf{5\sqrt{6}}$

product rule; prime factorization; commutativity of multiplication; product rule

b. $\sqrt{2x^3}\sqrt{6xy} = \sqrt{2\cdot 2\cdot 3x^4y} = \sqrt{4x^4}\cdot\sqrt{3y} = \mathbf{2x^2\sqrt{3y}}$

use commutativity of multiplication to isolate perfect square factors

Here the multiplication sign is assumed, even if it is not indicated.

c. $\frac{\sqrt{10x}}{\sqrt{5}} = \sqrt{\frac{10x}{5}} = \mathbf{\sqrt{2x}}$

quotient rule

d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}} = \sqrt[4]{\frac{32x^3}{2x}} = \sqrt[4]{16x^2} = \sqrt[4]{16}\cdot\sqrt[4]{x^2} = \mathbf{2\sqrt{x}}$

Recall that $\sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}$.

Caution! Remember to indicate the index of the radical for indexes higher than two.

The product and quotient rules are essential when simplifying radicals.

To **simplify a radical** means to:

1. Make sure that all **power factors of the radicand have exponents smaller than the index of the radical**.

   For example, $\sqrt[3]{2^4x^8y} = \sqrt[3]{2^3x^6}\cdot\sqrt[3]{2x^2y} = 2x^2\sqrt[3]{2x^2y}$.

2. Leave the radicand with **no fractions**.

   For example, $\sqrt{\frac{2x}{25}} = \frac{\sqrt{2x}}{\sqrt{25}} = \frac{\sqrt{2x}}{5}$.

3. **Rationalize** any **denominator**. (Make sure that denominators are **free from radicals,** see Section RD4.)

   For example, $\sqrt{\frac{4}{x}} = \frac{\sqrt{4}}{\sqrt{x}} = \frac{2\cdot\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}} = \frac{2\sqrt{x}}{x}$, provided that $x > 0$.

4. **Reduce the power** of the radicand **with the index** of the radical, if possible.

   For example, $\sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}$.

Example 2 ► **Simplifying Radicals**

Simplify each radical. Assume that all variables are positive.

a. $\sqrt[5]{96x^7y^{15}}$ **b.** $\sqrt[4]{\frac{a^{12}}{16b^4}}$ **c.** $\sqrt{\frac{25x^2}{8x^3}}$ **d.** $\sqrt[6]{27a^{15}}$

Solution ►

a. $\sqrt[5]{96x^7y^{15}} = \sqrt[5]{2^5 \cdot 3x^7y^{15}} = \mathbf{2xy^3\sqrt[5]{3x^2}}$

$\sqrt[5]{y^{15}} = y^3$

$\sqrt[5]{x^7} = x\sqrt[5]{x^2}$

Generally, to simplify $\sqrt[d]{x^a}$, we perform the division

$$a \div d = quotient\ \boldsymbol{q} + remainder\ \boldsymbol{r},$$

and then pull the $\boldsymbol{q}$-th power of x out of the radical, leaving the $\boldsymbol{r}$-th power of x under the radical. So, we obtain

$$\sqrt[d]{x^a} = x^q\sqrt[d]{x^r}$$

b. $\sqrt[4]{\frac{a^{12}}{16b^4}} = \frac{\sqrt[4]{a^{12}}}{\sqrt[4]{2^4b^4}} = \mathbf{\frac{a^3}{2b}}$

c. $\sqrt{\frac{25x^{2}}{8x^{3}}} = \sqrt{\frac{25}{2^3x}} = \frac{\sqrt{25}}{\sqrt{2^3x}} = \frac{5}{2\sqrt{2x}} \cdot \frac{\sqrt{2x}}{\sqrt{2x}} = \frac{5\sqrt{2x}}{2 \cdot 2x} = \mathbf{\frac{5\sqrt{2x}}{4x}}$

d. $\sqrt[6]{27a^{15}} = \sqrt[6]{3^3a^{15}} = a^2\sqrt[6]{3^3a^3} = a^2 \cdot \sqrt[6]{(3a)^3} = \mathbf{a^2\sqrt{3a}}$

Example 3 ► **Simplifying Expressions Involving Multiplication, Division, or Composition of Radicals with Different Indexes**

Simplify each expression. Leave your answer in simplified single radical form. Assume that all variables are positive.

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^2y}$ **b.** $\frac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}}$ **c.** $\sqrt[3]{x^2\sqrt{2x}}$

Solution ►

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^2y} = x^{\frac{1}{2}}y^{\frac{5}{2}} \cdot x^{\frac{2}{3}}y^{\frac{1}{3}} = x^{\frac{1\cdot 3}{2\cdot 3}+\frac{2\cdot 2}{3\cdot 2}}y^{\frac{5\cdot 3}{2\cdot 3}+\frac{1\cdot 2}{3\cdot 2}} = x^{\frac{7}{6}}y^{\frac{17}{6}} = (x^7y^{17})^{\frac{1}{6}}$

$= \sqrt[6]{x^7y^{17}} = \mathbf{xy^2\sqrt[6]{xy^5}}$

If radicals are of different indexes, convert them to exponential form.

b. $\frac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}} = \frac{a^{\frac{2}{4}}b^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}} = a^{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}b^{\frac{3\cdot 3}{4\cdot 3}-\frac{1\cdot 4}{3\cdot 4}} = a^{\frac{1\cdot 2}{6\cdot 2}}\,b^{\frac{5}{12}} = (a^2\,b^5)^{\frac{1}{12}} = \mathbf{\sqrt[12]{a^2\,b^5}}$

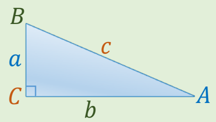
Bring the exponents to the LCD in order to leave the answer as a single radical.

c. $\sqrt[3]{x^2\sqrt{2x}} = x^{\frac{2}{3}} \cdot \left((2x)^{\frac{1}{2}}\right)^{\frac{1}{3}} = x^{\frac{2}{3}} \cdot 2^{\frac{1}{6}} \cdot x^{\frac{1}{6}} = x^{\frac{2\cdot 2}{3\cdot 2}+\frac{1}{6}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1}{6}}x^{\frac{5}{6}} = (2x^5)^{\frac{1}{6}} = \mathbf{\sqrt[6]{2x^5}}$

Pythagorean Theorem and Distance Formula

One of the most famous theorems in mathematics is the Pythagorean Theorem.

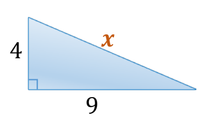
Pythagorean Theorem ► Suppose angle C in a triangle ABC is a $90°$ angle.
Then the **sum of the squares** of the lengths of the two **legs**, a and b, equals to the **square** of the length of the **hypotenuse** c:

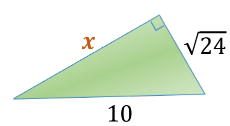


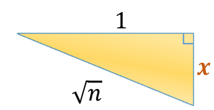
$$a^2 + b^2 = c^2$$

Example 4 ► **Using The Pythagorean Equation**

For the first two triangles, find the exact length x of the unknown side. For triangle (**c**), express length x in terms of the unknown n.

a.


b.


c.


Solution ►

Caution: Generally, $\sqrt{x^2} = |x|$ However, the length of a side of a triangle is positive. So, we can write $\sqrt{x^2} = x$

a. The length of the hypotenuse of the given right triangle is equal to x. So, the Pythagorean equation takes the form

$$x^2 = 4^2 + 9^2.$$

To solve it for x, we take a square root of each side of the equation. This gives us

$$\sqrt{x^2} = \sqrt{4^2 + 9^2}$$
$$x = \sqrt{16 + 81}$$
$$x = \boldsymbol{\sqrt{97}}$$

b. Since 10 is the length of the hypotenuse, we form the Pythagorean equation

$$10^2 = x^2 + \sqrt{24}^2.$$

To solve it for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we have

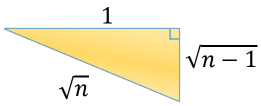
Customarily, we simplify each root, if possible.

$$10^2 - \sqrt{24}^2 = x^2$$
$$100 - 24 = x^2$$
$$x^2 = 76$$
$$x = \sqrt{76} = \sqrt{4 \cdot 19} = \boldsymbol{2\sqrt{19}}$$

c. The length of the hypotenuse is $\sqrt{n}$, so we form the Pythagorean equation as below.

$$\left(\sqrt{n}\right)^2 = 1^2 + x^2$$

To solve this equation for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we obtain



$$n = 1 + x^2$$
$$n - 1 = x^2$$
$$x = \sqrt{\boldsymbol{n-1}}$$

Note: Since the hypotenuse of length $\sqrt{n}$ must be longer than the leg of length 1, $n > 1$. This means that $n - 1 > 0$, and therefore $\sqrt{n-1}$ is a positive real number.

The Pythagorean Theorem allows us to find the distance between any two given points in a plane.

Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a coordinate plane. Then $|x_2 - x_1|$ represents the horizontal distance between A and B and $|y_2 - y_1|$ represents the vertical distance between A and B, as shown in *Figure 1*. Notice that by applying the absolute value operator to each difference of the coordinates we guarantee that the resulting horizontal and vertical distance is indeed a nonnegative number.

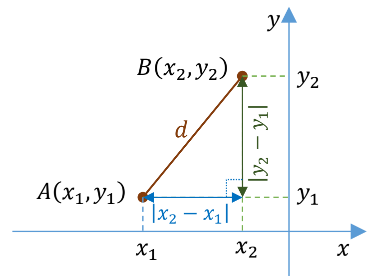


Figure 1

Applying the Pythagorean Theorem to the right triangle shown in *Figure 1*, we form the equation

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2,$$

where d is the distance between A and B.

Notice that $|x_2 - x_1|^2 = (x_2 - x_1)^2$ as a perfect square automatically makes the expression nonnegative. Similarly, $|y_2 - y_1|^2 = (y_2 - y_1)^2$. So, the Pythagorean equation takes the form

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

After solving this equation for d, we obtain the **distance formula**:

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Note: Observe that due to squaring the difference of the corresponding coordinates, **the distance between two points is the same regardless of which point is chosen as first**, (x_1, y_1), and second, (x_2, y_2).

Example 5 ► **Finding the Distance Between Two Points**

Find the exact distance between the points $(-2,4)$ and $(5,3)$.

Solution ► Let $(-2,4) = (x_1, y_1)$ and $(5,3) = (x_2, y_2)$. To find the distance d between the two points, we follow the distance formula:

$$d = \sqrt{\left(5-(-2)\right)^2 + (3-4)^2} = \sqrt{7^2 + (-1)^2} = \sqrt{49+1} = \sqrt{50} = \mathbf{5\sqrt{2}}$$

So, the points $(-2,4)$ and $(5,3)$ are $5\sqrt{2}$ units apart.

RD.3 Exercises

Multiply and simplify, if possible. Assume that all variables are positive.

1. $\sqrt{5}\cdot\sqrt{5}$
2. $\sqrt{18}\cdot\sqrt{2}$
3. $\sqrt{6}\cdot\sqrt{3}$
4. $\sqrt{15}\cdot\sqrt{6}$
5. $\sqrt{45}\cdot\sqrt{60}$
6. $\sqrt{24}\cdot\sqrt{75}$
7. $\sqrt{3x^3}\cdot\sqrt{6x^5}$
8. $\sqrt{5y^7}\cdot\sqrt{15a^3}$
9. $\sqrt{12x^3y}\,\sqrt{8x^4y^2}$
10. $\sqrt{30a^3b^4}\,\sqrt{18a^2b^5}$
11. $\sqrt[3]{4x^2}\,\sqrt[3]{2x^4}$
12. $\sqrt[4]{20a^3}\,\sqrt[4]{4a^5}$

Divide and simplify, if possible. Assume that all variables are positive.

13. $\frac{\sqrt{90}}{\sqrt{5}}$
14. $\frac{\sqrt{48}}{\sqrt{6}}$
15. $\frac{\sqrt{42a}}{\sqrt{7a}}$
16. $\frac{\sqrt{30x^3}}{\sqrt{10x}}$
17. $\frac{\sqrt{52ab^3}}{\sqrt{13a}}$
18. $\frac{\sqrt{56xy^3}}{\sqrt{8x}}$
19. $\frac{\sqrt{128x^2y}}{2\sqrt{2}}$
20. $\frac{\sqrt{48a^3b}}{2\sqrt{3}}$
21. $\frac{\sqrt[4]{80}}{\sqrt[4]{5}}$
22. $\frac{\sqrt[3]{108}}{\sqrt[3]{4}}$
23. $\frac{\sqrt[3]{96a^5b^2}}{\sqrt[3]{12a^2b}}$
24. $\frac{\sqrt[4]{48x^9y^{13}}}{\sqrt[4]{3xy^5}}$

Simplify each expression. Assume that all variables are positive.

25. $\sqrt{144x^4y^9}$
26. $-\sqrt{81m^8n^5}$
27. $\sqrt[3]{-125a^6b^9c^{12}}$
28. $\sqrt{50x^3y^4}$
29. $\sqrt[4]{\frac{1}{16}m^8n^{20}}$
30. $-\sqrt[3]{-\frac{1}{27}x^2y^7}$
31. $\sqrt{7a^7b^6}$
32. $\sqrt{75p^3q^4}$
33. $\sqrt[5]{64x^{12}y^{15}}$
34. $\sqrt[5]{p^{14}q^7r^{23}}$
35. $-\sqrt[4]{162a^{15}b^{10}}$
36. $-\sqrt[4]{32x^5y^{10}}$
37. $\sqrt{\frac{16}{49}}$
38. $\sqrt[3]{\frac{27}{125}}$
39. $\sqrt{\frac{121}{y^2}}$
40. $\sqrt{\frac{64}{x^4}}$
41. $\sqrt[3]{\frac{81a^5}{64}}$
42. $\sqrt{\frac{36x^5}{y^6}}$
43. $\sqrt[4]{\frac{16x^{12}}{y^4z^{16}}}$
44. $\sqrt[5]{\frac{32y^8}{x^{10}}}$
45. $\sqrt[4]{36}$
46. $\sqrt[6]{27}$
47. $-\sqrt[10]{x^{25}}$
48. $\sqrt[12]{x^{44}}$
49. $-\sqrt{\frac{1}{x^3y}}$
50. $\sqrt[3]{\frac{64x^{15}}{y^4z^5}}$
51. $\sqrt[6]{\frac{x^{13}}{y^6z^{12}}}$
52. $\sqrt[6]{\frac{p^9q^{24}}{r^{18}}}$

53. To simplify the radical $\sqrt{x^3+x^2}$, a student wrote $\sqrt{x^3+x^2} = x\sqrt{x} + x = x(\sqrt{x}+1)$. Is this correct? Justify your answer.

Perform operations. Leave the answer in simplified ***single radical*** *form. Assume that all variables are positive.*

54. $\sqrt{3}\cdot\sqrt[3]{4}$

55. $\sqrt{x}\cdot\sqrt[5]{x}$

56. $\sqrt[3]{x^2}\cdot\sqrt[4]{x}$

57. $\sqrt[3]{4}\cdot\sqrt[5]{8}$

58. $\frac{\sqrt[3]{a^2}}{\sqrt{a}}$

59. $\frac{\sqrt{x}}{\sqrt[4]{x}}$

60. $\frac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$

61. $\frac{\sqrt[5]{16a^2}}{\sqrt[3]{2a^2}}$

62. $\sqrt[3]{2\sqrt{x}}$

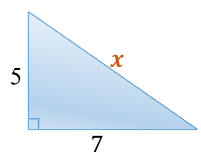
63. $\sqrt{x\sqrt[3]{2x^2}}$

64. $\sqrt[4]{3\sqrt[3]{9}}$

65. $\sqrt[3]{x^2\sqrt[4]{x^3}}$

For each right triangle, find length x. Simplify the answer if possible. In problems ***73*** *and* ***74,*** *expect the length x to be an expression in terms of n.*

66.



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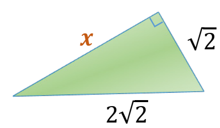

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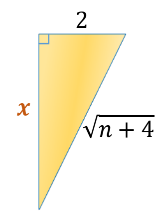

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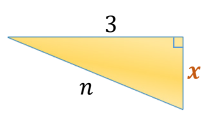

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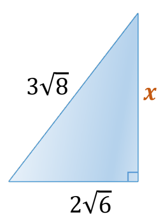
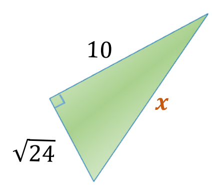


71.



Find the exact distance between each pair of points.

72. $(8,13)$ and $(2,5)$

73. $(-8,3)$ and $(-4,1)$

74. $(-6,5)$ and $(3,-4)$

75. $\left(\frac{5}{7},\frac{1}{14}\right)$ and $\left(\frac{1}{7},\frac{11}{14}\right)$

76. $(0,\sqrt{6})$ and $(\sqrt{7},0)$

77. $(\sqrt{2},\sqrt{6})$ and $(2\sqrt{2},-4\sqrt{6})$

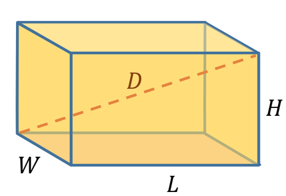
78. $(-\sqrt{5},6\sqrt{3})$ and $(\sqrt{5},\sqrt{3})$

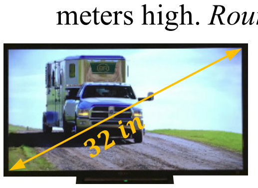
79. $(0,0)$ and (p,q)

80. $(x+h,y+h)$ and (x,y) (*assume that* $h>0$)

Solve each problem.

81. To find the diagonal of a box, we can use the formula $D=\sqrt{W^2+L^2+H^2}$, where W, L, and H are, respectively, the width, length, and height of the box. Find the diagonal D of a storage container that is 6.1 meters long, 2.4 meters wide, and 2.6 meters high. *Round your answer to the nearest centimeter.*



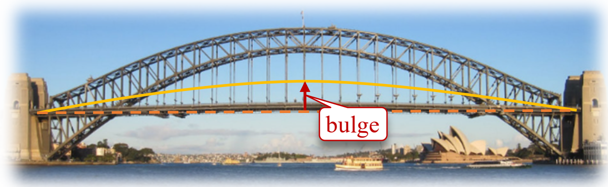


82. The screen of a 32-inch television is 27.9-inch wide. To the nearest tenth of an inch, what is the measure of its height? (Note: *TVs are measured diagonally, so a 32-inch television means that its screen measures diagonally 32 inches.*)

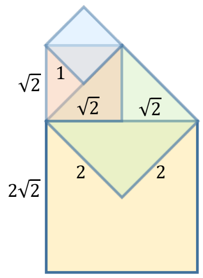
83. Suppose $A=(0,-3)$ and P is a point on the x-axis of a Cartesian coordinate system. Find all possible coordinates of P if $AP=5$.

84. Suppose $B = (1, 0)$ and P is a point on the y-axis of a Cartesian coordinate system. Find all possible coordinates of P if $BP = 2$.

85. Due to high temperatures, a 3-km bridge may expand up to 0.6 meters in length. If the maximum bulge occurs at the middle of the bridge, find the height of such a bulge. *The answer may be surprising. To avoid such situations, engineers design bridges with expansion spaces.*



RD4 Operations on Radical Expressions; Rationalization of Denominators



Unlike operations on fractions or decimals, sums and differences of many radicals cannot be simplified. For instance, we cannot combine $\sqrt{2}$ and $\sqrt{3}$, nor simplify expressions such as $\sqrt[3]{2}-1$. These types of radical expressions can only be approximated with the aid of a calculator.

However, some radical expressions can be combined (added or subtracted) and simplified. For example, the sum of $2\sqrt{2}$ and $\sqrt{2}$ is $3\sqrt{2}$, similarly as $2x + x = 3x$.

In this section, first, we discuss the addition and subtraction of radical expressions. Then, we show how to work with radical expressions involving a combination of the four basic operations. Finally, we examine how to rationalize denominators of radical expressions.

Addition and Subtraction of Radical Expressions

Recall that to perform addition or subtraction of two variable terms we need these terms to be **like**. This is because the addition and subtraction of terms are performed by factoring out the variable "like" part of the terms as a common factor. For example,

$$x^2 + 3x^2 = (1+3)x^2 = 4x^2$$

The same strategy works for addition and subtraction of the same types of radicals or **radical terms** (terms containing radicals).

Definition 4.1 ► Radical terms containing radicals with the same index and the same radicands are referred to as **like radicals** or **like radical terms**.

For example,

$\sqrt{5x}$ and $2\sqrt{5x}$ are **like** (the indexes and the radicands are the same)

while

$5\sqrt{2}$ and $2\sqrt{5}$ are **not like** (the radicands are different)

and

$\sqrt{x}$ and $\sqrt[3]{x}$ are **not like radicals** (the indexes are different).

To **add** or **subtract like radical expressions** we **factor out the common radical** and any other common factor, if applicable. For example,

$$4\sqrt{2} + 3\sqrt{2} = (4+3)\sqrt{2} = 7\sqrt{2},$$

and

$$4xy\sqrt{2} - 3x\sqrt{2} = (4y-3)x\sqrt{2}.$$

Caution! Unlike radical expressions cannot be combined. For example, we are unable to perform the addition $\sqrt{6}+\sqrt{3}$. Such a sum can only be approximated using a calculator.

Notice that unlike radicals may become like if we simplify them first. For example, $\sqrt{200}$ and $\sqrt{50}$ are not like, but $\sqrt{200} = 10\sqrt{2}$ and $\sqrt{50} = 5\sqrt{2}$. Since $10\sqrt{2}$ and $5\sqrt{2}$ are like radical terms, they can be combined. So, we can perform, for example, the addition:

$$\sqrt{200} + \sqrt{50} = 10\sqrt{2} + 5\sqrt{2} = 15\sqrt{2}$$

Example 1 ► **Adding and Subtracting Radical Expressions**

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

a. $5\sqrt{3} - 8\sqrt{3}$

b. $3\sqrt[5]{2} - 7x\sqrt[5]{2} + 6\sqrt[5]{2}$

c. $7\sqrt{45} + \sqrt{80} - \sqrt{12}$

d. $3\sqrt[3]{y^5} - 5y\sqrt[3]{y^2} + \sqrt[5]{32y^7}$

e. $\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}}$

f. $\sqrt{25x^2 - 25} - \sqrt{9x^2 - 9}$

Solution ►

a. To subtract like radicals, we combine their coefficients via factoring.

$$5\sqrt{3} - 8\sqrt{3} = (5 - 8)\sqrt{3} = \mathbf{-3\sqrt{3}}$$

The brackets are essential here.

b. $3\sqrt[5]{2} - 7x\sqrt[5]{2} + 6\sqrt[5]{2} = (3 - 7x + 6)\sqrt[5]{2} = \mathbf{(9 - 7x)\sqrt[5]{2}}$

Note: Even if not all coefficients are like, factoring the common radical is a useful strategy that allows us to combine like radical expressions.

c. The expression $7\sqrt{45} + \sqrt{80} - \sqrt{12}$ consists of unlike radical terms, so they cannot be combined in this form. However, if we simplify the radicals, some of them may become like and then become possible to combine.

$$7\sqrt{45} + \sqrt{80} - \sqrt{12} = 7\sqrt{9 \cdot 5} + \sqrt{16 \cdot 5} - \sqrt{4 \cdot 3} = 7 \cdot 3\sqrt{5} + 4\sqrt{5} - 2\sqrt{3}$$

$$= 21\sqrt{5} + 4\sqrt{5} - 2\sqrt{3} = \mathbf{25\sqrt{5} - 2\sqrt{3}}$$

d. As in the previous example, we simplify each radical expression before attempting to combine them.

$$3\sqrt[3]{y^5} - 5y\sqrt[3]{y^2} + \sqrt[5]{32y^7} = 3y\sqrt[3]{y^2} - 5y\sqrt[3]{y^2} + 2y\sqrt[5]{y^2}$$

$$= (3y - 5y)\sqrt[3]{y^2} + 2y\sqrt[5]{y^2} = \mathbf{-2y\sqrt[3]{y^2} + 2y\sqrt[5]{y^2}}$$

Remember to write the index with each radical.

Note: The last two radical expressions cannot be combined because of different indexes.

e. To perform the addition $\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}}$, we may simplify each radical expression first. Then, we add the expressions by bringing them to the least common denominator and finally, factor the common radical, as shown below.

$$\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}} = \frac{\sqrt{x}}{\sqrt{16}} + 2\frac{\sqrt{x^3}}{\sqrt{9}} = \frac{\sqrt{x}}{4} + 2\frac{\sqrt{x^3}}{3} = \frac{3\sqrt{x} + 2 \cdot 4 \cdot x\sqrt{x}}{12} = \mathbf{\left(\frac{3 + 8x}{12}\right)\sqrt{x}}$$

f. In an attempt to simplify radicals in the expression $\sqrt{25x^2-25}-\sqrt{9x^2-9}$, we factor each radicand first. So, we obtain

$$\sqrt{25x^2-25}-\sqrt{9x^2-9}=\sqrt{25(x^2-1)}-\sqrt{9(x^2-1)}=5\sqrt{x^2-1}-3\sqrt{x^2-1}$$
$$=\mathbf{2\sqrt{x^2-1}}$$

Caution! **The root of a sum <u>does not</u> equal the sum of the roots.** For example,

$$\sqrt{5}=\sqrt{\mathbf{1+4}}\neq\sqrt{\mathbf{1}}+\sqrt{\mathbf{4}}=1+2=3$$

So, radicals such as $\sqrt{25x^2-25}$ or $\sqrt{9x^2-9}$ can be simplified only via factoring a perfect square out of their radicals while $\sqrt{x^2-1}$ cannot be simplified any further.

Multiplication of Radical Expressions with More than One Term

Similarly as in the case of multiplication of polynomials, multiplication of radical expressions where at least one factor consists of more than one term is performed by applying the distributive property.

Example 2 ► **Multiplying Radical Expressions with More than One Term**

Multiply and then simplify each product. Assume that all variables represent positive real numbers.

a. $5\sqrt{2}(3\sqrt{2x}-\sqrt{6})$ **b.** $\sqrt[3]{x}(\sqrt[3]{3x^2}-\sqrt[3]{81x^2})$

c. $(2\sqrt{3}+\sqrt{2})(\sqrt{3}-3\sqrt{2})$ **d.** $(x\sqrt{x}-\sqrt{y})(x\sqrt{x}+\sqrt{y})$

e. $(3\sqrt{2}+2\sqrt[3]{x})(3\sqrt{2}-2\sqrt[3]{x})$ **f.** $(\sqrt{5y}+y\sqrt{y})^2$

Solution ► **a.**

$$5\sqrt{2}(3\sqrt{2x}-\sqrt{6})=15\sqrt{4x}-5\sqrt{2\cdot2\cdot3}=15\cdot2\sqrt{x}-5\cdot2\sqrt{3}=\mathbf{30\sqrt{x}-10\sqrt{3}}$$

$5\sqrt{2}\cdot3\sqrt{2x}=5\cdot3\sqrt{2\cdot2x}$

These are unlike terms. So, they cannot be combined.

b.

$$\sqrt[3]{x}(\sqrt[3]{3x^2}-\sqrt[3]{81x^2})=\sqrt[3]{3x^2\cdot x}-\sqrt[3]{81x^2\cdot x}=x\sqrt[3]{3}-3x\sqrt[3]{3}=\mathbf{-2x\sqrt[3]{3}}$$

distribution; simplification; combining like terms

c. To multiply two binomial expressions involving radicals we may use the **FOIL** method. Recall that the acronym **FOIL** refers to multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials.

F O I L

$$(2\sqrt{3}+\sqrt{2})(\sqrt{3}-3\sqrt{2}) = 2\cdot 3 - 6\sqrt{3\cdot 2} + \sqrt{2\cdot 3} - 3\cdot 2 = \cancel{6} - 6\sqrt{6} + \sqrt{6} - \cancel{6}$$

$$= \boldsymbol{-5\sqrt{6}}$$

d. To multiply two conjugate binomial expressions we follow the difference of squares formula, $(a-b)(a+b) = a^2 - b^2$. So, we obtain

$(\sqrt{x})^2 = x$

$$(x\sqrt{x}-\sqrt{y})(x\sqrt{x}+\sqrt{y}) = (x\sqrt{x})^2 - (\sqrt{y})^2 = x^2\cdot x - y = \boldsymbol{x^3 - y}$$

square each factor

e. Similarly as in the previous example, we follow the difference of squares formula.

$$(3\sqrt{2}+2\sqrt[3]{x})(3\sqrt{2}-2\sqrt[3]{x}) = (3\sqrt{2})^2 - (2\sqrt[3]{x})^2 = 9\cdot 2 - 4\sqrt[3]{x^2} = \boldsymbol{18 - 4\sqrt[3]{x^2}}$$

f. To multiply two identical binomial expressions we follow the perfect square formula, $(a+b)(a+b) = a^2 + 2ab + b^2$. So, we obtain

$$(\sqrt{5y}+y\sqrt{y})^2 = (\sqrt{5y})^2 + 2(\sqrt{5y})(y\sqrt{y}) + (y\sqrt{y})^2 = 5y + 2y\sqrt{5y^2} + y^2y$$

$$= \boldsymbol{5y + 2\sqrt{5}\,y^2 + y^3}$$

Rationalization of Denominators

As mentioned in *Section RD3*, the process of simplifying radicals involves rationalization of any emerging denominators. Similarly, a radical expression is not in its simplest form unless all its denominators are rational. This agreement originated before the days of calculators when computation was a tedious process performed by hand. Nevertheless, even in present time, the agreement of keeping denominators rational does not lose its validity, as we often work with variable radical expressions. For example, the expressions $\frac{2}{\sqrt{2}}$ and $\sqrt{2}$ are equivalent, as

$$\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Similarly, $\frac{x}{\sqrt{x}}$ is equivalent to $\sqrt{x}$, as

$$\frac{x}{\sqrt{x}} = \frac{x}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = \frac{x\sqrt{x}}{x} = \sqrt{x}$$

While one can argue that evaluating $\frac{2}{\sqrt{2}}$ is as easy as evaluating $\sqrt{2}$ when using a calculator, the expression $\sqrt{x}$ is definitely easier to use than $\frac{x}{\sqrt{x}}$ in any further algebraic manipulations.

Definition 4.2 ► The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalization** of the denominator.

Rationalization of denominators is carried out by multiplying the given fraction by a factor of 1, as shown in the next two examples.

Example 3 ► **Rationalizing Monomial Denominators**

Simplify, if possible. Leave the answer with a rational denominator. Assume that all variables represent positive real numbers.

a. $\frac{-1}{3\sqrt{5}}$ **b.** $\frac{5}{\sqrt[3]{32x}}$ **c.** $\sqrt[4]{\frac{81x^5}{y}}$

Solution ►

a. Notice that $\sqrt{5}$ can be converted to a rational number by multiplying it by another $\sqrt{5}$. Since the denominator of a fraction cannot be changed without changing the numerator in the same way, we multiply both, the numerator and denominator of $\frac{-1}{3\sqrt{5}}$ by $\sqrt{5}$. So, we obtain

$$\frac{-1}{3\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{-\sqrt{5}}{3 \cdot 5} = -\frac{\sqrt{\mathbf{5}}}{\mathbf{15}}$$

b. First, we may want to simplify the radical in the denominator. So, we have

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{\sqrt[3]{8 \cdot 4x}} = \frac{5}{2\sqrt[3]{4x}}$$

Then, notice that since $\sqrt[3]{4x} = \sqrt[3]{2^2x}$, it is enough to multiply it by $\sqrt[3]{2x^2}$ to nihilate the radical. This is because $\sqrt[3]{2^2x} \cdot \sqrt[3]{2x^2} = \sqrt[3]{2^3x^3} = 2x$. So, we proceed

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{2\sqrt[3]{4x}} \cdot \frac{\sqrt[3]{2x^2}}{\sqrt[3]{2x^2}} = \frac{5\sqrt[3]{2x^2}}{2 \cdot 2x} = \frac{\mathbf{5}\sqrt[3]{\mathbf{2x^2}}}{\mathbf{4x}}$$

Caution: A common mistake in the rationalization of $\sqrt[3]{4x}$ is the attempt to multiply it by a copy of $\sqrt[3]{4x}$. However, $\sqrt[3]{4x} \cdot \sqrt[3]{4x} = \sqrt[3]{16x^2} = 2\sqrt[3]{3x^2}$ is still not rational. This is because we work with a cubic root, not a square root. So, to rationalize $\sqrt[3]{4x}$ we must look for 'filling' the radicand to a perfect cube. This is achieved by multiplying $4x$ by $2x^2$ to get $8x^3$.

c. To simplify $\sqrt[4]{\frac{81x^5}{y}}$, first, we apply the quotient rule for radicals, then simplify the radical in the numerator, and finally, rationalize the denominator. So, we have

$$\sqrt[4]{\frac{81x^5}{y}} = \frac{\sqrt[4]{81x^5}}{\sqrt[4]{y}} = \frac{3x\sqrt[4]{x}}{\sqrt[4]{y}} \cdot \frac{\sqrt[4]{y^3}}{\sqrt[4]{y^3}} = \frac{\mathbf{3x}\sqrt[4]{\mathbf{xy^3}}}{\mathbf{y}}$$

To rationalize a binomial containing square roots, such as $2-\sqrt{x}$ or $\sqrt{2}-\sqrt{3}$, we need to find a way to square each term separately. This can be achieved through multiplying by a conjugate binomial, in order to benefit from the difference of squares formula. In particular, we can rationalize denominators in expressions below as follows:

$$\frac{1}{2-\sqrt{x}}=\frac{1}{(2-\sqrt{x})}\cdot\frac{(2+\sqrt{x})}{(2+\sqrt{x})}=\frac{\mathbf{2+\sqrt{x}}}{\mathbf{4-x}}$$

Apply the difference of squares formula: $(a-b)(a+b)=a^2-b^2$

or

$$\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}}=\frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})}\cdot\frac{(\sqrt{2}-\sqrt{3})}{(\sqrt{2}-\sqrt{3})}=\frac{2-\sqrt{6}}{2-3}=\frac{2-\sqrt{6}}{-1}=\mathbf{\sqrt{6}-2}$$

Example 4 ► **Rationalizing Binomial Denominators**

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\frac{1-\sqrt{3}}{1+\sqrt{3}}$

b. $\frac{\sqrt{xy}}{2\sqrt{x}-\sqrt{y}}$

Solution ►

a. $\frac{1-\sqrt{3}}{1+\sqrt{3}}\cdot\frac{(1-\sqrt{3})}{(1-\sqrt{3})}=\frac{1-2\sqrt{3}+3}{1-3}=\frac{4-2\sqrt{3}}{-2}=\frac{\cancel{-2}(-2+\sqrt{3})}{\cancel{-2}}=\mathbf{\sqrt{3}-2}$ (factor)

b. $\frac{\sqrt{xy}}{2\sqrt{x}-\sqrt{y}}\cdot\frac{(2\sqrt{x}+\sqrt{y})}{(2\sqrt{x}+\sqrt{y})}=\frac{2x\sqrt{y}+y\sqrt{x}}{4x-y}$

Some of the challenges in algebraic manipulations involve simplifying quotients with radical expressions, such as $\frac{4-2\sqrt{3}}{-2}$, which appeared in the solution to *Example 4a*. The key concept that allows us to simplify such expressions is **factoring**, as only common factors can be reduced.

Example 5 ► **Writing Quotients with Radicals in Lowest Terms**

Write each quotient in lowest terms.

a. $\frac{15-6\sqrt{5}}{6}$

b. $\frac{3x+\sqrt{8x^2}}{6x}$

Solution ► **a.** To reduce this quotient to the lowest terms we may factor the numerator first,

$$\frac{15-6\sqrt{5}}{6}=\frac{\cancel{3}(5-2\sqrt{5})}{\cancel{6}_2}=\mathbf{\frac{5-2\sqrt{5}}{2}},$$

or alternatively, rewrite the quotient into two fractions and then simplify,

$$\frac{15-6\sqrt{5}}{6}=\frac{15}{6}-\frac{6\sqrt{5}}{6}=\mathbf{\frac{5}{2}-\sqrt{5}}.$$

Caution: Here are the common errors to avoid:

~~$\frac{15-\cancel{6}\sqrt{5}}{\cancel{6}}=15-\sqrt{5}$~~ - only common factors can be reduced!

~~$\frac{15-6\sqrt{5}}{6}=\frac{\cancel{9}^3\sqrt{5}}{\cancel{6}_2}$~~ $=\frac{3\sqrt{5}}{2}$ - subtraction is performed after multiplication!

b. To reduce this quotient to the lowest terms, we simplify the radical and factor the numerator first. So,

$$\frac{3x+\sqrt{8x^2}}{6x}=\frac{3x+2x\sqrt{2}}{6x}=\frac{\cancel{x}(3+2\sqrt{2})}{6\cancel{x}}=\mathbf{\frac{3+2\sqrt{2}}{6}}$$

This expression cannot be simplified any further.

RD.4 Exercises

1. A student claims that $24-4\sqrt{x}=20\sqrt{x}$ because for $x=1$ both sides of the equation equal to 20. Is this a valid justification? Explain.

2. Generally, $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$. For example, if $a=b=1$, we have $\sqrt{1+1}=\sqrt{2}\neq 2=1+1=\sqrt{1}+\sqrt{1}$. Can you think of a situation when $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$?

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

3. $2\sqrt{3}+5\sqrt{3}$

4. $6\sqrt[3]{x}-4\sqrt[3]{x}$

5. $9y\sqrt{3x}+4y\sqrt{3x}$

6. $12a\sqrt{5b}-4a\sqrt{5b}$

7. $5\sqrt{32}-3\sqrt{8}+2\sqrt{3}$

8. $-2\sqrt{48}+4\sqrt{75}-\sqrt{5}$

9. $\sqrt[3]{16}+3\sqrt[3]{54}$

10. $\sqrt[4]{32}-3\sqrt[4]{2}$

11. $\sqrt{5a}+2\sqrt{45a^3}$

12. $\sqrt[3]{24x}-\sqrt[3]{3x^4}$

13. $4\sqrt{x^3}-2\sqrt{9x}$

14. $7\sqrt{27x^3}+\sqrt{3x}$

15. $6\sqrt{18x}-\sqrt{32x}+2\sqrt{50x}$

16. $2\sqrt{128a}-\sqrt{98a}+2\sqrt{72a}$

17. $\sqrt[3]{6x^4} + \sqrt[3]{48x} - \sqrt[3]{6x}$

18. $9\sqrt{27y^2} - 14\sqrt{108y^2} + 2\sqrt{48y^2}$

19. $3\sqrt{98n^2} - 5\sqrt{32n^2} - 3\sqrt{18n^2}$

20. $-4y\sqrt{xy^3} + 7x\sqrt{x^3y}$

21. $6a\sqrt{ab^5} - 9b\sqrt{a^3b}$

22. $\sqrt[3]{-125p^9} + p\sqrt[3]{-8p^6}$

23. $3\sqrt[4]{x^5y} + 2x\sqrt[4]{xy}$

24. $\sqrt{125a^5} - 2\sqrt[3]{125a^4}$

25. $x\sqrt[3]{16x} + \sqrt{2} - \sqrt[3]{2x^4}$

26. $\sqrt{9a-9} + \sqrt{a-1}$

27. $\sqrt{4x+12} - \sqrt{x+3}$

28. $\sqrt{x^3-x^2} - \sqrt{4x-4}$

29. $\sqrt{25x-25} - \sqrt{x^3-x^2}$

30. $\frac{4\sqrt{3}}{3} - \frac{2\sqrt{3}}{9}$

31. $\frac{\sqrt{27}}{2} - \frac{3\sqrt{3}}{4}$

32. $\sqrt{\frac{49}{x^4}} + \sqrt{\frac{81}{x^8}}$

33. $2a\sqrt[4]{\frac{a}{16}} - 5a\sqrt[4]{\frac{a}{81}}$

34. $-4\sqrt[3]{\frac{4}{y^9}} + 3\sqrt[3]{\frac{9}{y^{12}}}$

35. A student simplifies the below expression as follows:

$$\begin{aligned}\sqrt{8} + \sqrt[3]{16} &\stackrel{?}{=} \sqrt{4\cdot 2} + \sqrt[3]{8\cdot 2}\\ &\stackrel{?}{=} \sqrt{4}\cdot\sqrt{2} + \sqrt[3]{8}\cdot\sqrt[3]{2}\\ &\stackrel{?}{=} 2\sqrt{2} + 2\sqrt[3]{2}\\ &\stackrel{?}{=} 4\sqrt{4}\\ &\stackrel{?}{=} 8\end{aligned}$$

Check each equation for correctness and disscuss any errors that you can find. What would you do differently and why?

36. Match each expression from ***Column I*** with the equivalent expression in ***Column II***. Assume that A and B represent positive real numbers.

Column I	***Column II***
A. $(A+\sqrt{B})(A-\sqrt{B})$	**a.** $A - B$
B. $(\sqrt{A}+B)(\sqrt{A}-B)$	**b.** $A + 2B\sqrt{A} + B^2$
C. $(\sqrt{A}+\sqrt{B})(\sqrt{A}-\sqrt{B})$	**c.** $A - B^2$
D. $(\sqrt{A}+\sqrt{B})^2$	**d.** $A - 2\sqrt{AB} + B$
E. $(\sqrt{A}-\sqrt{B})^2$	**e.** $A^2 - B$
F. $(\sqrt{A}+B)^2$	**f.** $A + 2\sqrt{AB} + B$

Multiply, and then simplify each product. Assume that all variables represent positive real numbers.

37. $\sqrt{5}(3-2\sqrt{5})$ **38.** $\sqrt{3}(3\sqrt{3}-\sqrt{2})$ **39.** $\sqrt{2}(5\sqrt{2}-\sqrt{10})$

40. $\sqrt{3}(-4\sqrt{3}+\sqrt{6})$ **41.** $\sqrt[3]{2}(\sqrt[3]{4}-2\sqrt[3]{32})$ **42.** $\sqrt[3]{3}(\sqrt[3]{9}+2\sqrt[3]{21})$

43. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$ **44.** $(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})$ **45.** $(2\sqrt{3}+5)(2\sqrt{3}-5)$

46. $(6+3\sqrt{2})(6-3\sqrt{2})$ **47.** $(5-\sqrt{5})^2$ **48.** $(\sqrt{2}+3)^2$

49. $(\sqrt{a}+5\sqrt{b})(\sqrt{a}-5\sqrt{b})$ **50.** $(2\sqrt{x}-3\sqrt{y})(2\sqrt{x}+3\sqrt{y})$ **51.** $(\sqrt{3}+\sqrt{6})^2$

52. $(\sqrt{5}-\sqrt{10})^2$ **53.** $(2\sqrt{5}+3\sqrt{2})^2$ **54.** $(2\sqrt{3}-5\sqrt{2})^2$

55. $(4\sqrt{3}-5)(\sqrt{3}-2)$ **56.** $(4\sqrt{5}+3\sqrt{3})(3\sqrt{5}-2\sqrt{3})$ **57.** $(\sqrt[3]{2y}-5)(\sqrt[3]{2y}+1)$

58. $(\sqrt{x+5}-3)(\sqrt{x+5}+3)$ **59.** $(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})$ **60.** $(\sqrt{x+2}+\sqrt{x-2})^2$

Given $f(x)$ and $g(x)$, find $(f+g)(x)$ and $(fg)(x)$.

61. $f(x)=5x\sqrt{20x}$ and $g(x)=3\sqrt{5x^3}$ **62.** $f(x)=2x\sqrt[4]{64x}$ and $g(x)=-3\sqrt[4]{4x^5}$

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

63. $\frac{\sqrt{5}}{2\sqrt{2}}$ **64.** $\frac{3}{5\sqrt{3}}$ **65.** $\frac{12}{\sqrt{6}}$

66. $-\frac{15}{\sqrt{24}}$ **67.** $-\frac{10}{\sqrt{20}}$ **68.** $\sqrt{\frac{3x}{20}}$

69. $\sqrt{\frac{5y}{32}}$ **70.** $\frac{\sqrt[3]{7a}}{\sqrt[3]{3b}}$ **71.** $\frac{\sqrt[3]{2y^4}}{\sqrt[3]{6x^4}}$

72. $\frac{\sqrt[3]{3n^4}}{\sqrt[3]{5m^2}}$ **73.** $\frac{pq}{\sqrt[4]{p^3q}}$ **74.** $\frac{2x}{\sqrt[5]{18x^8}}$

75. $\frac{17}{6+\sqrt{2}}$ **76.** $\frac{4}{3-\sqrt{5}}$ **77.** $\frac{2\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

78. $\frac{6\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$ **79.** $\frac{3}{3\sqrt{5}+2\sqrt{3}}$ **80.** $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+5\sqrt{2}}$

81. $\frac{m-4}{\sqrt{m}+2}$ **82.** $\frac{4}{\sqrt{x}-2\sqrt{y}}$ **83.** $\frac{\sqrt{3}+2\sqrt{x}}{\sqrt{3}-2\sqrt{x}}$

84. $\frac{\sqrt{x}-2}{3\sqrt{x}+\sqrt{y}}$ **85.** $\frac{2\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ **86.** $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Write each quotient in lowest terms. Assume that all variables represent positive real numbers.

87. $\frac{10-20\sqrt{5}}{10}$ **88.** $\frac{12+6\sqrt{3}}{6}$ **89.** $\frac{12-9\sqrt{72}}{18}$

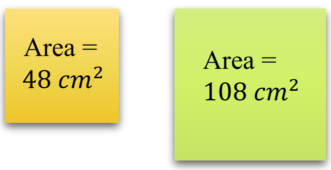
90. $\frac{2x+\sqrt{8x^2}}{2x}$ **91.** $\frac{6p-\sqrt{24p^3}}{3p}$ **92.** $\frac{9x+\sqrt{18}}{15}$

93. When solving one of the trigonometry problems, a student come up with the answer $\frac{\sqrt{3}-1}{1+\sqrt{3}}$. The textbook answer to this problem was $2-\sqrt{3}$. Was the student's answer equivalent to the textbook answer?

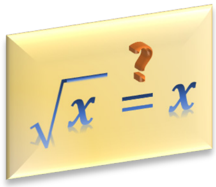
Solve each problem.

94. The base of the second tallest of the Pyramids of Giza is a square with an area of 46,225 m^2. What is its perimeter?

95. The areas of two types of square wall tiles sold at the local Home Depot store are 48 cm^2 and 108 cm^2, respectively. What is the difference in the length of sides of the two tiles? *Give the exact answer in a simplified radical form and its approximation to the nearest tenth.*



RD5 Radical Equations



In this section, we discuss techniques for solving radical equations. These are equations containing at least one radical expression with a variable, such as $\sqrt{3x-2}=x$, or a variable expression raised to a fractional exponent, such as $(2x)^{\frac{1}{3}}+1=5$.
At the end of this section, we revisit working with formulas involving radicals as well as application problems that can be solved with the use of radical equations.

Radical Equations

Definition 5.1 ► A **radical equation** is an equation in which a variable appears in one or more radicands. This includes radicands 'hidden' under fractional exponents.
For example, since $(x-1)^{\frac{1}{2}}=\sqrt{x-1}$, then the base $x-1$ is, in fact, the 'hidden' radicand.

Some examples of radical equations are

$$x=\sqrt{2x},\quad \sqrt{x}+\sqrt{x-2}=5,\quad (x-4)^{\frac{3}{2}}=8,\quad \sqrt[3]{3+x}=5$$

Note that $x=\sqrt{2}$ is not a radical equation since there is no variable under the radical sign.

The process of solving radical equations involves clearing radicals by raising both sides of an equation to an appropriate power. This method is based on the following property of equality.

Power Rule: For any **odd** natural number $\boldsymbol{n}$, the equation $\boldsymbol{a}=\boldsymbol{b}$ is equivalent to the equation $\boldsymbol{a^n}=\boldsymbol{b^n}$.

For any **even** natural number $\boldsymbol{n}$, if an equation $\boldsymbol{a}=\boldsymbol{b}$ is true, then $\boldsymbol{a^n}=\boldsymbol{b^n}$ is true.

When rephrased, the power rule for odd powers states that the solution sets to both equations, $a=b$ and $a^n=b^n$, are exactly the same.

However, the power rule for even powers states that the solutions to the original equation $a=b$ are among the solutions to the 'power' equation $a^n=b^n$.

Unfortunately, the reverse implication does not hold for even numbers n. We cannot conclude that $a=b$ from the fact that $a^n=b^n$ is true. For instance, $3^2=(-3)^2$ is true but $3\neq-3$. This means that not all solutions of the equation $a^n=b^n$ are in fact true solutions to the original equation $a=b$. Solutions that do not satisfy the original equation are called **extraneous solutions** or **extraneous roots**. Such solutions must be rejected.

For example, to solve $\sqrt{2-x}=x$, we may square both sides of the equation to obtain the quadratic equation

$$2-x=x^2.$$

Then, we solve it via factoring and the zero-product property:

$$x^2+x-2=0$$
$$(x+2)(x-1)=0$$

So, the possible solutions are $x = -2$ and $x = 1$.

Notice that $x = 1$ satisfies the original equation, as $\sqrt{2-1} = 1$ is true. However, $x = -2$ does not satisfy the original equation as its left side equals to $\sqrt{2-(-2)} = \sqrt{4} = 2$, while the right side equals to -2. Thus, $x = -2$ is the extraneous root and as such, it does not belong to the solution set of the original equation. So, the solution set of the original equation is $\{1\}$.

Caution: When the power rule for **even powers** is used to solve an equation, **every solution** of the 'power' equation **must be checked in the original equation**.

Example 1 ► **Solving Equations with One Radical**

Solve each equation.

a. $\sqrt{3x+4} = 4$

b. $\sqrt{2x-5} + 4 = 0$

c. $2\sqrt{x+1} = x - 7$

d. $\sqrt[3]{x-8} + 2 = 0$

Solution ►

a. Since the radical in $\sqrt{3x+4} = 4$ is isolated on one side of the equation, squaring both sides of the equation allows for clearing (reversing) the square root. Then, by solving the resulting polynomial equation, one can find the possible solution(s) to the original equation.

$(\sqrt{a})^2 = \left(a^{\frac{1}{2}}\right)^2 = a$

$$\left(\sqrt{3x+4}\right)^2 = (4)^2$$
$$3x + 4 = 16$$
$$3x = 12$$
$$x = 4$$

To check if 4 is a true solution, it is enough to check whether or not $x = 4$ satisfies the original equation.

$$\sqrt{3 \cdot 4 + 4} \stackrel{?}{=} 4$$
$$\sqrt{16} \stackrel{?}{=} 4$$
$$4 = 4 \checkmark$$

true

Since $x = 4$ satisfies the original equation, the solution set is $\{\mathbf{4}\}$.

b. To solve $\sqrt{2x-5} + 4 = 0$, it is useful to isolate the radical on one side of the equation. So, consider the equation

$$\sqrt{2x-5} = -4$$

Notice that the left side of the above equation is nonnegative for any x-value while the right side is constantly negative. Thus, such an equation cannot be satisfied by any x-value. Therefore, this equation has **no solution**.

c. Squaring both sides of the equation gives us

$$\left(2\sqrt{x+1}\right)^2 = (x-7)^2$$

$$4(x+1) = x^2 - 14x + 49$$

the bracket is essential here

apply the **perfect square** formula $(a-b)^2 = a^2 - 2ab + b^2$

$$4x + 4 = x^2 - 14x + 49$$

$$x^2 - 18x + 45 = 0$$

$$(x-3)(x-15) = 0$$

So, the possible solutions are $x = 3$ or $x = 15$. We check each of them by substituting them into the original equation.

If $x = 3$, then

$$2\sqrt{3+1} \stackrel{?}{=} 3 - 7$$

$$2\sqrt{4} \stackrel{?}{=} -4$$

$$4 \neq -4$$ ✗ false

So $x = 3$ is an **extraneous** root.

If $x = 15$, then

$$2\sqrt{15+1} \stackrel{?}{=} 15 - 7$$

$$2\sqrt{16} \stackrel{?}{=} 8$$

$$8 = 8$$ ✓ true

Since only 15 satisfies the original equation, the solution set is $\{15\}$.

d. To solve $\sqrt[3]{x-8} + 2 = 0$, we first isolate the radical by subtracting 2 from both sides of the equation.

$$\sqrt[3]{x-8} = -2$$

Then, to clear the cube root, we raise both sides of the equation to the third power.

$$\left(\sqrt[3]{x-8}\right)^3 = (-2)^3$$

So, we obtain

$$x - 8 = -8$$

$$x = 0$$

Since we applied the power rule for odd powers, the obtained solution is the true solution. So the solution set is $\{0\}$.

Observation: When using the power rule for odd powers checking the obtained solutions against the original equation is not necessary. This is because there is no risk of obtaining extraneous roots when applying the power rule for odd powers.

To solve radical equations with more than one radical term, we might need to apply the power rule repeatedly until all radicals are cleared. In an efficient solution, each application of the power rule should cause clearing of at least one radical term. For that reason, it is a good idea to isolate a single radical term on one side of the equation before each application of the power rule. For example, to solve the equation

$$\sqrt{x-3}+\sqrt{x+5}=4,$$

we isolate one of the radicals before squaring both sides of the equation. So, we have

$$\left(\sqrt{x-3}\right)^2=\left(4-\sqrt{x+5}\right)^2$$

$$\cancel{x}-3=\underbrace{16}_{a^2}-\underbrace{8\sqrt{x+5}}_{2ab}+\underbrace{\cancel{x}+5}_{b^2}$$

Remember that the **perfect square** formula consists of **three terms**.

Then, we isolate the remaining radical term and simplify, if possible. This gives us

$$8\sqrt{x+5}=24$$

$$\sqrt{x+5}=3$$

Squaring both sides of the last equation gives us

$$x+5=9$$

$$x=4$$

The reader is encouraged to check that $x=\mathbf{4}$ is the true solution to the original equation.

A general strategy for solving radical equations, including those with two radical terms, is as follows.

Summary of Solving a Radical Equation

- **Isolate one of the radical terms**. Make sure that one radical term is alone on one side of the equation.
- **Apply an appropriate power rule.** Raise each side of the equation to a power that is the same as the index of the isolated radical.
- **Solve the resulting equation.** If it still contains a radical, repeat the first two steps.
- **Check** all proposed solutions in the original equation.
- **State the solution set** to the original equation.

Example 2 ► **Solving Equations Containing Two Radical Terms**

Solve each equation.

a. $\sqrt{3x+1}-\sqrt{x+4}=1$ **b.** $\sqrt[3]{4x-5}=2\sqrt[3]{x+1}$

Solution ► **a.** We start solving the equation $\sqrt{3x+1}-\sqrt{x+4}=1$ by isolating one radical on one side of the equation. This can be done by adding $\sqrt{x+4}$ to both sides of the equation. So, we have

$$\sqrt{3x+1}=1+\sqrt{x+4}$$

which after squaring give us

$$\left(\sqrt{3x+1}\right)^2 = \left(1+\sqrt{x+4}\right)^2$$
$$3x+1 = 1+2\sqrt{x+4}+x+4$$
$$2x-4 = 2\sqrt{x+4}$$
$$x-2 = \sqrt{x+4}.$$

To clear the remaining radical, we square both sides of the above equation again.

$$(x-2)^2 = \left(\sqrt{x+4}\right)^2$$
$$x^2-4x+4 = x+4$$
$$x^2-5x = 0.$$

The resulting polynomial equation can be solved by factoring and applying the zero-product property. Thus,

$$x(x-5) = 0.$$

So, the possible roots are $x = 0$ or $x = 5$.

We check each of them by substituting to the original equation.

If $x = 0$, then

$$\sqrt{3\cdot 0+1}-\sqrt{0+4} \stackrel{?}{=} 1$$
$$\sqrt{1}-\sqrt{4} \stackrel{?}{=} 1$$
$$1-2 \stackrel{?}{=} 1$$
$$-1 \neq 1$$

false

If $x = 5$, then

$$\sqrt{3\cdot 5+1}-\sqrt{5+4} \stackrel{?}{=} 1$$
$$\sqrt{16}-\sqrt{9} \stackrel{?}{=} 1$$
$$4-3 \stackrel{?}{=} 1$$
$$1 = 1$$

true

Since $x = 0$ is the **extraneous** root, it does not belong to the solution set.

Only 5 satisfies the original equation. So, the solution set is $\{\mathbf{5}\}$.

b. To solve the equation $\sqrt[3]{4x-5} = 2\sqrt[3]{x+1}$, we would like to clear the cubic roots. This can be done by cubing both of its sides, as shown below.

$$\left(\sqrt[3]{4x-5}\right)^3 = \left(2\sqrt[3]{x+1}\right)^3$$
$$4x-5 = 2^3(x+1)$$

the bracket is essential here

$$4x-5 = 8x+8$$
$$-13 = 4x$$
$$x = -\frac{\mathbf{13}}{\mathbf{4}}$$

Since we applied the power rule for cubes, the obtained root is the true solution of the original equation.

Formulas Containing Radicals

Many formulas involve radicals. For example, the period T, in seconds, of a pendulum of length L, in feet, is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Sometimes, we might need to solve a radical formula for a specified variable. In addition to all the strategies for solving formulas for a variable, discussed in *Sections L2*, *F4*, and *RT6*, we may need to apply the power rule to clear the radical(s) in the formula.

Example 3 ► **Solving Radical Formulas for a Specified Variable**

Solve each formula for the indicated variable.

a. $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for $\boldsymbol{a}$ **b.** $r = \sqrt[3]{\frac{A}{P}} - 1$ for $\boldsymbol{P}$

Solution ►

a. Since $\boldsymbol{a}$ appears in the radicand, to solve $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for $\boldsymbol{a}$, we may want to clear the radical by squaring both sides of the equation. So, we have

$$N^2 = \left(\frac{1}{2\pi}\sqrt{\frac{a}{r}}\right)^2$$

$$N^2 = \frac{1}{(2\pi)^2} \cdot \frac{a}{r}$$

$$\boldsymbol{4\pi^2 N^2 r = a}$$

Note: We could also first multiply by 2π and then square both sides of the equation.

b. First, observe the position of $\boldsymbol{P}$ in the equation $r = \sqrt[3]{\frac{A}{P}} - 1$. It appears in the denominator of the radical. Therefore, to solve for $\boldsymbol{P}$, we may plan to isolate the cube root first, cube both sides of the equation to clear the radical, and finally bring $\boldsymbol{P}$ to the numerator. So, we have

$$r = \sqrt[3]{\frac{A}{P}} - 1$$

$$(r+1)^3 = \left(\sqrt[3]{\frac{A}{P}}\right)^3$$

$$(r+1)^3 = \frac{A}{\boldsymbol{P}}$$

$$\boldsymbol{P} = \frac{\boldsymbol{A}}{(\boldsymbol{r}+\mathbf{1})^{\mathbf{3}}}$$

Radicals in Applications

Many application problems in sciences, engineering, or finances translate into radical equations.

Example 4 ► **Finding the Velocity of a Skydiver**

After d meters of a free fall from an airplane, a skydiver's velocity v, in kilometers per hour, can be estimated according to the formula $v = 15.9\sqrt{d}$. Approximately how far, in meters, does a skydiver need to fall to attain the velocity of 100 km/h?

Solution ► We may substitute $v = 100$ into the equation $v = 15.9\sqrt{d}$ and solve it for d, as below.

$$100 = 15.9\sqrt{d}$$

$$6.3 \approx \sqrt{d}$$

$$\mathbf{40} \approx \boldsymbol{d}$$

Thus, a skydiver falls at 100 kph approximately after 40 meters of free falling.

RD.5 Exercises

True or false.

1. $\sqrt{2}x = x^2 - \sqrt{5}$ is a radical equation.

2. When raising each side of a radical equation to a power, the resulting equation is equivalent to the original equation.

3. $\sqrt{3x+9} = x$ cannot have negative solutions.

4. -9 is a solution to the equation $\sqrt{x} = -3$.

Solve each equation.

5. $\sqrt{7x-3}=6$

6. $\sqrt{5y+2}=7$

7. $\sqrt{6x}+1=3$

8. $\sqrt{2k}-4=6$

9. $\sqrt{x+2}=-6$

10. $\sqrt{y-3}=-2$

11. $\sqrt[3]{x}=-3$

12. $\sqrt[3]{a}=-1$

13. $\sqrt[4]{y-3}=2$

14. $\sqrt[4]{n+1}=3$

15. $5=\frac{1}{\sqrt{a}}$

16. $\frac{1}{\sqrt{y}}=3$

17. $\sqrt{3r+1}-4=0$

18. $\sqrt{5x-4}-9=0$

19. $4-\sqrt{y-2}=0$

20. $9-\sqrt{4a+1}=0$

21. $x-7=\sqrt{x-5}$

22. $x+2=\sqrt{2x+7}$

23. $2\sqrt{x+1}-1=x$

24. $3\sqrt{x-1}-1=x$

25. $y-4=\sqrt{4-y}$

26. $x+3=\sqrt{9-x}$

27. $x=\sqrt{x^2+4x-20}$

28. $x=\sqrt{x^2+3x+9}$

29. Discuss the validity of the following solution:

$$\sqrt{2x+1}=4-x$$
$$2x+1=16+x^2$$
$$x^2-2x+15=0$$
$$(x-5)(x+3)=0$$
$$\text{so} \quad x=5 \text{ or } x=-3$$

30. Discuss the validity of the following solution:

$$\sqrt{3x+1}-\sqrt{x+4}=1$$
$$(3x+1)-(x+4)=1$$
$$2x-3=1$$
$$2x=4$$
$$x=2$$

Solve each equation.

31. $\sqrt{5x+1}=\sqrt{2x+7}$

32. $\sqrt{5y-3}=\sqrt{2y+3}$

33. $\sqrt[3]{p+5}=\sqrt[3]{2p-4}$

34. $\sqrt[3]{x^2+5x+1}=\sqrt[3]{x^2+4x}$

35. $2\sqrt{x-3}=\sqrt{7x+15}$

36. $\sqrt{6x-11}=3\sqrt{x-7}$

37. $3\sqrt{2t+3}-\sqrt{t+10}=0$

38. $2\sqrt{y-1}-\sqrt{3y-1}=0$

39. $\sqrt{x-9}+\sqrt{x}=1$

40. $\sqrt{y-5}+\sqrt{y}=5$

41. $\sqrt{3n}+\sqrt{n-2}=4$

42. $\sqrt{x+5}-2=\sqrt{x-1}$

43. $\sqrt{14-n}=\sqrt{n+3}+3$

44. $\sqrt{p+15}-\sqrt{2p+7}=1$

45. $\sqrt{4a+1}-\sqrt{a-2}=3$

46. $4-\sqrt{a+6}=\sqrt{a-2}$

47. $\sqrt{x-5}+1=-\sqrt{x+3}$

48. $\sqrt{3x-5}+\sqrt{2x+3}+1=0$

49. $\sqrt{2m-3}+2-\sqrt{m+7}=0$

50. $\sqrt{x+2}+\sqrt{3x+4}=2$

51. $\sqrt{6x+7}-\sqrt{3x+3}=1$

52. $\sqrt{4x+7}-4=\sqrt{4x-1}$

53. $\sqrt{5y+4}-3=\sqrt{2y-2}$

54. $\sqrt{2\sqrt{x+11}}=\sqrt{4x+2}$

55. $\sqrt{1+\sqrt{24+10x}}=\sqrt{3x+5}$

56. $(2x-9)^{\frac{1}{2}}=2+(x-8)^{\frac{1}{2}}$

57. $(3k+7)^{\frac{1}{2}}=1+(k+2)^{\frac{1}{2}}$

58. $(x+1)^{\frac{1}{2}}-(x-6)^{\frac{1}{2}}=1$

59. $\sqrt{(x^2-9)^{\frac{1}{2}}}=2$

60. $\sqrt{\sqrt{x}+4}=\sqrt{x}-2$

61. $\sqrt{a^2+30a}=a+\sqrt{5a}$

62. Discuss how to evaluate the expression $\sqrt{5+3\sqrt{3}}-\sqrt{5-3\sqrt{3}}$ without the use of a calculator.

Solve each formula for the indicated variable.

63. $Z=\sqrt{\frac{L}{C}}$ for L

64. $V=\sqrt{\frac{2K}{m}}$ for K

65. $V=\sqrt{\frac{2K}{m}}$ for m

66. $r=\sqrt{\frac{Mm}{F}}$ for M

67. $r=\sqrt{\frac{Mm}{F}}$ for F

68. $Z=\sqrt{L^2+R^2}$ for R

69. $F=\frac{1}{2\pi\sqrt{LC}}$ for C

70. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for a

71. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for r

Solve each problem.

72. One of Einstein's special relativity principles states that time passes faster for bodies that travel with greater speed. The ratio of the time that passes for a body that moves with a speed v to the elapsed time that passes on Earth is called the ***aging rate*** and can be calculated by using the formula $\boldsymbol{r}=\frac{\sqrt{c^2-v^2}}{\sqrt{c^2}}$, where c is the speed of light, and v is the speed of the travelling body. For example, the aging rate of 0.5 means that one year for the person travelling at the speed v corresponds to two years spent on Earth.

a. Find the aging rate for a person travelling at 80% of the speed of light.

b. Find the elapsed time on Earth for 20 days of travelling time at 60% of the speed of light.

73. Assume that the formula $\boldsymbol{BSA}=\sqrt{\frac{11wh}{18000}}$ can be used to calculate the ***Body Surface Area***, in square meters, of a person with the weight w, in kilograms, and the height h, in centimeters. Greg weighs 78 kg and has a BSA of 3 m^2. To the nearest centimeter, how tall is he?

74. The distance d, in kilometers, to the horizon for an object h kilometers above the Earth's surface can be approximated by using the equation $\boldsymbol{d = \sqrt{12800h + h^2}}$. Estimate the distance between a satellite that is 1000 km above the Earth's surface and the horizon.

75. The formula $\boldsymbol{S = \frac{24}{5}\sqrt{10fL}}$, where f is the drag factor of the road surface, and L is the length of a skid mark, in meters, allows for calculating the speed S, in kilometers per hour, of a car before it started skidding to a stop. To the nearest meter, calculate the length of the skid marks left by a stopping car on a road surface with a drag factor of 0.5, if the car was travelling at 50 km/h at the time of applying the brakes.

RD6 Complex Numbers

Have you wondered if there's a solution to an equation like $x^2 = -4$? We know there is no solution in the set of real numbers since the square of any real number is positive; however, a solution does exist in the set of *complex numbers*. Complex numbers allow us to work with square roots of negative numbers and solve equations like $x^2 = -4$. This is important because equations with complex solutions arise frequently in mathematics, physics, engineering, electronics, and many other fields.
In this section, we introduce the imaginary unit and use it to perform operations with complex numbers.

Imaginary and Complex Numbers

Definition 6.1 ► The **imaginary unit** i is the number whose square is -1,

$$i^2 = -1 \quad \text{and} \quad i = \sqrt{-1}$$

The imaginary unit can be used to simplify the square roots of negative numbers,

$$\sqrt{-p} = \sqrt{p}\, i,$$

where p is a positive real number.

Note: The i **multiplies** the radical and is **not** part of the radicand.

Example 1 ► **Rewriting Square Roots of Negative Numbers Using *i***

Write each expression in terms of i and simplify if possible.

a. $\sqrt{-25}$ **b.** $\sqrt{-7}$

c. $\sqrt{-72}$ **d.** $-\sqrt{-60}$

Solution ►

a. We use *Definition 6.1* to rewrite the expression and simplify:

$$\sqrt{-25} = \sqrt{25}\, i = \mathbf{5}\boldsymbol{i}$$

b. $\sqrt{-7} = \boldsymbol{\sqrt{7}\, i}$ since the radicand 7 has no perfect square factors.

c. $\sqrt{-72} = \sqrt{72}\, i = \sqrt{36 \cdot 2}\, i = \boldsymbol{6\sqrt{2}\, i}$

d. $-\sqrt{-60} = -\sqrt{60}\, i = -\sqrt{4 \cdot 15}\, i = \boldsymbol{-2\sqrt{15}\, i}$

the leading negative remains unchanged

Definition 6.2 ► A **complex number** in standard form is $a + bi$, where a and b are real numbers.

$$a + bi$$

real part (a), *imaginary part* (b)

Observation: When the real part of a complex number is zero, $a = 0$, the number is imaginary (bi). When the imaginary part is zero, $b = 0$, the number is real (a). So there are three types of complex numbers - real numbers, imaginary numbers, and numbers that have both a real part and an imaginary part.

Addition and Subtraction of Complex Numbers

Now we are ready to perform some operations on complex numbers.

To add or subtract complex numbers, combine the real parts together and the imaginary parts together, as in the example below:

$$(1+2i)+(7-3i) = 1+7+2i-3i = (1+7)+(2-3)i = 8-i$$

real parts *imaginary parts*

Caution: If the complex numbers are not already in standard form, convert them **before** performing operations.

Example 2 ► **Adding and Subtracting Complex Numbers**

Perform operations and simplify, if possible.

a. $(9-4i)-(2+6i)$ **b.** $\sqrt{-81}+\sqrt{-1}$

c. $\sqrt{-72}-3\sqrt{-2}$ **d.** $\left(10-2\sqrt{-3}\right)+\left(5+6\sqrt{-27}\right)$

Solution ► **a.** First rewrite the subtraction to release the brackets, then combine like terms.

$$9-4i-2-6i = (9-2)+(-4i-6i) = \mathbf{7-10}\boldsymbol{i}$$

b. Use $\sqrt{-1} = i$ to rewrite in standard form **before** performing the addition

$$\sqrt{-81}+\sqrt{-1} = 9i+i = \mathbf{10}\boldsymbol{i}$$

remember to simplify any radicals with perfect square factors

c. $\sqrt{-72}-3\sqrt{-2} = \sqrt{72}\,i-3\sqrt{2}\,i = 6\sqrt{2}\,i-3\sqrt{2}\,i = \mathbf{3\sqrt{2}}\,\boldsymbol{i}$

d. $\left(10-2\sqrt{-3}\right)+\left(5+6\sqrt{-27}\right) = \left(10-2\sqrt{3}\,i\right)+\left(5+6\cdot 3\sqrt{3}\,i\right)$

$$= \left(10-2\sqrt{3}\,i\right)+\left(5+18\sqrt{3}\,i\right)$$

$$= \mathbf{15+16\sqrt{3}}\,\boldsymbol{i}$$

Multiplication of Complex Numbers

Most of our algebraic rules for real numbers hold for complex numbers. One notable exception is that the product rule for radicals, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, **is not true** if a and b are **both negative**. To illustrate this, recall our definition of i

$$i^2 = -1 \text{ and } \sqrt{-1} = i$$

Now calculate $i^2 = \sqrt{-1} \cdot \sqrt{-1}$ using $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$:

$$\sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1 \cdot -1} = \sqrt{1} = 1$$

This contradicts our original definition that $i^2 = -1$ and therefore $\sqrt{a} \cdot \sqrt{b} \neq \sqrt{ab}$ for negative a and b.

To avoid accidentally using an invalid rule, we always change $\sqrt{-1}$ to i first, then carry out our calculations. This way, the order of operations will be applied correctly.

Example 3 ► **Multiplying Complex Numbers**

Multiply.

a. $\sqrt{-8} \cdot \sqrt{-2}$

b. $2i \cdot 7i$

c. $(4 + 3i)(1 - 5i)$

d. $(5 + \sqrt{-6})(3 - \sqrt{-2})$

Solution ►

a. Rewrite in standard form **before** simplifying

$$\sqrt{-8} \cdot \sqrt{-2} = 2\sqrt{2}\, i \cdot \sqrt{2}\, i = 2\sqrt{4}\, i^2 = 2 \cdot 2 \cdot (-1) = \mathbf{-4}$$

product rule valid here since $2 > 0$

replace i^2 with its value of -1 when it appears

b. Multiply imaginary numbers like monomials then use $i^2 = -1$ where appropriate

$$2i \cdot 7i = 14i^2 = 14(-1) = \mathbf{-14}$$

c. Use distribution to multiply complex numbers in the same way as binomials

$$(4 + 3i)(1 - 5i) = 4 - 20i + 3i - 15i^2 = 4 - 17i - 15(-1) = \mathbf{19 - 17i}$$

collect like terms

d. $(5 + \sqrt{-6})(3 - \sqrt{-2}) = (5 + \sqrt{6}\, i)(3 - \sqrt{2}\, i) = 15 - 5\sqrt{2}\, i + 3\sqrt{6}\, i - \sqrt{12}\, i^2$

$$= 15 + (-5\sqrt{2} + 3\sqrt{6})i - 2\sqrt{3}\,(-1)$$

$$= \mathbf{15 + 2\sqrt{3} + (-5\sqrt{2} + 3\sqrt{6})i}$$

Since complex numbers behave like binomials when multiplied, patterns can help us simplify some products more efficiently than using distribution or FOIL. In the following example, we use the perfect squares formula, $(a \pm b)^2 = a^2 \pm 2ab + b^2$, and the difference of squares formula, $(a+b)(a-b)=a^2-b^2$, which were introduced in *Section P2*.

Example 4 ► **Multiplying Complex Numbers Using Patterns**

Multiply.

a. $(4+i)^2$

b. $(7-6i)^2$

c. $(1+2i)(1-2i)$

d. $(-3+i)(-3-i)$

Solution ►

a. We recognize the perfect squares pattern with $a=4$ and $b=i$.

$$(4+i)^2=4^2+2\cdot 4\cdot i+i^2=16+8i+(-1)=\mathbf{15+8i}$$

b. Here $a=7, b=6i$ in the perfect squares formula. The binomial is a difference, so the formula uses the subtraction sign for the middle term.

$$(7-6i)^2=7^2-2\cdot 7\cdot(6i)+(6i)^2=49-84i+36i^2=49-84i-36$$
$$=\mathbf{13-84i}$$

c. This is a product of conjugates, so we use the difference of squares formula with $a=1, b=2i$.

$$(1+2i)(1-2i)=1^2-(2i)^2=1-4i^2=1+4=\mathbf{5}$$

d. Apply the difference of squares formula with $a=-3, b=i$.

$$(-3+i)(-3-i)=(-3)^2-i^2=9+1=\mathbf{10}$$

Notice in *Example 4c* and *4d* that the product of conjugate pairs resulted in a real value. In general, we have

$$(a+bi)(a-bi)=a^2-(bi)^2=a^2-b^2i^2=a^2-b^2(-1)=a^2+b^2$$

Therefore, the product of any complex conjugate pair, $a+bi$ and $a-bi$, is real and equal to a^2+b^2. This new pattern is referred to as the **product of complex conjugates formula**

$$\boldsymbol{(a+bi)(a-bi)=a^2+b^2}.$$

Powers of i

We simplify complex expressions by treating i much like a variable; however, it is important to remember that i has a **constant value** of $\sqrt{-1}$. This means we can simplify powers of i further than we would be able to simplify powers of variables. In fact, any power of i can be simplified to one of four values: i, -1, $-i$, or 1. Look for the pattern in the first several powers:

$$\begin{aligned}
&i = i\\
&i^2 = -1\\
&i^3 = i \cdot i^2 = -i\\
&i^4 = (i^2)^2 = (-1)^2 = 1\\
&i^5 = i^4 \cdot i = 1 \cdot i = i\\
&i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1\\
&i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i\\
&i^8 = (i^4)^2 = (1)^2 = 1
\end{aligned}$$

As we go to higher powers, the pattern $i, -1, -i, 1$ repeats over and over as above.

In the evaluation of the 5th to 8th powers above, the power i^4 was used repeatedly to rewrite the original power. This is because $i^4 = 1$, which is a very nice number to work with. When simplifying powers of i, it is easiest and most efficient to rewrite the power in terms of i^4 using exponent rules, as in the example below.

Example 5 ► **Simplifying Powers of *i***

Simplify.

a. i^{12} **b.** i^{33}

c. i^{42} **d.** i^{63}

Solution ►

a. Since $12 \div 4 = 3$, we can rewrite i^{12} as $i^{4 \cdot 3}$, which is equivalent to $(i^4)^3 = (1)^3 = \mathbf{1}$

b. This time, $33 \div 4 = 8$ with remainder 1. So we can write $33 = 4 \cdot 8 + 1$, which is used in our simplification as

$$i^{4 \cdot 8 + 1} = (i^4)^8 \cdot i = (1)^8 \cdot i = \boldsymbol{i}$$

c. Here, $42 \div 4 = 10$ with a remainder of 2:

$$i^{42} = i^{40} i^2 = (1)(-1) = \mathbf{-1}$$

d. 63 has a remainder of 3 when divided by 4:

$$i^{63} = i^{60} i^3 = (1)(-i) = -\boldsymbol{i}$$

Observation: Since any perfect 4th power of i simplifies to 1, when simplifying higher powers of i, we can divide the exponent by 4, note any remainder, r, and replace the power with i^r.

Division of Complex Numbers

Dividing complex numbers is very similar to rationalizing denominators. We get rid of any imaginary numbers in the denominator by using the product of complex conjugates formula, then simplify to standard form.

Example 6 ► **Dividing Complex Numbers**

Simplify.

a. $\frac{2+3i}{1+2i}$ **b.** $\frac{3}{i}$

c. $\frac{5+2i}{5-2i}$ **d.** $(2-i) \div (i-3)$

Solution ►

a. We identify the complex conjugate of the denominator as $1-2i$, then multiply numerator and denominator by this value:

$$\frac{2+3i}{1+2i} = \frac{2+3i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{2-4i+3i-6i^2}{1^2+2^2} = \frac{2-i+6}{5} = \frac{8-i}{5} = \mathbf{\frac{8}{5} - \frac{1}{5}i}$$

$(a+bi)(a-bi) = a^2+b^2$

standard form, $a+bi$

b. The denominator is $i = 0 + 1i$, so the complex conjugate is $0 - 1i = -i$

$$\frac{3}{i} \cdot \frac{-i}{-i} = \frac{-3i}{-i^2} = \frac{-3i}{-(-1)} = -3i$$

Alternatively, we could multiply numerator and denominator by i and obtain the same result (but that's only the case when the denominator is a purely imaginary number)

$$\frac{3}{i} \cdot \frac{i}{i} = \frac{3i}{i^2} = \frac{3i}{-1} = -3i$$

c. Here, the complex conjugate of the denominator, $5-2i$, is $5+2i$

$$\frac{5+2i}{5-2i} \cdot \frac{5+2i}{5+2i} = \frac{5^2+20i+(2i)^2}{5^2+2^2} = \frac{25+20i-4}{25+4} = \frac{21+20i}{29} = \frac{21}{29} + \frac{20}{29}i$$

d. Rewrite the division as a fraction and change the denominator into standard form, $(2-i) \div (i-3) = \frac{2-i}{-3+i}$, then multiply the numerator and denominator by the complex conjugate of the the denominator, $-3-i$

$$\frac{2-i}{-3+i} \cdot \frac{-3-i}{-3-i} = \frac{-6-2i+3i+i^2}{(-3)^2+1^2} = \frac{-6+i-1}{9+1} = \frac{-7+i}{10} = -\mathbf{\frac{7}{10}} + \mathbf{\frac{1}{10}i}$$

RD.6 Exercises

Find the mistake.

1. $\sqrt{-3} \cdot \sqrt{-15} = \sqrt{-3 \cdot -15} = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$

Match each number in Column I to its complex conjugate in Column II.

2.

Column I	Column II
a. $8 + 21i$	**A.** $8 + 21i$
b. $-8 - 21i$	**B.** $8 - 21i$
c. $8 - 21i$	**C.** $-8 - 21i$
d. $-8 + 21i$	**D.** $-8 + 21i$

3. Quinn says the solution to $x^2 = -64$ is $8i$, while Finn says the solution is $-8i$. Who is correct?

Complete the indicated operation(s) and simplify.

4. $\sqrt{-81}$

5. $\sqrt{-100}$

6. $\sqrt{-72}$

7. $\sqrt{-98}$

8. $\sqrt{-5} \cdot \sqrt{-5}$

9. $\sqrt{-7} \cdot \sqrt{-7}$

10. $\sqrt{-10} \cdot \sqrt{-5}$

11. $\sqrt{-7} \cdot \sqrt{-21}$

12. $\sqrt{-75} + \sqrt{-108}$

13. $\sqrt{-32} - \sqrt{-128}$

14. $2\sqrt{-45} - 7\sqrt{-80}$

15. $-3\sqrt{-40} + 6\sqrt{-250}$

16. $(4 + \sqrt{-64})(7 + 2\sqrt{-16})$

17. $(-2 + 9\sqrt{-1})(5 + \sqrt{-49})$

18. $(6 - \sqrt{-8})(3 + \sqrt{-50})$

19. $(3 - \sqrt{-75})(5 - \sqrt{-147})$

20. $(1 + \sqrt{-18})(1 - \sqrt{-18})$

21. $(8 + \sqrt{-48})(8 - \sqrt{-48})$

22. $3i(4 + 7i)$

23. $2i(1 - 9i)$

24. $(1 + 4i)(5 - 6i)$

25. $(8 - i)(2 - 10i)$

26. $(5 - 3i)^2$

27. $(6 + 7i)^2$

28. $(8 + 5i)(8 - 5i)$

29. $(10 - 9i)(10 + 9i)$

30. i^{45}

31. i^{56}

32. i^{103}

33. i^{201}

34. i^{90}

35. i^{79}

36. $\frac{6+\sqrt{-60}}{2}$

37. $\frac{6-\sqrt{-504}}{3}$

38. $\frac{5-3\sqrt{-525}}{10}$

39. $\frac{8+\sqrt{-624}}{40}$

40. $\frac{5}{i}$

41. $\frac{6}{5i}$

42. $\frac{7}{4+i}$

43. $\frac{3}{5+7i}$

44. $\frac{3-2i}{3+2i}$

45. $\frac{4+3i}{4-3i}$

46. $\frac{8-9i}{1-6i}$

47. $\frac{7+5i}{11+4i}$

Determine if the complex number is a solution to the equation given.

48. $5i\,;\ x^2 + 25 = 0$

49. $-2i\,;\ x^2 = -4$

50. $1 + 2i\,;\ x^2 - 2x + 5 = 0$

51. $3 - 2i\,;\ x^2 - 6x + 13 = 0$

52. $4 - 3i\,;\ x^2 - 3x + 10 = 0$

53. $5 + i\,;\ x^2 + 5x + 60 = 0$

Attributions