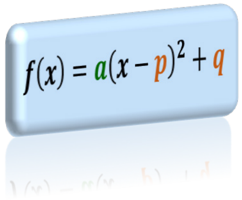
# Q3 Properties and Graphs of Quadratic Functions



In this section, we explore an alternative way of graphing quadratic functions. It turns out that if a quadratic function is given in vertex form, $\boldsymbol{f(x)=a(x-p)^2+q}$, its graph can be obtained by transforming the shape of the basic **parabola**, $f(x)=x^2$, by applying a **vertical dilation** by the factor of $a$, as well as a **horizontal translation** by $p$ units and **vertical translation** by $q$ units. This approach makes the graphing process easier than when using a table of values.

In addition, the vertex form allows us to identify the main characteristics of the corresponding graph such as **shape**, **opening**, **vertex**, and **axis of symmetry**. Then, the additional properties of a quadratic function, such as **domain** and **range**, or where the function increases or decreases can be determined by observing the obtained graph.

## Properties and Graph of the Basic Parabola $f(x)=x^2$

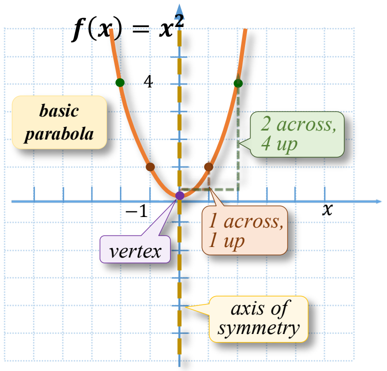
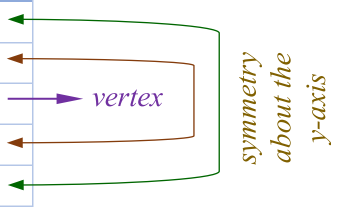


**Figure 3.1**

Recall the shape of the **basic parabola**, $f(x)=x^2$, as discussed in *Section P4*.

| $x$ | $x^2$ |
|---|---|
| $-2$ | $4$ |
| $-1$ | $1$ |
| $0$ | $0$ |
| $1$ | $1$ |
| $2$ | $4$ |



Observe the relations between the points listed in the table above. If we start with plotting the **vertex** $(\mathbf{0},\mathbf{0})$, then the next pair of points, $(\mathbf{1},\mathbf{1})$ and $(-\mathbf{1},\mathbf{1})$, is plotted **1 unit across** from the vertex (both ways) and **1 unit up**. The following pair, $(\mathbf{2},\mathbf{4})$ and $(-\mathbf{2},\mathbf{4})$, is plotted **2 units across** from the vertex and **4 units up**. The graph of the parabola is obtained by connecting these 5 main points by a curve, as illustrated in *Figure 3.1*.

The graph of this parabola is symmetric in the $y$-axis, so the equation of the **axis of symmetry** is $\boldsymbol{x=0}$.

The **domain** of the basic parabola is the set of all real numbers, $\mathbb{R}$, as $f(x)=x^2$ is a polynomial, and polynomials can be evaluated for any real $x$-value.

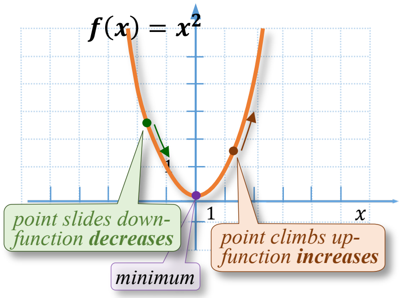


**Figure 3.2**

The **arms** of the parabola are directed **upwards**, which means that the vertex is the lowest point of the graph. Hence, the **range** of the basic parabola function, $f(x)=x^2$, is the interval $[\mathbf{0},\infty)$, and the **minimum value** of the function is **0**.

Suppose a point 'lives' on the graph and travels from left to right. Observe that in the case of the basic parabola, if $x$-coordinates of the 'travelling' point are smaller than 0, the point slides down along the graph. Similarly, if $x$-coordinates are larger than 0, the point climbs up the graph. (See *Figure 3.2*) To describe this property in mathematical language, we say that the function $f(x)=x^2$ **decreases** in the interval $(-\infty,0]$ and **increases** in the interval $[0,\infty)$.

## Properties and Graphs of a Dilated Parabola $f(x) = ax^2$

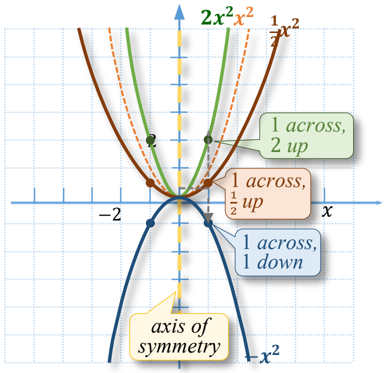


**Figure 3.3**

*Figure 3.3* shows graphs of several functions of the form $f(x) = ax^2$. Observe how the shapes of these parabolas change for various values of $a$ in comparison to the shape of the basic parabola $y = x^2$.

The common point for all of these parabolas is the vertex $(0,0)$. Additional points, essential for graphing such parabolas, are shown in the table below.

| $x$ | $ax^2$ |
|---|---|
| $-2$ | $4a$ |
| $-1$ | $a$ |
| $0$ | $0$ |
| $1$ | $a$ |
| $2$ | $4a$ |

→ *vertex*

*1 unit apart from zero, $a$ units up*

*2 units apart from zero, $4a$ units up*

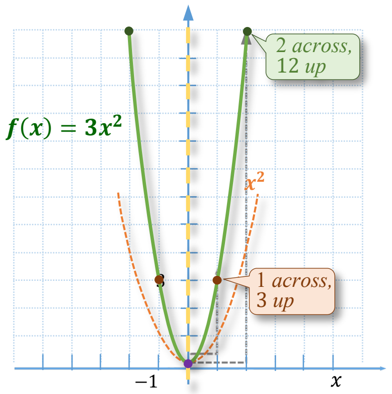


**Figure 3.4**

For example, to graph $f(x) = 3x^2$, it is convenient to plot the **vertex** first, which is at the point $(0,0)$. Then, we may move the pen **1** unit **across** from the vertex (either way) and **3** units **up** to plot the points $(-1,3)$ and $(1,3)$. If the grid allows, we might want to plot the next two points, $(-2,12)$ and $(2,12)$, by moving the pen **2** units **across** from the vertex and $4 \cdot 3 = \mathbf{12}$ units **up**, as in *Figure 3.4*.

Notice that the obtained shape (in solid green) is **narrower** than the shape of the basic parabola (in dashed orange). However, similarly as in the case of the basic parabola, the shape of the dilated function is still **symmetrical about the $y$-axis, $x = 0$**.

Now, suppose we want to graph the function $f(x) = -\frac{1}{2}x^2$. As before, we may start by plotting the vertex at $(0,0)$. Then, we move the pen **1** unit **across** from the vertex (either way) and **half** a unit **down** to plot the points $\left(-1, -\frac{1}{2}\right)$ and $\left(1, -\frac{1}{2}\right)$, as in *Figure 3.5*. The next pair of points can be plotted by moving the pen **2** units **across** from the vertex and **2** units **down**, as the ordered pairs $(-2,-2)$ and $(2,-2)$ satisfy the equation $f(x) = -\frac{1}{2}x^2$.

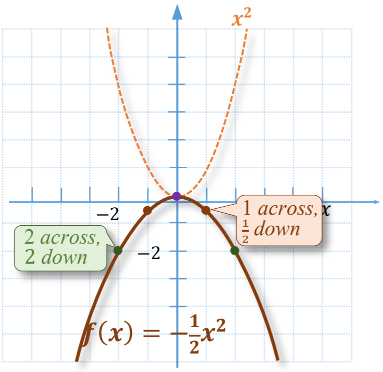


**Figure 3.5**

Notice that this time the obtained shape (in solid brown) is **wider** than the shape of the basic parabola (in dashed orange). Also, as a result of the **negative $a$-value**, the parabola opens **down**, and the **range** of this function is $(-\infty, \mathbf{0}]$.

Generally, the **shape** of a quadratic function of the form $\boldsymbol{f(x) = ax^2}$ is

- **narrower** than the shape of the basic parabola, if $|a| > \mathbf{1}$;
- **wider** than the shape of the basic parabola, if $\mathbf{0} < |a| < \mathbf{1}$; and
- **the same** as the shape of the **basic parabola**, $y = x^2$, if $|a| = \mathbf{1}$.

The parabola opens **up**, for $\boldsymbol{a > 0}$, and **down**, for $\boldsymbol{a < 0}$.
Thus the **vertex** becomes the **lowest point** of the graph, if $\boldsymbol{a > 0}$, and the **highest point** of the graph, if $\boldsymbol{a < 0}$.

The **range** of $\boldsymbol{f(x) = ax^2}$ is $[\mathbf{0}, \infty)$, if $\boldsymbol{a > 0}$, and $(-\infty, \mathbf{0}]$, if $\boldsymbol{a < 0}$.

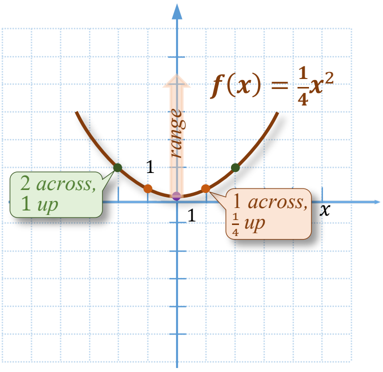
The **axis of symmetry** of the dilated parabola $\boldsymbol{f(x) = ax^2}$ remains the same as that of the basic parabola, which is $\boldsymbol{x = 0}$.

### Example 1 ► Graphing a Dilated Parabola and Describing Its Shape, Opening, and Range

For each quadratic function, describe its shape and opening. Then graph it and determine its range.

**a.** $f(x) = \frac{1}{4}x^2$ **b.** $g(x) = -2x^2$

**Solution ►**



**a.** Since the leading coefficient of the function $f(x) = \frac{1}{4}x^2$ is positive, the parabola **opens up**. Also, since $0 < \frac{1}{4} < 1$, we expect the shape of the parabola to be **wider** than that of the basic parabola.

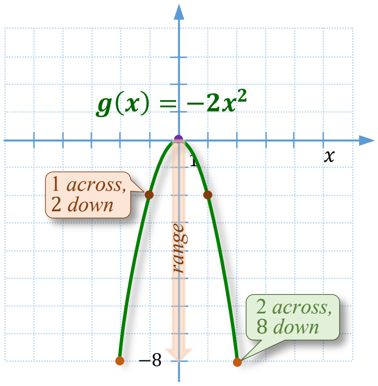
To graph $f(x) = \frac{1}{4}x^2$, first we plot the vertex at $(0,0)$ and then points $\left(\pm 1, \frac{1}{4}\right)$ and $\left(\pm 2, \frac{1}{4} \cdot 4\right) = (\pm 2, 1)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the $y$-axis, we observe that the range of the function is $[0, \infty)$.

**b.** Since the leading coefficient of the function $g(x) = -2x^2$ is negative, the parabola **opens down**. Also, since $|-2| > 1$, we expect the shape of the parabola to be **narrower** than that of the basic parabola.

To graph $g(x) = -2x^2$, first we plot the vertex at $(0,0)$ and then points $(\pm 1, -2)$ and $(\pm 2, -2 \cdot 4) = (\pm 2, -8)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the $y$-axis, we observe that the range of the function is $(-\infty, 0]$.



## Properties and Graphs of the Basic Parabola with Shifts

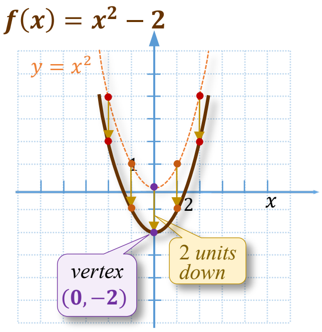


**Figure 3.6**

Suppose we would like to graph the function $f(x) = x^2 - 2$. We could do this via a table of values, but there is an easier way if we already know the shape of the basic parabola $y = x^2$.

Observe that for every $x$-value, the value of $x^2 - 2$ is obtained by subtracting 2 from the value of $x^2$. So, to graph $f(x) = x^2 - 2$, it is enough to **move each point** $(x, x^2)$ of the basic parabola by **two units down**, as indicated in *Figure 3.6*.

The shift of $y$-values by 2 units down causes the **range** of the new function, $f(x) = x^2 - 2$, to become $\boldsymbol{[-2, \infty)}$. Observe that this vertical shift also changes the minimum value of this function, from 0 to $-2$.

The **axis of symmetry** remains unchanged, and it is $\boldsymbol{x = 0}$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = x^2 + q}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{q}$ **steps up**, if $\boldsymbol{q > 0}$;
- **shifting** the graph of the basic parabola $\boldsymbol{|q|}$ **steps down**, if $\boldsymbol{q < 0}$.

The **vertex** of such parabola is at $\boldsymbol{(0, q)}$. The **range** of it is $\boldsymbol{[q, \infty)}$.
The **minimum** (lowest) **value** of the function is $\boldsymbol{q}$.
The **axis of symmetry** is $\boldsymbol{x = 0}$.

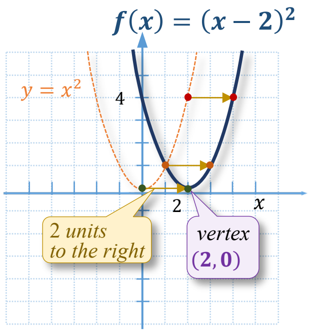


**Figure 3.7**

Now, suppose we wish to graph the function $f(x) = (x-2)^2$. We can graph it by joining the points calculated in the table below.

| $x$ | $(x-2)^2$ |
|---|---|
| **0** | **4** |
| **1** | **1** |
| **2** | **0** |
| **3** | **1** |
| **4** | **4** |

Observe that the parabola $f(x) = (x-2)^2$ assumes its lowest value at the vertex. The lowest value of the perfect square $(x-2)^2$ is zero, and it is attained at the $x$-value of 2. Thus, the vertex of this parabola is (2,0).

Notice that the **vertex** $\boldsymbol{(2, 0)}$ of $f(x) = (x-2)^2$ is positioned 2 units to the right from the vertex (0,0) of the basic parabola.

This suggests that the graph of the function $f(x) = (x-2)^2$ can be obtained without the aid of a table of values. It is enough to shift the graph of the basic parabola **2 units** to the **right**, as shown in *Figure 3.7*.

Observe that the horizontal shift does not influence the **range** of the new parabola $f(x) = (x-2)^2$. It is still $\boldsymbol{[0, \infty)}$, the same as for the basic parabola. However, the **axis of symmetry** has changed to $\boldsymbol{x = 2}$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = (x-p)^2}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{p}$ **steps** to the **right**, if $\boldsymbol{p > 0}$;
- **shifting** the graph of the basic parabola $\boldsymbol{|p|}$ **steps** to the **left**, if $\boldsymbol{p < 0}$.

The **vertex** of such a parabola is at $\boldsymbol{(p, 0)}$. The **range** of it is $\boldsymbol{[0, \infty)}$.

The **minimum value** of the function is $\boldsymbol{0}$.

The **axis of symmetry** is $\boldsymbol{x = p}$.

## Example 2 ► Graphing Parabolas and Observing Transformations of the Basic Parabola

Graph each parabola by plotting its vertex and following the appropriate opening and shape. Then describe transformations of the basic parabola that would lead to the obtained graph. Finally, state the range and the equation of the axis of symmetry.

**a.** $f(x) = (x+3)^2$ **b.** $g(x) = -x^2 + 1$

**Solution** ►

**a.** The perfect square $(x+3)^2$ attains its lowest value at $x=-3$. So, the **vertex** of the parabola $f(x)=(x+3)^2$ is $(\mathbf{-3,0})$. Since the leading coefficient is 1, the parabola takes the shape of $y=x^2$, and its **arms open up**.

The graph of the function $f$ can be obtained by **shifting** the graph of the basic parabola **3 units to the left**, as shown in *Figure 3.8*.

The **range** of function $f$ is $[\mathbf{0},\infty)$, and the equation of the **axis of symmetry** is $\boldsymbol{x=-3}$.

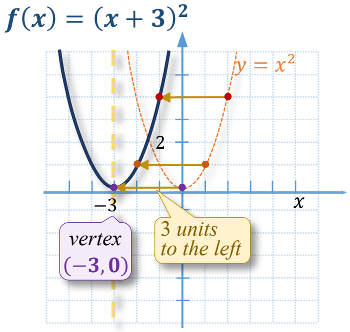


**Figure 3.8**

**b.** The expression $-x^2+1$ attains its highest value at $x=0$. So, the **vertex** of the parabola $g(x)=-x^2+1$ is $(\mathbf{0,1})$. Since the leading coefficient is $-1$, the parabola takes the shape of $y=x^2$, but its **arms open down**.

The graph of the function $g$ can be obtained by:

- first, **flipping the graph** of the basic parabola **over the $x$-axis**, and then

- **shifting** the graph of $y=-x^2$ **1 unit up**, as shown in *Figure 3.9*.

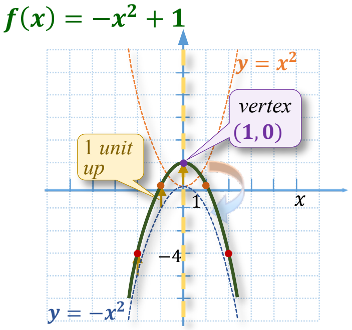


**Figure 3.9**

The **range** of the function $g$ is $(-\infty,\mathbf{1}]$, and the equation of the **axis of symmetry** is $\boldsymbol{x=0}$.

***Note:*** The order of transformations in the above example is essential. The reader is encouraged to check that **shifting** the graph of $y=x^2$ by 1 unit up first and then **flipping** it over the $x$-axis results in a different graph than the one in *Figure 3.9*.

## Properties and Graphs of Quadratic Functions Given in the Vertex Form $f(x)=a(x-p)^2+q$

So far, we have discussed properties and graphs of quadratic functions that can be obtained from the graph of the basic parabola by applying mainly a single transformation. These transformations were: dilations (including flips over the $x$-axis), and horizontal and vertical shifts. Sometimes, however, we need to apply more than one transformation. We have already encountered such a situation in *Example 2b*, where a flip and a vertical shift were applied. Now, we will look at properties and graphs of any function of the form $\boldsymbol{f(x)=a(x-p)^2+q}$, referred to as the **vertex form** of a quadratic function.

Suppose we wish to graph $f(x) = 2(x+1)^2 - 3$. This can be accomplished by connecting the points calculated in a table of values, such as the one below, or by observing the coordinates of the vertex and following the shape of the graph of $y = 2x^2$. Notice that the vertex of our parabola is at $(-1, -3)$. This information can be taken directly from the equation $f(x) = 2(x+1)^2 - 3 = 2(x-(-1))^2 - 3$,

*opposite to the number in the bracket*

*the same last number*

without the aid of a table of values.

| $x$ | $2(x+1)^2-3$ |
|---|---|
| $-3$ | $5$ |
| $-2$ | $-1$ |
| $-1$ | $-3$ |
| $0$ | $-1$ |
| $1$ | $5$ |

*1 unit apart from zero, 2 units up*

*vertex*

The rest of the points follow the pattern of the shape for the $y = 2x^2$ parabola: 1 across, 2 up; 2 across, $4 \cdot 2 = 8$ up. So, we connect the points as in *Figure 3.10*.

Notice that the graph of function $f$ could also be obtained as a result of translating the graph of $y = 2x^2$ by 1 unit left and 3 units down, as indicated in *Figure 3.10* by the blue vectors.

Here are the main properties of the graph of function $f$:
- It has a **shape** of $y = \mathbf{2x^2}$;
- It is a parabola that **opens up**;
- It has a **vertex** at $(\mathbf{-1}, \mathbf{-3})$;
- It is **symmetrical** about the line $\boldsymbol{x = -1}$;
- Its **minimum value** is **– 3**, and this minimum is attained at $x = -1$;
- Its domain is the set of all real numbers, and its **range** is the interval $[\mathbf{-3}, \infty)$;
- It **decreases** for $x \in (-\infty, \mathbf{-1}]$ and **increases** for $x \in [\mathbf{-1}, \infty)$.

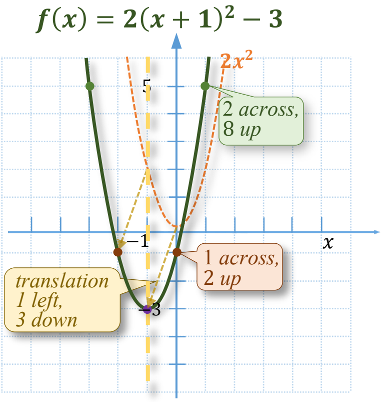


**Figure 3.10**

The above discussion of properties and graphs of a quadratic function given in vertex form leads us to the following general observations:

### Characteristics of Quadratic Functions Given in Vertex Form $f(x) = a(x-p)^2 + q$

**1.** The graph of a quadratic function given in **vertex form**

$$f(x) = a(x-p)^2 + q, \text{ where } a \neq 0,$$

is a **parabola** with **vertex** $(p, q)$ and **axis of symmetry** $x = p$.

**2.** The graph **opens up** if $a$ is **positive** and **down** if $a$ is **negative**.

**3.** If $a > 0$, $q$ is the **minimum value**. If $a < 0$, $q$ is the **maximum value**.

**3.** The graph is **narrower** than that of $y = x^2$ if $|a| > 1$.
The graph is **wider** than that of $y = x^2$ if $0 < |a| < 1$.

**4.** The **domain** of function $f$ is the set of real numbers, $\mathbb{R}$.
The **range** of function $f$ is $[q, \infty)$ if $a$ is **positive** and $(-\infty, q]$ if $a$ is **negative**.

## Example 3 ► Identifying Properties and Graphing Quadratic Functions Given in Vertex Form $\boldsymbol{f(x)=a(x-p)^2+q}$

For each function, identify its **vertex**, **opening**, **axis of symmetry**, and **shape**. Then graph the function and state its **domain** and **range**. Finally, describe **transformations** of the basic parabola that would lead to the obtained graph.

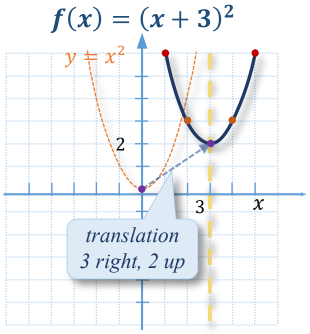
**a.** $f(x)=(x-3)^2+2$ **b.** $g(x)=-\frac{1}{2}(x+1)^2+3$

**Solution** ►

**a.** The vertex of the parabola $f(x)=(x-3)^2+2$ is $\boldsymbol{(3,2)}$; the graph **opens up**, and the equation of the axis of symmetry is $\boldsymbol{x=3}$. To graph this function, we can plot the vertex first and then follow the shape of the basic parabola $y=\boldsymbol{x^2}$.

The domain of function $f$ is $\mathbb{R}$, and the range is $[\mathbf{2},\infty)$.

The graph of $f$ can be obtained by shifting the graph of the basic parabola **3 units to the right** and **2 units up**.



**b.** The vertex of the parabola $g(x)=-\frac{1}{2}(x+1)^2+3$ is $(\mathbf{-1,3})$; the graph **opens down**, and the equation of the axis of symmetry is $\boldsymbol{x=-1}$. To graph this function, we can plot the vertex first and then follow the shape of the parabola $y=-\frac{1}{2}\boldsymbol{x^2}$. This means that starting from the vertex, we move the pen one unit across (both ways) and drop half a unit to plot the next two points, $\left(0,\frac{5}{2}\right)$ and symmetrically $\left(-2,\frac{5}{2}\right)$. To plot the following two points, again, we start from the vertex and move our pen two units across and 2 units down (as $-\frac{1}{2}\cdot 4=-2$). So, the next two points are $(1,1)$ and symmetrically $(-4,1)$, as indicated in *Figure 3.11*.

The domain of function $g$ is $\mathbb{R}$, and the range is $(-\infty,\mathbf{3}]$.

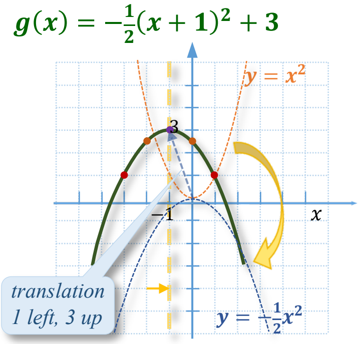


**Figure 3.11**

The graph of $g$ can be obtained from the graph of the basic parabola in two steps:

1. **Dilate** the basic parabola by multiplying its $y$-values by the factor of $-\frac{1}{2}$.
2. Shift the graph of the dilated parabola $y=-\frac{1}{2}\boldsymbol{x^2}$, **1 unit to the left** and **3 units up**, as indicated in *Figure 3.11*.

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Aside from the main properties such as vertex, opening and shape, we are often interested in $x$- and $y$-intercepts of the given parabola. The next example illustrates how to find these intercepts from the vertex form of a parabola.

## Example 4 ► Finding the Intercepts from the Vertex Form $\boldsymbol{f(x)=a(x-p)^2+q}$

Find the $x$- and $y$-intercepts of each parabola.

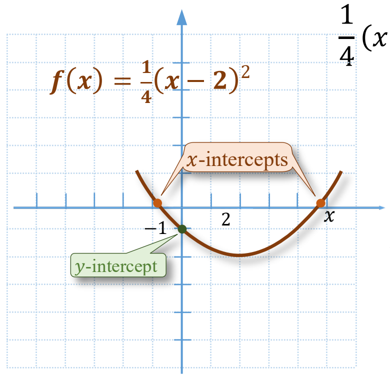
**a.** $f(x) = \frac{1}{4}(x-2)^2 - 2$ **b.** $g(x) = -2(x+1)^2 - 3$

**Solution** ► **a.** To find the $y$-intercept, we evaluate the function at zero. Since

$$f(0) = \frac{1}{4}(-2)^2 - 2 = 1 - 2 = -1,$$

then the $y$-intercept is $(\mathbf{0}, -\mathbf{1})$.

To find $x$-intercepts, we set $f(x) = 0$. So, we need to solve the equation

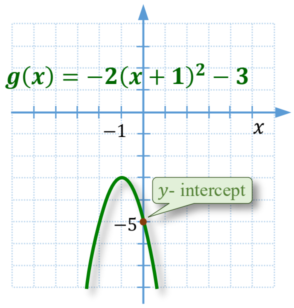


$$\begin{aligned} \frac{1}{4}(x-2)^2 - 2 &= 0 \\ \frac{1}{4}(x-2)^2 &= 2 \\ (x-2)^2 &= 8 \\ \sqrt{(x-2)^2} &= \sqrt{8} \\ |x-2| &= 2\sqrt{2} \\ x - 2 &= \pm 2\sqrt{2} \\ x = 2 \pm 2 &= \begin{cases} 2 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{cases} \end{aligned}$$

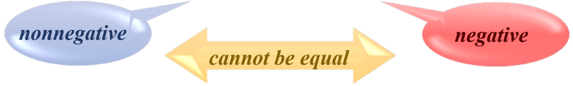
Hence, the two $x$-intercepts are: $\left(\mathbf{2} - \mathbf{2}\sqrt{\mathbf{2}}, \mathbf{0}\right)$ and $\left(\mathbf{2} + \mathbf{2}\sqrt{\mathbf{2}}, \mathbf{0}\right)$.

**b.** Since $g(0) = -2(1)^2 - 3 = -5$, then the $y$-intercept is $(\mathbf{0}, -\mathbf{5})$.

To find $x$-intercepts, we attempt to solve the equation



$$\begin{aligned} -2(x+1)^2 - 3 &= 0 \\ -2(x+1)^2 &= 3 \\ (x+1)^2 &= -\frac{3}{2} \end{aligned}$$



However, since the last equation doesn't have any solution, we conclude that function $g(x)$ has no $x$-intercepts.

## Q.3 Exercises

**1.** Match each quadratic function **a.-d.** with its graph **I-IV**.

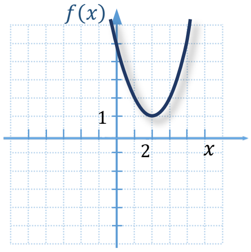
**a.** $f(x) = (x-2)^2 - 1$

**b.** $f(x) = (x-2)^2 + 1$

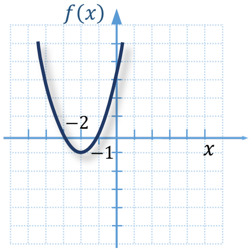
**c.** $f(x) = (x+2)^2 + 1$

**d.** $f(x) = (x+2)^2 - 1$

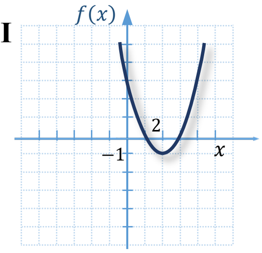
**I**



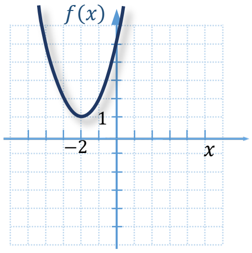
**II**



**III**



**IV**



**2.** Match each quadratic function **a.-d.** with its graph **I-IV**.

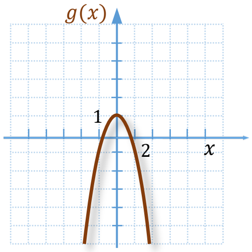
**a.** $g(x) = -(x-2)^2 + 1$

**b.** $g(x) = x^2 - 1$

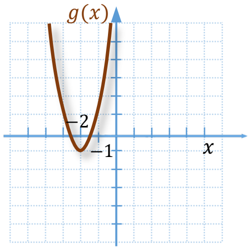
**c.** $g(x) = -2x^2 + 1$

**d.** $g(x) = 2(x+2)^2 - 1$

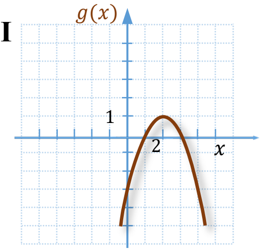
**I**



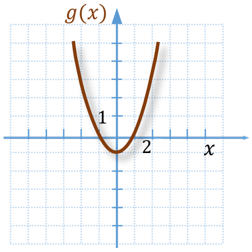
**II**



**III**



**IV**



**3.** Match each quadratic function with the characteristics of its parabolic graph.

| | | | |
|---|---|---|---|
| **a.** | $f(x) = 5(x-3)^2 + 2$ | **I** | vertex $(3,2)$, opens down |
| **b.** | $f(x) = -4(x+2)^2 - 3$ | **II** | vertex $(3,2)$, opens up |
| **c.** | $f(x) = -\frac{1}{2}(x-3)^2 + 2$ | **III** | vertex $(-2,-3)$, opens down |
| **d.** | $f(x) = \frac{1}{4}(x+2)^2 - 3$ | **IV** | vertex $(-2,-3)$, opens up |

*For each quadratic function, describe the* **shape** *(as* **wider**, **narrower**, *or the* **same** *as the shape of* $y = x^2$*) and* **opening (up** *or* **down)** *of its graph. Then* **graph it** *and determine its* **range**.

**4.** $f(x) = 3x^2$

**5.** $f(x) = -\frac{1}{2}x^2$

**6.** $f(x) = -\frac{3}{2}x^2$

**7.** $f(x) = \frac{5}{2}x^2$

**8.** $f(x) = -x^2$

**9.** $f(x) = \frac{1}{3}x^2$

***Graph*** *each parabola by plotting its vertex, and following its shape and opening. Then,* **describe transformations** *of the basic parabola that would lead to the obtained graph. Finally, state the* **domain** *and* **range**, *and the equation of the* ***axis of symmetry***.

**10.** $f(x) = (x-3)^2$

**11.** $f(x) = -x^2 + 2$

**12.** $f(x) = x^2 - 5$

**13.** $f(x) = -(x+2)^2$

**14.** $f(x) = -2x^2 - 1$

**15.** $f(x) = \frac{1}{2}(x+2)^2$

*For each parabola, state its* **vertex**, **shape**, **opening**, *and* **x- and y-intercepts**. *Then,* **graph** *the function and describe* **transformations** *of the basic parabola that would lead to the obtained graph.*

**16.** $f(x) = 3x^2 - 1$

**17.** $f(x) = -\frac{3}{4}x^2 + 3$

**18.** $f(x) = -\frac{1}{2}(x+4)^2 + 2$

**19.** $f(x) = \frac{5}{2}(x-2)^2 - 4$

**20.** $f(x) = 2(x-3)^2 + \frac{3}{2}$

**21.** $f(x) = -3(x+1)^2 + 5$

**22.** $f(x) = -\frac{2}{3}(x+2)^2 + 4$

**23.** $f(x) = \frac{4}{3}(x-3)^2 - 2$

**24.** Four students, **A**, **B**, **C**, and **D**, tried to graph the function $f(x) = -2(x+1)^2 - 3$ by transforming the graph of the basic parabola, $y = x^2$. Here are the transformations that each student applied

*Student* ***A****:*
- shift 1 unit left and 3 units down
- dilation of $y$-values by the factor of $-2$

*Student* ***B****:*
- dilation of $y$-values by the factor of $-2$
- shift 1 unit left
- shift 3 units down

*Student* ***C****:*
- flip over the $x$-axis
- shift 1 unit left and 3 units down
- dilation of $y$-values by the factor of 2

*Student* ***D****:*
- shift 1 unit left
- dilation of $y$-values by the factor of 2
- shift 3 units down
- flip over the $x$-axis

With the assumption that all transformations were properly applied, discuss whose graph was correct and what went wrong with the rest of the graphs. Is there any other sequence of transformations that would result in a correct graph?

*For each parabola, state the coordinates of its* **vertex** *and then* **graph** *it. Finally, state the* **extreme value (maximum** *or* **minimum**, *whichever applies) and the* **range** *of the function.*

**25.** $f(x) = 3(x-1)^2$

**26.** $f(x) = -\frac{5}{2}(x+3)^2$

**27.** $f(x) = (x+2)^2 - 3$

**28.** $f(x) = -3(x+4)^2 + 5$

**29.** $f(x) = -2(x-5)^2 - 2$

**30.** $f(x) = 2(x-4)^2 + 1$

**31.** $f(x) = \frac{1}{2}(x+1)^2 + \frac{3}{2}$

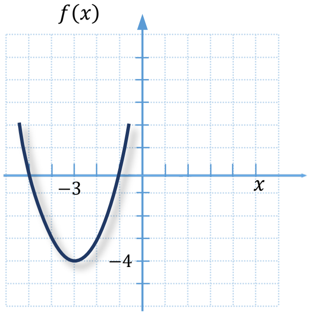
**32.** $f(x) = -\frac{1}{2}(x-1)^2 - 3$

**33.** $f(x) = -\frac{1}{4}(x-3)^2 + 4$

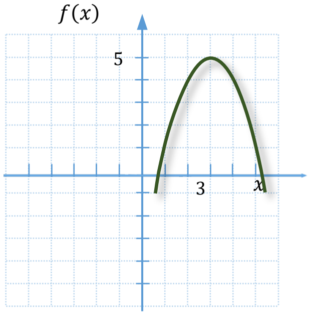
**34.** $f(x) = \frac{3}{4}\left(x+\frac{5}{2}\right)^2 - \frac{3}{2}$

*Given the graph of a parabola, state the most probable* ***equation*** *of the corresponding function. Hint: Use the vertex form of a quadratic function.*

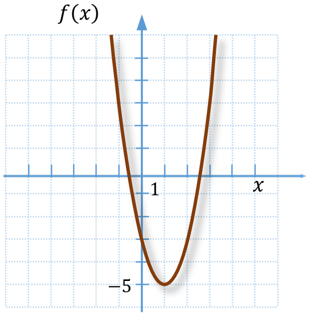
**35.**



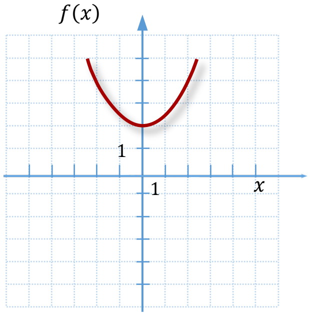
**36.**



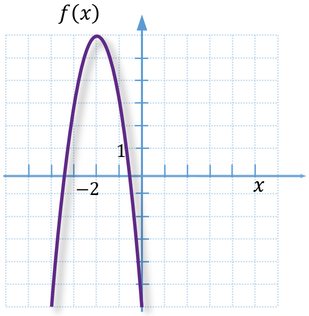
**37.**



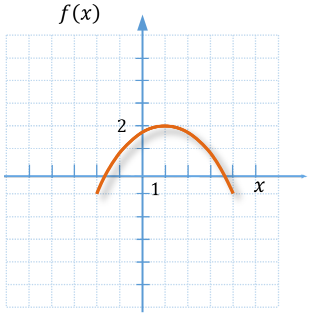
**38.**



**39.**



**40.**



# Q4 Properties of Quadratic Functions and Optimization Problems

In the previous section, we examined how to graph and read the characteristics of the graph of a quadratic function given in vertex form, $f(x) = a(x-p)^2 + q$. In this section, we discuss the ways of **graphing** and reading the **characteristics** of the graph of a quadratic function given in **standard form**, $\boldsymbol{f(x) = ax^2 + bx + c}$. One of these ways is to convert standard form of the function to vertex form by **completing the square** so that the information from the vertex form may be used for graphing. The other handy way of graphing and reading properties of a quadratic function is to **factor** the defining trinomial and use the **symmetry** of a parabolic function.

At the end of this section, we apply properties of quadratic functions to solve certain **optimization problems**. To solve these problems, we look for the **maximum** or **minimum** of a particular quadratic function satisfying specified conditions called **constraints**. Optimization problems often appear in geometry, calculus, business, computer science, etc.

## Graphing Quadratic Functions Given in the Standard Form $f(x) = ax^2 + bx + c$

To graph a quadratic function given in standard form, $f(x) = ax^2 + bx + c$, we can use one of the following methods:

1. constructing a **table of values** (this would always work, but it could be cumbersome);
2. converting to **vertex form** by using the technique of completing the square (see *Examples 1-3*);
3. **factoring** and employing the properties of a parabolic function. (this is a handy method if the function can be easily factored – see *Examples 4 and 5*)

The table of values approach can be used for any function, and it was already discussed on various occasions throughout this textbook.

Converting to **vertex form** involves completing the square. For example, to convert the function $f(x) = 2x^2 + x - 5$ to its vertex form, we might want to start by dividing both sides of the equation by the leading coefficient 2, and then complete the square for the polynomial on the right side of the equation, as below.

$$\frac{f(x)}{2} = x^2 + \frac{1}{2}x - \frac{5}{2}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5 \cdot 8}{2 \cdot 8}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{41}{16}$$

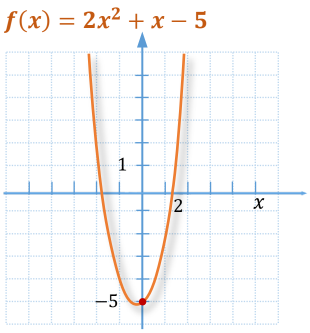


**Figure 4.1**

Finally, the vertex form is obtained by multiplying both sides of the equation back by 2. So, we have

$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{41}{8}$$

This form lets us identify the vertex, $\left(-\frac{1}{4}, -\frac{41}{8}\right)$, and the shape, $y = 2x^2$, of the parabola, which is essential for graphing it. To create an approximate graph of function $f$, we may want to round the vertex to approximately $(-0.25, -5.1)$ and evaluate $f(0) = 2 \cdot 0^2 + 0 - 5 = -5$. So, the graph is as in *Figure 4.1*.

## **Example 1** ► Converting the Standard Form of a Quadratic Function to the Vertex Form

Rewrite each function in its vertex form. Then, identify the vertex.

**a.** $f(x) = -3x^2 + 2x$ **b.** $g(x) = \frac{1}{2}x^2 + x + 3$

**Solution** ►

**a.** To convert $f$ to its vertex form, we follow the completing the square procedure. After dividing the equation by the leading coefficient,

$$f(x) = -3x^2 + 2x,$$

we have

$$\frac{f(x)}{-3} = x^2 - \frac{2}{3}x$$

Then, we complete the square for the right side of the equation,

$$\frac{f(x)}{-3} = \left(x - \frac{1}{3}\right)^2 - \frac{1}{9},$$

and finally, multiply back by the leading coefficient,

$$f(x) = -3\left(x - \frac{1}{3}\right)^2 + \frac{1}{3}.$$

Therefore, the vertex of this parabola is at the point $\boldsymbol{\left(\frac{1}{3}, \frac{1}{3}\right)}$.

**b.** As in the previous example, to convert $g$ to its vertex form, we first wish to get rid of the leading coefficient. This can be achieved by multiplying both sides of the equation $g(x) = \frac{1}{2}x^2 + x + 3$ by 2. So, we obtain

$$2g(x) = x^2 + 2x + 6$$
$$2g(x) = (x + 1)^2 - 1 + 6$$
$$2g(x) = (x + 1)^2 + 5,$$

which can be solved back for $g$,

$$g(x) = \frac{1}{2}(x + 1)^2 + \frac{5}{2}.$$

Therefore, the vertex of this parabola is at the point $\boldsymbol{\left(-1, \frac{5}{2}\right)}$.

Completing the square allows us to derive a formula for the vertex of the graph of any quadratic function given in its standard form, $f(x) = ax^2 + bx + c$, where $a \neq 0$. Applying the same procedure as in *Example 1*, we calculate

$$f(x) = ax^2 + bx + c$$

$$\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}$$

***Recall:*** This is the **discriminant** **Δ!**

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{-(b^2 - 4ac)}{4a}$$

Thus, the coordinates of the vertex $(\boldsymbol{p}, \boldsymbol{q})$ are $\boldsymbol{p} = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$ and $\boldsymbol{q} = \frac{-(\boldsymbol{b^2 - 4ac})}{\boldsymbol{4a}} = \frac{-\Delta}{\boldsymbol{4a}}$ .

***Observation:*** Notice that the expression for $q$ can also be found by evaluating $f$ at $x = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$.

So, the vertex of the parabola can also be expressed as $\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$.

VERTEX FORMULA

Summarizing, the **vertex** of a parabola defined by $f(x) = \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$, where $\boldsymbol{a} \neq 0$, can be calculated by following one of the formulas:

$$\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-(\boldsymbol{b^2 - 4ac})}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-\Delta}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$$

**Example 2** ► **Using the Vertex Formula to Find the Vertex of a Parabola**

Use the vertex formula to find the vertex of the graph of $f(x) = -x^2 - x + 1$.

**Solution** ► The first coordinate of the vertex is equal to $-\frac{b}{2a} = -\frac{-1}{2\cdot(-1)} = -\frac{\boldsymbol{1}}{\boldsymbol{2}}$.

The second coordinate can be calculated by following the formula

$$\frac{-\Delta}{4a} = \frac{-((-1)^2 - 4\cdot(-1)\cdot 1)}{4\cdot(-1)} = \frac{\boldsymbol{5}}{\boldsymbol{4}},$$

or by evaluating $f\left(-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 1 = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{\boldsymbol{5}}{\boldsymbol{4}}$.

So, the vertex is $\left(-\frac{\boldsymbol{1}}{\boldsymbol{2}}, \frac{\boldsymbol{5}}{\boldsymbol{4}}\right)$.

### Example 3 ► Graphing a Quadratic Function Given in Standard Form

Graph each function.

**a.** $g(x) = \frac{1}{2}x^2 + x + 3$ **b.** $f(x) = -x^2 - x + 1$

**Solution** ►

**a.** The shape of the graph of function $g$ is the same as that of $y = \frac{1}{2}x^2$. Since the leading coefficient is positive, the arms of the parabola **open up**.
The **vertex**, $\left(-\mathbf{1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, was found in *Example 1b* as a result of completing the square. Since the vertex is in quadrant II and the graph opens up, we do not expect any $x$-intercepts. However, without much effort, we can find the $y$-intercept by evaluating $g(0) = \mathbf{3}$. Furthermore, since $(0,3)$ belongs to the graph, then by symmetry, $(-2,3)$ must also belong to the graph. So, we graph function $g$ is as in *Figure 4.2*.

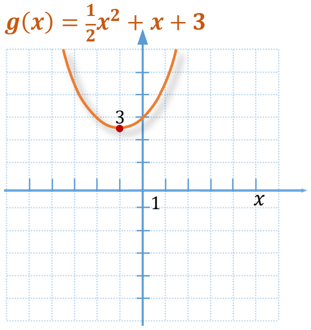


**Figure 4.2**

When plotting points with fractional coordinates, round the values to one place value.

**b.** The graph of function $f$ has the shape of the basic parabola. Since the leading coefficient is negative, the arms of the parabola **open down**.
The **vertex**, $\left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{4}}\right)$, was found in *Example 2* by using the vertex formula. Since the vertex is in quadrant II and the graph opens down, we expect two $x$-intercepts. Their values can be found via the quadratic formula applied to the equation $-x^2 - x + 1 = 0$. So, the $x$-intercepts are $x_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \approx \mathbf{-1.6}$ *or* $\mathbf{0.6}$. In addition, the $y$-intercept of the graph is $f(0) = \mathbf{1}$.
Using all this information, we graph function $f$, as in *Figure 4.3*.

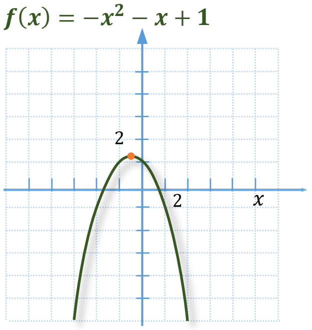


**Figure 4.3**

## Graphing Quadratic Functions Given in the Factored Form $f(x) = a(x - r_1)(x - r_2)$

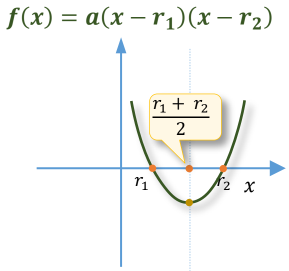


**Figure 4.4**

What if a quadratic function is given in factored form? Do we have to change it to vertex or standard form in order to find the vertex and graph it?

The factored form, $f(x) = a(x - r_1)(x - r_2)$, allows us to find the roots (or $x$-intercepts) of such a function. These are $r_1$ and $r_2$. A parabola is symmetrical about the axis of symmetry, which is the vertical line passing through its vertex. So, the first coordinate of the vertex is the same as the first coordinate of the midpoint of the line segment connecting the roots, $r_1$ with $r_2$, as indicated in *Figure 4.4*. Thus, the first coordinate of the vertex is the average of the two roots, $\frac{r_1 + r_2}{2}$. Then, the second coordinate of the vertex can be found by evaluating $f\left(\frac{r_1 + r_2}{2}\right)$.

### Example 4 Graphing a Quadratic Function Given in a Factored Form

Graph function $g(x) = -(x-2)(x+1)$.

**Solution**

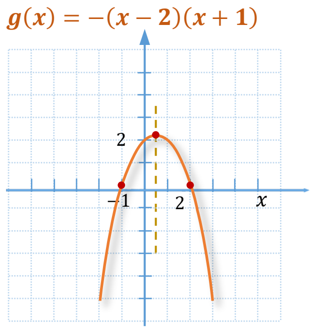


**Figure 4.5**

First, observe that the graph of function $g$ has the same shape as the graph of the basic parabola, $f(x) = x^2$. Since the leading coefficient is negative, the arms of the parabola **open down**. Also, the graph intersects the $x$-axis at 2 and $-1$. So, the first coordinate of the vertex is the average of 2 and $-1$, which is $\frac{1}{2}$. The second coordinate is

$$g\left(\frac{1}{2}\right) = -\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} + 1\right) = -\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$$

Therefore, function $g$ can be graphed by connecting the vertex, $\left(\frac{1}{2}, \frac{9}{4}\right)$, and the $x$-intercepts, $(-1,0)$ and $(2,0)$, with a parabolic curve, as in *Figure 4.5*. For a more precise graph, we may additionally plot the $y$-intercept, $g(0) = 2$, and the symmetrical point $g(1) = 2$.

### Example 5 Using Complete Factorization to Graph a Quadratic Function

Graph function $f(x) = 4x^2 - 2x - 6$.

**Solution**

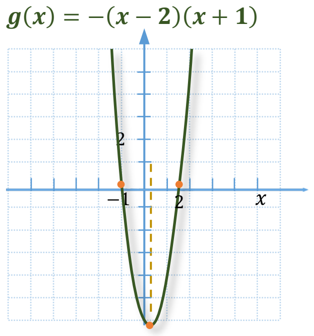


**Figure 4.6**

Since the discriminant $\Delta = (-2)^2 - 4 \cdot 4 \cdot (-6) = 4 + 96 = 100$ is a perfect square number, the defining trinomial is factorable. So, to graph function $f$, we may want to factor it first. Notice that the GCF of all the terms is 2. So, $f(x) = 2(2x^2 - x - 3)$. Then, using factoring techniques discussed in *Section F2*, we obtain $f(x) = 2(2x-3)(x+1)$. This form allows us to identify the roots (or zeros) of function $f$, which are $\frac{3}{2}$ and $-1$. So, the first coordinate of the vertex is the average of $\frac{3}{2} = 1.5$ and $-1$, which is $\frac{1.5+(-1)}{2} = \frac{0.5}{2} = 0.25$. The second coordinate can be calculated by evaluating

$$f(0.25) = 2(2 \cdot 0.25 - 3)(0.25 + 1) = 2(0.5 - 3)(1.25) = 2(-2.5)(1.25) = -6.25$$

So, we can graph function $f$ by connecting its vertex, $(\mathbf{0.25}, \mathbf{-6.25})$, and its $x$-intercepts, $(-1,0)$ and $(1.5,0)$, with a parabolic curve, as in *Figure 4.6*. For a more precise graph, we may additionally plot the $y$-intercept, $f(0) = -6$, and by symmetry, $f(0.5) = -6$.

***Observation:*** Since $x$-intercepts of a parabola are the solutions (zeros) of its equation, the equation of a parabola with $x$-intercepts at $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ can be written as

$$\boldsymbol{y = a(x - r_1)(x - r_2)},$$

for some real coefficient $\boldsymbol{a} \neq 0$.

### Example 6 ► Finding an Equation of a Quadratic Function Given Its Solutions

**a.** Find an equation of a quadratic function whose graph passes the $x$-axis at $-1$ and 3.

**b.** Find an equation of a quadratic function whose graph passes the $x$-axis at $-1$ and 3 and the $y$-axis at $-4$.

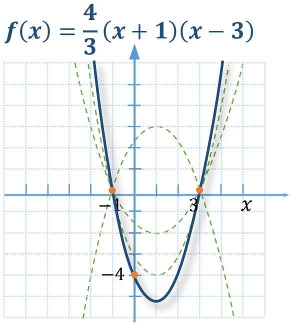
**c.** Write a quadratic equation with integral coefficients knowing that the solutions of this equation are $\frac{1}{2}$ and $-\frac{2}{3}$.

**Solution**

► **a.** $x$-intercepts of a function are the zeros of this function. So, $-1$ and 3 are the zeros of the quadratic function. This means that the defining formula for such function should include factors $\big(x-(-1)\big)$ and $(x-3)$. So, it could be

$$f(x) = (x+1)(x-3).$$

Notice that this is indeed a quadratic function with $x$-intercepts at $-1$ and 3. Hence, it satisfies the conditions of the problem.



**b.** Using the solution to *Example 6a*, notice that any function of the form

$$f(x) = a(x+1)(x-3),$$

where $a$ is a nonzero real number, is a quadratic function with $x$-intercepts at $-1$ and 3. To guarantee that the graph of our function passes through the point $(0,-4)$, we need to find the particular value of the coefficient $a$. This can be done by substituting $x = 0$ and $f(x) = -4$ into the function's equation and solving it for $a$. Thus,

$$-4 = a(0+1)(0-3)$$

$$-4 = -3a$$

$$a = \frac{4}{3},$$

and the desired function is $f(x) = \frac{4}{3}(x+1)(x-3)$.

**c.** First, observe that $\frac{1}{2}$ is a solution to the linear equation $2x - 1 = 0$. Similarly, $-\frac{2}{3}$ is a solution to the equation $3x + 2 = 0$. Multiplying these two equations side by side, we obtain a quadratic equation
$(2x-1)(3x+2) = 0$

that satisfies the conditions of the problem.

***Note:*** Here, we could create the desired equation by writing

$$\left(x-\frac{1}{2}\right)\left(x-\left(-\frac{2}{3}\right)\right) = 0$$

and then multiplying it by the $LCD = \mathbf{6} = \mathbf{2}\cdot\mathbf{3}$

$$\mathbf{2}\left(x-\frac{1}{2}\right)\mathbf{3}\left(x+\frac{2}{3}\right) = \mathbf{6}\cdot 0$$

$$(2x-1)(3x+2) = 0$$

## Optimization Problems

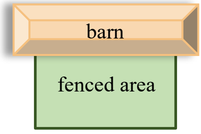
In many applied problems we are interested in **maximizing** or **minimizing** some quantity under specific conditions, called **constraints**. For example, we might be interested in finding the greatest area that can be fenced in by a given length of fence, or minimizing the cost of producing a container of a given shape and volume. These types of problems are called **optimization problems**.

Since the vertex of the graph of a quadratic function is either the highest or the lowest point of the parabola, it can be used in solving optimization problems that can be modeled by a quadratic function.

The vertex of a parabola provides the following information.

- The $y$-value of the vertex gives the maximum or minimum value of $y$.
- The $x$-value tells where the maximum or minimum occurs.

### Example 7 ► Maximizing Area of a Rectangular Region



John has 60 meters of fencing to enclose a rectangular field by his barn. Assuming that the barn forms one side of the rectangle, find the maximum area he can enclose and the dimensions of the enclosed field that yield this area.

**Solution** ►

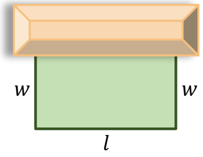


**Figure 4.7**

Let $l$ and $w$ represent the length and width of the enclosed area correspondingly, as indicated in *Figure 4.7*. The 60 meters of fencing is used to cover the distance of twice along the width and once along the length. So, we can form the constraint equation

$$2w + l = 60 \qquad (1)$$

To analyse the area of the field,

$$A = lw, \qquad (2)$$

we would like to express it as a function of one variable, for example $w$. To do this, we can solve the constraint equation (1) for $l$ and substitute the obtained expression into the equation of area, (2). Since $l = 60 - 2w$, then

$$A = lw = (60 - 2w)w$$

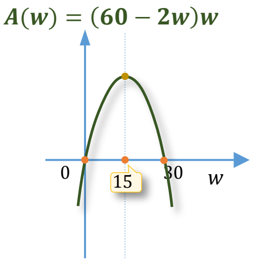


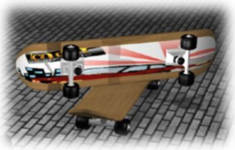
**Figure 4.8**

Observe that the graph of the function $A(w) = (60 - 2w)w$ is a parabola that opens down and intersects the $x$-axis at 0 and 30. This is because the leading coefficient of $(60 - 2w)w$ is negative and the roots to the equation $(60 - 2w)w = 0$ are 0 and 30. These roots are symmetrical in the axis of symmetry, which also passes through the vertex of the parabola, as illustrated in *Figure 4.8*. So, the first coordinate of the vertex is the average of the two roots, which is $\frac{0+30}{2} = 15$. Thus, the width that would maximize the enclosed area is $w_{max} = $ **15 meters**. Consequently, the length that would maximize the enclosed area is $l_{max} = 60 - 2w_{max} = 60 - 2 \cdot 15 =$ **30 meters**. The maximum area represented by the second coordinate of the vertex can be obtained by evaluating the function of area at the width of 15 meters.

$$A(15) = (60 - 2 \cdot 15)15 = 30 \cdot 15 = 450 \text{ m}^2$$

So, the maximum area that can be enclosed by 60 meters of fencing is **450 square meters**, and the dimensions of this rectangular area are **15 by 30 meters**.

**Example 8** ► **Minimizing Average Unit Cost**



A company producing skateboards has determined that when $x$ hundred skateboards are produced, the average cost of producing one skateboard can be modelled by the function

$$C(x) = 0.15x^2 - 0.75x + 1.5,$$

where $C(x)$ is in hundreds of dollars. How many skateboards should be produced to minimize the average cost of producing one skateboard? What would this cost be?

**Solution** ► Since $C(x)$ is a quadratic function, to find its minimum, it is enough to find the vertex of the parabola $C(x) = 0.15x^2 - 0.75x + 1.5$. This can be done either by completing the square or by using the formula for the vertex, $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$. We will do the latter. So, the vertex is

$$\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = \left(\frac{0.75}{0.3}, \frac{-(0.75^2 - 4 \cdot 0.15 \cdot 1.5)}{0.6}\right) = \left(2.5, \frac{-(0.5625 - 1.35)}{0.6}\right)$$
$$= \left(2.5, \frac{0.3375}{0.6}\right) = (\mathbf{2.5, 0.5625}).$$

This means that the lowest average unit cost can be achieved when 250 skateboards are produced, and that the lowest average cost of producing a skateboard would be $56.25.

## Q.4 Exercises

*Convert each quadratic function to its **vertex form**. Then, state the coordinates of the **vertex**.*

**1.** $f(x) = x^2 + 6x + 10$

**2.** $f(x) = x^2 - 4x - 5$

**3.** $f(x) = x^2 + x - 3$

**4.** $f(x) = x^2 - x + 7$

**5.** $f(x) = -x^2 + 7x + 3$

**6.** $f(x) = 2x^2 - 4x + 1$

**7.** $f(x) = -3x^2 + 6x + 12$

**8.** $f(x) = -2x^2 - 8x + 10$

**9.** $f(x) = \frac{1}{2}x^2 + 3x - 1$

*Use the vertex formula, $\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$, to find the coordinates of the **vertex** of each parabola.*

**10.** $f(x) = x^2 + 6x + 3$

**11.** $f(x) = -x^2 + 3x - 5$

**12.** $f(x) = \frac{1}{2}x^2 - 4x - 7$

**13.** $f(x) = -3x^2 + 6x + 5$

**14.** $f(x) = 5x^2 - 7x$

**15.** $f(x) = 3x^2 + 6x - 20$

*For each parabola, state its **vertex**, **opening** and **shape**. Then **graph** it and state the **domain** and **range**.*

**16.** $f(x) = x^2 - 5x$

**17.** $f(x) = x^2 + 3x$

**18.** $f(x) = x^2 - 2x - 5$

**19.** $f(x) = -x^2 + 6x - 3$

**20.** $f(x) = -x^2 - 3x + 2$

**21.** $f(x) = 2x^2 + 12x + 18$

**22.** $f(x) = -2x^2 + 3x - 1$ **23.** $f(x) = -2x^2 + 4x + 1$ **24.** $f(x) = 3x^2 + 4x + 2$

*For each quadratic function, state its* ***zeros*** *(roots), coordinates of the* ***vertex****,* ***opening*** *and* ***shape****. Then* ***graph*** *it and identify its* ***extreme*** *(minimum or maximum)* ***value*** *as well as where it occurs.*

**25.** $f(x) = (x - 2)(x + 2)$ **26.** $f(x) = -(x + 3)(x - 1)$ **27.** $f(x) = x^2 - 4x$

**28.** $f(x) = x^2 + 5x$ **29.** $f(x) = x^2 - 8x + 16$ **30.** $f(x) = -x^2 - 4x - 4$

**31.** $f(x) = -3(x^2 - 1)$ **32.** $f(x) = \frac{1}{2}(x + 3)(x - 4)$ **33.** $f(x) = -\frac{3}{2}(x - 1)(x - 5)$

*Find an equation of a quadratic function satisfying the given conditions.*

**34.** passes the $x$-axis at $-2$ and 5 **35.** has $x$-intercepts at 0 and $\frac{2}{5}$

**36.** passes the $x$-axis at $-3$ and $-1$ and $y$-axis at 2 **37.** $f(1) = 0, f(4) = 0, f(0) = 3$

*Write a quadratic equation with the indicated solutions using only integral coefficients.*

**38.** $-5$ and 6 **39.** 0 and $\frac{1}{3}$ **40.** $-\frac{2}{5}$ and $\frac{3}{4}$ **41.** 2

**42.** Suppose the $x$-intercepts of the graph of a parabola are $(x_1, 0)$ and $(x_2, 0)$. What is the equation of the axis of symmetry of this graph?

**43.** How can we determine the number of $x$-intercepts of the graph of a quadratic function without graphing the function?

*True or false? Explain.*

**44.** The domain and range of a quadratic function are both the set of real numbers.

**45.** The graph of every quadratic function has exactly one $y$-intercept.

**46.** The graph of $y = -2(x - 1)^2 - 5$ has no $x$-intercepts.

**47.** The maximum value of $y$ in the function $y = -4(x - 1)^2 + 9$ is 9.

**48.** The value of the function $f(x) = x^2 - 2x + 1$ is at its minimum when $x = 0$.

**49.** The graph of $f(x) = 9x^2 + 12x + 4$ has one $x$-intercept and one $y$-intercept.

**50.** If a parabola opens down, it has two $x$-intercepts.

*Solve each problem.*

**51.** A ball is projected from the ground straight up with an initial velocity of 24.5 m/sec. The function $h(t) = -4.9t^2 + 24.5t$ allows for calculating the height $h(t)$, in meters, of the ball above the ground after $t$ seconds.

What is the maximum height reached by the ball? In how many seconds should we expect the ball to come back to the ground?

**52.** A firecracker is fired straight up and explodes at its maximum height above the ground. The function $h(t) = -4.9t^2 + 98t$ allows for calculating the height $h(t)$, in meters, of the firecracker above the ground $t$ seconds after it was fired. In how many seconds after firing should we expect the firecracker to explode and at what height?

**53.** Antonio prepares and sells his favourite desserts at a market stand. Suppose his daily cost, C, in dollars, to sell $n$ desserts can be modelled by the function $C(n) = 0.5n^2 - 30n + 350$. How many of these desserts should he sell to minimize the cost and what is the minimum cost?

**54.** Chris has a hot-dog stand. His daily cost, C, in dollars, to sell n hot-dogs can be modelled by the function $C(n) = 0.1n^2 - 15n + 700$. How many hotdogs should he sell to minimize the cost and what is the minimum cost?

**55.** Find two positive numbers with a sum of 32 that would produce the maximum product.

**56.** Find two numbers with a difference of 32 that would produce the minimum product.

**57.** Luke uses 16 meters of fencing to enclose a rectangular area for his baby goats. The enclosure shares one side with a large barn, so only 3 sides need to be fenced. If Luke wishes to enclose the greatest area, what should the dimensions of the enclosure be?

**58.** Ryan uses 60 meters of fencing to enclose a rectangular area for his livestock. He plans to subdivide the area by placing additional fence down the middle of the rectangle to separate different types of livestock. What dimensions of the overall rectangle will maximize the total area of the enclosure?

**59.** Julia works as a tour guide. She charges \$58 for an individual tour. When more people come for a tour, she charges \$2 less per person for each additional person, up to 25 people.

**a.** Express the price per person $P$ as a function of the number of people $n$, for $n \in \{1,2, \dots ,25\}$.

**b.** Express her revenue, $R$, as a function of the number of people on tour.

**c.** How many people on tour would maximize Julia's revenue?

**d.** What is the highest revenue she can achieve?

**60.** One-day adult passes for The Mission Folk Festival cost \$50. At this price, the organizers of the festival expect about 1300 people to purchase the pass. Suppose that the organizers observe that every time they increase the cost per pass by 5\$, the number of passes sold decrease by about 100.

**a.** Express the number of passes sold, $N$, as a function of the cost, $c$, of a one-day pass.

**b.** Express the revenue, $R$, as a function of the cost, $c$, of a one-day pass.

**c.** How much should a one-day pass costs to maximize the revenue?

**d.** What is the maximum revenue?

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