Polynomials and Polynomial Functions

One of the simplest types of algebraic expressions is a polynomial. Polynomials are formed only by addition and multiplication of variables and constants. Since both addition and multiplication produce unique values for any given inputs, polynomials are in fact functions. Some of the simplest polynomial functions are linear functions, such as $P(x) = 2x + 1$, and quadratic functions, such as $Q(x) = x^2 + x - 6$. Due to their comparably simple form, polynomial functions appear in a wide variety of areas of mathematics and science, economics, business, and many other areas of life. Polynomial functions are often used to model various natural phenomena, such as the shape of a mountain, the distribution of temperature, the trajectory of projectiles, etc. The shape and properties of polynomial functions are helpful when constructing such structures as roller coasters or bridges, solving optimization problems, or even analysing stock market prices.

In this chapter, we will introduce polynomial terminology, perform operations on polynomials, and evaluate and compose polynomial functions.

P1 Addition and Subtraction of Polynomials

Terminology of Polynomials

Recall that products of constants, variables, or expressions are called **terms** (see *Section R3, Definition 3.1*). **Terms** that are **products** of only **numbers** and **variables** are called **monomials**. Examples of monomials are $-2x$, xy^2, $\frac{2}{3}x^3$, etc.

Definition 1.1 ► A **polynomial** is a sum of monomials.

A **polynomial** in a single variable is the sum of terms of the form ax^n, where a is a **numerical coefficient**, x is the variable, and n is a whole number.

An ***n*-th degree polynomial** in x-variable has the form

$$\boldsymbol{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0},$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$.

Note: A polynomial can always be considered as a sum of monomial terms even though there are negative signs when writing it.
For example, polynomial $x^2 - 3x - 1$ can be seen as the sum of signed terms

$$x^2 + -3x + -1$$

Definition 1.2 ► The **degree of a monomial** is the sum of exponents of all its variables.

For example, the degree of $5x^3y$ is 4, as the sum of the exponent of x^3, which is 3 and the exponent of y, which is 1. To record this fact, we write $\deg(5x^3y) = 4$.

The **degree of a polynomial** is the highest degree out of all its terms.

For example, the degree of $2x^2y^3 + 3x^4 - 5x^3y + 7$ is 5 because $\deg(2x^2y^3) = 5$ and the degrees of the remaining terms are not greater than 5.

Polynomials that are sums of two terms, such as $x^2 - 1$, are called **binomials**.
Polynomials that are sums of three terms, such as $x^2 + 5x - 6$ are called **trinomials**.

The **leading term** of a polynomial is the highest degree term.
The **leading coefficient** is the numerical coefficient of the leading term.
So, the leading term of the polynomial $1 - x - x^2$ is $-x^2$, even though it is not the first term. The leading coefficient of the above polynomial is -1, as $-x^2$ can be seen as $(-1)x^2$.

A first degree term is often referred to as a **linear term**. A second degree term can be referred to as a **quadratic term**. A zero degree term is often called a **constant** or a **free term**.

Below are the parts of an n-th degree polynomial in a single variable x:

leading coefficient

$$\underbrace{a_n x^n}_{\substack{\text{leading}\\ \text{term}}} + a_{n-1}x^{n-1} + \cdots + \underbrace{a_2 x^2}_{\substack{\text{quadratic}\\ \text{term}}} + \underbrace{a_1 x}_{\substack{\text{linear}\\ \text{term}}} + \underbrace{a_0}_{\substack{\text{constant}\\ (\text{free})\\ \text{term}}}$$

Note: Single variable polynomials are usually arranged in descending powers of the variable. Polynomials in more than one variable are arranged in decreasing degrees of terms. If two terms are of the same degree, they are arranged with respect to the descending powers of the variable that appers first in alphabetical order.

For example, polynomial $x^2 + x - 3x^4 - 1$ is customarily arranged as follows

$$-3x^4 + x^2 + x - 1,$$

while polynomial $3x^3y^2 + 2y^6 - y^2 + 4 - x^2y^3 + 2xy$ is usually arranged as below.

$$\underbrace{2y^6}_{\substack{\text{6th}\\ \text{degree}\\ \text{term}}} \underbrace{+3x^3y^2 - x^2y^3}_{\substack{\text{5th degree terms}\\ \text{arranged}\\ \text{with respect to } x}} \underbrace{+2xy - y^2}_{\substack{\text{2nd degree}\\ \text{terms arranged}\\ \text{with respect to } x}} \underbrace{+4}_{\substack{\text{zero}\\ \text{degree}\\ \text{term}}}$$

Example 1 ► **Writing Polynomials in Descending Order and Identifying Parts of a Polynomial**

Suppose $P = x - 6x^3 - x^6 + 4x^4 + 2$ and $Q = 2y - 3xyz - 5x^2 + xy^2$. For each polynomial:

a. Write the polynomial in descending order.
b. State the degree of the polynomial and the number of its terms.
c. Identify the leading term, the leading coefficient, the coefficient of the linear term, the coefficient of the quadratic term, and the free term of the polynomial.

Solution ► **a.** After arranging the terms in descending powers of x, polynomial P becomes

$$\boldsymbol{-x^6 + 4x^4 - 6x^3 + x + 2},$$

while polynomial Q becomes

$$\boldsymbol{xy^2 - 3xyz - 5x^2 + 2y}.$$

Notice that the first two terms, xy^2 and $-3xyz$, are both of the same degree. So, to decide which one should be written first, we look at powers of x. Since these powers are again the same, we look at powers of y. This time, the power of y in xy^2 is higher than the power of y in $-3xyz$. So, the term xy^2 should be written first.

b. The polynomial P has **5 terms**. The highest power of x in P is 6, so the **degree** of the polynomial P is **6**.
The polynomial Q has **4 terms**. The highest degree terms in Q are xy^2 and $-3xyz$, both third degree. So the **degree** of the polynomial Q is **3**.

$x = 1 \cdot x$
$-x = (-1)x$

c. The leading term of the polynomial $P = -x^6 + 4x^4 - 6x^3 + x - 2$ is $-x^6$, so the **leading coefficient** equals $\mathbf{-1}$.
The linear term of P is x, so the **coefficient of the linear term** equals **1**.
P doesn't have any quadratic term so the coefficient of the quadratic term equals **0**.
The **free term** of P equals $\mathbf{-2}$.

The leading term of the polynomial $Q = xy^2 - 3xyz - 5x^2 + 2y$ is xy^2, so the **leading coefficient** is equal to **1**.
The linear term of Q is $2y$, so the **coefficient of the linear term** equals **2**.
The quadratic term of Q is $-5x^2$, so the **coefficient of the quadratic term** equals $\mathbf{-5}$.
The polynomial Q does not have a free term, so the **free term** equals **0**.

Example 2 ► **Classifying Polynomials**

Describe each polynomial as a *constant*, *linear*, *quadratic*, or *n-th degree* polynomial. Decide whether it is a *monomial*, *binomial*, or *trinomial*, if applicable.

a. $x^2 - 9$ **b.** $-3x^7y$
c. $x^2 + 2x - 15$ **d.** π
e. $4x^5 - x^3 + x - 7$ **f.** $x^4 + 1$

Solution ►

a. $x^2 - 9$ is a second degree polynomial with two terms, so it is a **quadratic binomial**.

b. $-3x^7y$ is an **8-th degree monomial**.

c. $x^2 + 2x - 15$ is a second degree polynomial with three terms, so it is a **quadratic trinomial**.

d. π is a 0-degree term, so it is a **constant monomial**.

e. $4x^5 - x^3 + x - 7$ is a **5-th degree polynomial**.

f. $x^4 + 1$ is a **4-th degree binomial**.

Polynomials as Functions and Evaluation of Polynomials

Each term of a polynomial in one variable is a product of a number and a power of the variable. The polynomial itself is either one term or a sum of several terms. Since taking a power of a given value, multiplying, and adding given values produce unique answers, polynomials are also functions. While f, g, or h are the most commonly used letters to represent functions, other letters can also be used. To represent polynomial functions, we customarily use capital letters, such as P, Q, R, etc.

Any polynomial function P of degree n, has the form

$$\boldsymbol{P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$, and $n \in \mathbb{W}$.

Since polynomials are functions, they can be evaluated for different x-values.

Example 3 ► **Evaluating Polynomials**

Given $P(x) = 3x^3 - x^2 + 4$, evaluate the following expressions:

a. $P(0)$ **b.** $P(-1)$
c. $2 \cdot P(1)$ **d.** $P(a)$

Solution ►

a. $P(0) = 3 \cdot 0^3 - 0^2 + 4 = \mathbf{4}$

b. $P(-1) = 3 \cdot (-1)^3 - (-1)^2 + 4 = 3 \cdot (-1) - 1 + 4 = -3 - 1 + 4 = \mathbf{0}$

When evaluating at negative x-values, it is essential to use brackets in place of the variable before substituting the desired value.

c. $2 \cdot P(1) = 2 \cdot \underbrace{(3 \cdot 1^3 - 1^2 + 4)}_{this\ is\ P(1)} = 2 \cdot (3 - 1 + 4) = 2 \cdot 6 = \mathbf{12}$

d. To find the value of $P(a)$, we replace the variable x in $P(x)$ with a. So, this time the final answer,

$$P(a) = \mathbf{3a^3 - a^2 + 4},$$

is an expression in terms of a rather than a specific number.

Since polynomials can be evaluated at any real x-value, then the **domain** (*see Section G3, Definition 5.1*) of any polynomial is the set $\mathbb{R}$ of all real numbers.

Addition and Subtraction of Polynomials

Recall that terms with the same variable part are referred to as **like terms** (see *Section R3, Definition 3.1*). Like terms can be **combined** by adding their coefficients. For example,

$$\underbrace{2x^2y - 5x^2y = (2-5)x^2y}_{\substack{by\ distributive\ property \\ (factoring)}} = -3x^2y$$

In practice, this step is not necessary to write.

Unlike terms, such as $2x^2$ and $3x$, **cannot be combined**.

Example 4 ► Simplifying Polynomial Expressions

Simplify each polynomial expression.

a. $5x - 4x^2 + 2x + 7x^2$ **b.** $8p - (2 - 3p) + (3p - 6)$

Solution ►

a. To simplify $5x - 4x^2 + 2x + 7x^2$, we combine like terms, starting from the highest degree terms. It is suggested to underline the groups of like terms, using different type of underlining for each group, so that it is easier to see all the like terms and not to miss any of them. So,

$$5x - 4x^2 + 2x + 7x^2 = \mathbf{3x^2 + 7x}$$

Remember that the sign in front of a term belongs to this term.

b. To simplify $8p - (2 - 3p) + (3p - 6)$, first we remove the brackets using the distributive property of multiplication and then we combine like terms. So, we have

$-(2 - 3p)$
$= (-1)(2 - 3p)$

$$8p - (2 - 3p) + (3p - 6)$$
$$= 8p - 2 + 3p + 3p - 6$$
$$= \mathbf{14p - 8}$$

Example 5 ► Adding or Subtracting Polynomials

Perform the indicated operations.

a. $(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$
b. $(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$
c. $[9p - (3p - 2)] - [4p - (3 - 7p) + p]$

Solution ►

a. To add polynomials, combine their like terms. So,

remove any bracket preceeded by a "+" sign

$$(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$$
$$= 6a^5 - 4a^3 + 3a - 1 + 2a^4 + a^2 - 5a + 9$$
$$= \mathbf{6a^5 + 2a^4 - 4a^3 + a^2 - 2a + 8}$$

b. To subtract a polynomial, add its opposite. In practice, remove any bracket preceeded by a negative sign by reversing the signs of all the terms of the polynomial inside the bracket. So,

$$(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$$
$$= 4y^3 - 3y^2 + y + 6 - y^3 - 3y + 2$$
$$= \mathbf{3y^3 - 3y^2 - 2y + 8}$$

To remove a bracket preceeded by a "–" sign, reverse each sign inside the bracket.

c. First, perform the operations within the square brackets and then subtract the resulting polynomials. So,

$$[9p - (3p - 2)] - [4p - (3 - 7p) + p]$$
$$= [9p - 3p + 2] - [4p - 3 + 7p + p]$$
$$= [6p + 2] - [12p - 3]$$
$$= 6p + 2 - 12p + 3$$
$$= \mathbf{-6p + 5}$$

collect like terms before removing the next set of brackets

Addition and Subtraction of Polynomial Functions

Similarly as for polynomials, addition and subtraction can also be defined for general functions.

Definition 1.3 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **sum function** $\boldsymbol{f + g}$ is defined as

$$(\boldsymbol{f} + \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})$$

and the **difference function** $\boldsymbol{f - g}$ is defined as

$$(\boldsymbol{f} - \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the sum or difference function is the intersection $\boldsymbol{D_f \cap D_g}$ of the domains of the two functions.

A frequently used application of a sum or difference of polynomial functions comes from the business area. The fact that profit P equals revenue R minus cost C can be recorded using function notation as

$$P(x) = (R - C)(x) = R(x) - C(x),$$

where x is the number of items produced and sold. Then, if $R(x) = 6.5x$ and $C(x) = 3.5x + 900$, the profit function becomes

$$\boldsymbol{P}(\boldsymbol{x}) = R(x) - C(x) = 6.5x - (3.5x + 900) = 6.5x - 3.5x - 900 = \mathbf{3x - 900}.$$

Example 6 ► **Adding or Subtracting Polynomial Functions**

Suppose $P(x) = x^2 - 6x + 4$ and $Q(x) = 2x^2 - 1$. Find the following:

a. $(P + Q)(x)$ and $(P + Q)(2)$
b. $(P - Q)(x)$ and $(P - Q)(-1)$
c. $(P + Q)(k)$
d. $(P - Q)(2a)$

Solution ► **a.** Using the definition of the sum of functions, we have

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{x}) = P(x) + Q(x) = \underbrace{x^2 - 6x + 4}_{P(x)} + \underbrace{2x^2 - 1}_{Q(x)} = \boldsymbol{3x^2 - 6x + 3}$$

Therefore, $(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{2}) = 3 \cdot 2^2 - 6 \cdot 2 + 3 = 12 - 12 + 3 = \boldsymbol{3}$.

Alternatively, $(P+Q)(2)$ can be calculated without refering to the function $(P+Q)(x)$, as shown below.

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{2}) = P(2) + Q(2) = \underbrace{2^2 - 6 \cdot 2 + 4}_{P(2)} + \underbrace{2 \cdot 2^2 - 1}_{Q(2)}$$

$$= 4 - 12 + 4 + 8 - 1 = \boldsymbol{3}.$$

b. Using the definition of the difference of functions, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{x}) = P(x) - Q(x) = \underbrace{x^2 - 6x + 4}_{P(x)} - \underbrace{(2x^2 - 1)}_{Q(x)}$$

$$= x^2 - 6x + 4 - 2x^2 + 1 = \boldsymbol{-x^2 - 6x + 5}$$

To evaluate $(P-Q)(-1)$, we will take advantage of the difference function calculated above. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{-1}) = -(-1)^2 - 6(-1) + 5 = -1 + 6 + 5 = \boldsymbol{10}.$$

c. By *Definition 1.3*,

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = P(k) + Q(k) = k^2 - 6k + 4 + 2k^2 - 1 = \boldsymbol{3k^2 - 6k + 3}$$

Alternatively, we could use the sum function already calculated in the solution to *Example 6a.* Then, the result is instant: $(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = \boldsymbol{3k^2 - 6k + 3}$.

d. To find $(P-Q)(2a)$, we will use the difference function calculated in the solution to *Example 6b*. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{2a}) = -(2a)^2 - 6(2a) + 5 = \boldsymbol{-4a^2 - 12a + 5}.$$

P.1 Exercises

Determine whether the expression is a monomial.

1. $-\pi x^3 y^2$ **2.** $5x^{-4}$ **3.** $5\sqrt{x}$ **4.** $\sqrt{2}x^4$

Identify the degree and coefficient.

5. xy^3 **6.** $-x^2 y$ **7.** $\sqrt{2}xy$ **8.** $-3\pi x^2 y^5$

Arrange each polynomial in descending order of powers of the variable. Then, identify the degree and the leading coefficient of the polynomial.

9. $5 - x + 3x^2 - \frac{2}{5}x^3$

10. $7x + 4x^4 - \frac{4}{3}x^3$

11. $8x^4 + 2x^3 - 3x + x^5$

12. $4y^3 - 8y^5 + y^7$

13. $q^2 + 3q^4 - 2q + 1$

14. $3m^2 - m^4 + 2m^3$

State the degree of each polynomial and identify it as a monomial, binomial, trinomial, or n-term polynomial.

15. $7n - 5$

16. $4z^2 - 11z + 2$

17. 25

18. $-6p^4q + 3p^3q^2 - 2pq^3 - p^4$

19. $-mn^6$

20. $16k^2 - 9p^2$

Let $P(x) = -2x^2 + x - 5$ and $Q(x) = 2x - 3$. Evaluate each expression.

21. $P(-1)$

22. $P(0)$

23. $2P(1)$

24. $P(a)$

25. $Q(-1)$

26. $Q(5)$

27. $Q(a)$

28. $Q(3a)$

29. $3Q(-2)$

30. $3P(a)$

31. $3Q(a)$

32. $Q(a+1)$

Simplify each polynomial expression.

33. $5x + 4y - 6x + 9y$

34. $4x^2 + 2x - 6x^2 - 6$

35. $6xy + 4x - 2xy - x$

36. $3x^2y + 5xy^2 - 3x^2y - xy^2$

37. $9p^3 + p^2 - 3p^3 + p - 4p^2 + 2$

38. $n^4 - 2n^3 + n^2 - 3n^4 + n^3$

39. $4 - (2 + 3m) + 6m + 9$

40. $2a - (5\mathrm{a} - 3) - (7a - 2)$

41. $6 + 3x - (2x + 1) - (2x + 9)$

42. $4y - 8 - (-3 + y) - (11y + 5)$

Perform the indicated operations.

43. $(x^2 - 5y^2 - 9z^2) + (-6x^2 + 9y^2 - 2z^2)$

44. $(7x^2y - 3xy^2 + 4xy) + (-2x^2y - xy^2 + xy)$

45. $(-3x^2 + 2x - 9) - (x^2 + 5x - 4)$

46. $(8y^2 - 4y^3 - 3y) - (3y^2 - 9y - 7y^3)$

47. $(3r^6 + 5) + (-7r^2 + 2r^6 - r^5)$

48. $(5x^{2a} - 3x^a + 2) + (-x^{2a} + 2x^a - 6)$

49. $(-5a^4 + 8a^2 - 9) - (6a^3 - a^2 + 2)$

50. $(3x^{3a} - x^a + 7) - (-2x^{3a} + 5x^{2a} - 1)$

51. $(10xy - 4x^2y^2 - 3y^3) - (-9x^2y^2 + 4y^3 - 7xy)$

52. Subtract $(-4x + 2z^2 + 3m)$ from the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$.

53. Subtract the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$ from $(-4x + 2z^2 + 3m)$.

54. $[2p - (3p - 6)] - \left[\left(5p - (8 - 9p)\right) + 4p\right]$

55. $-[3z^2 + 5z - (2z^2 - 6z)] + \left[\left(8z^2 - (5z - z^2)\right) + 2z^2\right]$

56. $5k - (5k - [2k - (4k - 8k)]) + 11k - (9k - 12k)$

For each pair of functions, find ***a)*** $(f+g)(x)$ *and* ***b)*** $(f-g)(x)$.

57. $f(x)=5x-6,\ \ g(x)=-2+3x$

58. $f(x)=x^2+7x-2,\ \ g(x)=6x+5$

59. $f(x)=3x^2-5x,\ \ g(x)=-5x^2+2x+1$

60. $f(x)=2x^n-3x-1,\ \ g(x)=5x^n+x-6$

61. $f(x)=2x^{2n}-3x^n+3,\ \ g(x)=-8x^{2n}+x^n-4$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x+5}$, *and* $\boldsymbol{R(x)=x-2}$. *Find each of the following.*

62. $(P+R)(-1)$

63. $(P-Q)(-2)$

64. $(Q-R)(3)$

65. $(R-Q)(0)$

66. $(R-Q)(k)$

67. $(P+Q)(a)$

68. $(Q-R)(a+1)$

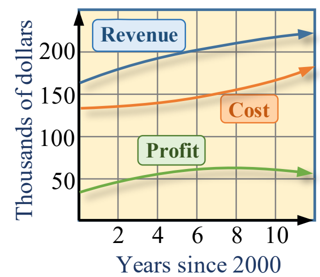
69. $(P+R)(2k)$

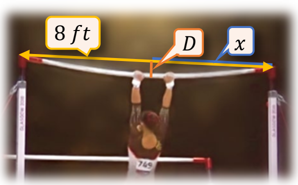
Solve each problem.

70. Suppose that during the years 2000-2012 the revenue R and the cost C of a particular business are modelled by the polynomials

$$R(t)=-0.296t^2+9.72t+164 \ \text{ and } \ C(t)=0.154t^2+2.15t+135,$$

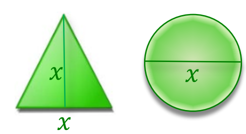
where t represents the number of years since 2000 and both $R(t)$ and $C(t)$ are in thousands of dollars. Write a polynomial that models the profit $P(t)$ of this business during the years 2000-2012.





71. Suppose that the deflection D of an 8 feet-long gymnastic bar can be approximated by the polynomial function $D(x)=0.037x^4-0.59x^3+2.35x^2$, where x is the distance in feet from one end of the bar and D is in centimeters. To the nearest tenth of a centimeter, determine the maximum deflection for this bar, assuming that it occurs at the middle of the bar.

72. Write a polynomial that can be used to calculate the sum of areas of a triangle with the base and height of length x and a circle with diameter x. Determine the total area of the two shapes for $x=5$ centimeters. *Round to the nearest square centimeter.*



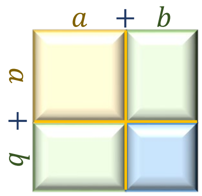
73. Suppose the cost in dollars of sewing n dresses is given by $C(n)=32n+1500$. If the dresses can be sold for \$56 each, complete the following.

a. Write a function $R(n)$ that gives the revenue for selling n dresses.

b. Write a formula $P(n)$ for the profit. *Recall that profit is defined as the difference between revenue and cost.*

c. Evaluate $P(100)$ and interpret the answer.

P2 Multiplication of Polynomials



As shown in the previous section, addition and subtraction of polynomials results in another polynomial. This means that the **set of polynomials** is **closed under** the operation of **addition** and **subtraction**. In this section, we will show that the set of polynomials is also closed under the operation of **multiplication**, meaning that a product of polynomials is also a polynomial.

Properties of Exponents

Since multiplication of polynomials involves multiplication of powers, let us review properties of exponents first.

Recall:

power *exponent*

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$$

base

For example, $x^4 = x \cdot x \cdot x \cdot x$ and we read it "x to the fourth power" or shorter "x to the fourth". If $n = 2$, the power x^2 is customarily read "x squared". If $n = 3$, the power x^3 is often read "x cubed".

Let $a \in \mathbb{R}$, and $m, n \in \mathbb{W}$. The table below shows basic exponential rules with some examples justifying each rule.

Power Rules for Exponents

General Rule	Description	Example
$a^m \cdot a^n = a^{m+n}$	To **multiply powers** of the same bases, keep the base and **add** the **exponents**.	$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x) = x^{2+3} = x^5$
$\frac{a^m}{a^n} = a^{m-n}$	To **divide powers** of the same bases, keep the base and **subtract** the **exponents**.	$\frac{x^5}{x^2} = \frac{(x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x})}{(\cancel{x} \cdot \cancel{x})} = x^{5-2} = x^3$
$(a^m)^n = a^{mn}$	To raise a **power to a power, multiply the exponents.**	$(x^2)^3 = (x \cdot x)(x \cdot x)(x \cdot x) = x^{2 \cdot 3} = x^6$
$(ab)^n = a^n b^n$	To raise a **product to a power**, raise **each factor to that power**.	$(2x)^3 = 2^3 x^3$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	To raise a **quotient to a power**, raise the **numerator and** the **denominator to that power**.	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2}$
$a^0 = 1$ for $a \neq 0$ 0^0 is undefined	A **nonzero number** raised **to the power of zero** equals **one**.	$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1$

Example 1 ► **Simplifying Exponential Expressions**

Simplify.

a. $(-3xy^2)^4$ **b.** $(5p^3q)(-2pq^2)$

c. $\left(\frac{-2x^5}{x^2y}\right)^3$ **d.** $x^{2a}x^a$

Solution ►

a. To simplify $(-3xy^2)^4$, we apply the fourth power to each factor in the bracket. So,

$$(-3xy^2)^4 = \underbrace{(-3)^4}_{\substack{even\ power \\ of\ a\ negative \\ is\ a\ positive}} \cdot x^4 \cdot \underbrace{(y^2)^4}_{\substack{multiply \\ exponents}} = \mathbf{3^4x^4y^8}$$

$p = p^1$

b. To simplify $(5p^3q)(-2pq^2)$, we multiply numbers, powers of p, and powers of q. So,

$$(5p^3q)(-2pq^2) = (-2)\cdot 5 \cdot \underbrace{p^3 \cdot p}_{\substack{add \\ exponents}} \cdot \underbrace{q \cdot q^2}_{\substack{add \\ exponents}} = \mathbf{-10p^4q^3}$$

c. To simplify $\left(\frac{-2x^5}{x^2y}\right)^3$, first we reduce the common factors and then we raise every factor of the numerator and denominator to the third power. So, we obtain

$$\left(\frac{-2x^{\cancel{5}\,3}}{\cancel{x^2}y}\right)^3 = \left(\frac{-2x^3}{y}\right)^3 = \frac{(-2)^3(x^3)^3}{y^3} = \frac{\mathbf{-8x^9}}{\mathbf{y^3}}$$

d. When multiplying powers with the same bases, we add exponents, so $x^{2a}x^a = \mathbf{x^{3a}}$

Multiplication of Polynomials

Multiplication of polynomials involves finding products of monomials. To multiply monomials, we use the commutative property of multiplication and the product rule of powers.

Example 2 ► **Multiplying Monomials**

Find each product.

a. $(3x^4)(5x^3)$ **b.** $(5b)(-2a^2b^3)$ **c.** $-4x^2(3xy)(-x^2y)$

Solution ►

a. $(3x^4)(5x^3) = 3 \cdot \underbrace{x^4 \cdot 5}_{\text{commutative property}} \cdot x^3 = 3 \cdot 5 \cdot \underbrace{x^4 \cdot x^3}_{\text{product rule of powers}} = \mathbf{15x^7}$

To find the product of monomials, find the following:

- the final **sign**,
- the **number**,
- the **power**.

The intermediate steps are not necessary to write. The final answer is immediate if we follow the order: **sign**, **number**, **power** of each variable.

b. $(5b)(-2a^2b^3) = 5(-2)a^2bb^3 = \mathbf{-10a^2b^4}$

c. $-4x^2(3xy)(-x^2y) = \underbrace{(-4)\cdot 3 \cdot (-1)}_{\text{multiply coefficients}} \underbrace{x^2xx^2\,yy}_{\text{apply product rule of powers}} = \mathbf{12x^5y^2}$

To multiply polynomials by a monomial, we use the distributive property of multiplication.

Example 3 ► **Multiplying Polynomials by a Monomial**

Find each product.

a. $-2x(3x^2 - x + 7)$ **b.** $(5b - ab^3)(3ab^2)$

Solution ►

a. To find the product $-2x(3x^2 - x + 7)$, we distribute the monomial $-2x$ to each term inside the bracket. So, we have

$$-2x(3x^2 - x + 7) = \underbrace{-2x(3x^2) - 2x(-x) - 2x(7)}_{\textit{this step can be done mentally}} = \mathbf{-6x^3 + 2x^2 - 14x}$$

b. $$(5b - ab^3)(3ab^2) = \underbrace{5b(3ab^2) - ab^3(3ab^2)}_{\textit{this step can be done mentally}}$$

arranged in decreasing order of powers

$$= 15ab^3 - 3a^2b^5 = \mathbf{-3a^2b^5 + 15ab^3}$$

When multiplying polynomials by polynomials we **multiply each term of the first polynomial by each term of the second polynomial**. This process can be illustrated with finding areas of a rectangle whose sides represent each polynomial. For example, we multiply $(2x + 3)(x^2 - 3x + 1)$ as shown below

	x^2	$-3x$	$+1$
$2x$	$2x^3$	$-6x^2$	$2x$
$+3$	$3x^2$	$-9x$	3

So, $$\begin{array}{rrrrr} (2x+3)(x^2-3x+1) = & 2x^3 & -6x^2 & +2x & \\ & & +3x^2 & -9x & +3 \\ \hline = & \mathbf{2x^3} & \mathbf{-3x^2} & \mathbf{-7x} & \mathbf{+3} \end{array}$$

line up like terms to combine them

Example 4 ► **Multiplying Polynomials by Polynomials**

Find each product.

a. $(3y^2 - 4y - 2)(5y - 7)$ **b.** $4a^2(2a - 3)(3a^2 + a - 1)$

Solution ►

a. To find the product $(3y^2 - 4y - 2)(5y - 7)$, we can distribute the terms of the second bracket over the first bracket and then collect the like terms. So, we have

$$\begin{array}{rrrrr} (3y^2 - 4y - 2)(5y - 7) = & 15y^3 & -20y^2 & -10y & \\ & & -21y^2 & +28y & +14 \\ \hline = & \mathbf{15y^3} & \mathbf{-41y^2} & \mathbf{+18y} & \mathbf{+14} \end{array}$$

b. To find the product $4a^2(2a - 3)(3a^2 + a - 1)$, we will multiply the two brackets first, and then multiply the resulting product by $4a^2$. So,

$$4a^2(2a-3)(3a^2+a-1) = 4a^2\underbrace{\left(6a^3 + 2a^2 - 2a - 9a^2 - 3a + 3\right)}_{\text{collect like terms before removing the bracket}}$$

$$= 4a^2(6a^3 - 7a^2 - 5a + 3) = \mathbf{24a^5 - 28a^4 - 20a^3 + 12a^2}$$

In multiplication of binomials, it might be convenient to keep track of the multiplying terms by following the **FOIL** mnemonic, which stands for multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials. Here is how it works:

First Last

$$(2x-3)(x+5) = 2x^2 \underbrace{+10x - 3x}_{\text{the sum of the Outer and Inner terms becomes the middle term}} - 15 = \mathbf{2x^2 + 7x - 15}$$

Inner

Outer

Example 5 ► **Using the FOIL Method in Binomial Multiplication**

Find each product.

a. $(x+3)(x-4)$ **b.** $(5x-6)(2x+3)$

Solution ► **a.** To find the product $(x+3)(x-4)$, we may follow the **FOIL** method

x^2 -12

$$(x+3)(x-4) = \mathbf{x^2 - x - 12}$$

$3x$

$-4x$

To find the **linear (middle)** term try to add the inner and outer products mentally.

b. Observe that the linear term of the product $(5x-6)(2x+3)$ is equal to the sum of $-12x$ and $15x$, which is $3x$. So, we have

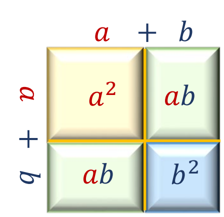
$$(5x-6)(2x+3) = \mathbf{10x^2 + 3x - 18}$$

Special Products

Suppose we want to find the product $(a+b)(a+b)$. This can be done via the FOIL method

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b}) = a^2 + ab + ab + b^2 = \boldsymbol{a^2 + 2ab + b^2},$$

or via the geometric visualization:



Observe that since the products of the inner and outer terms are both equal to ab, we can use a shortcut and write the middle term of the final product as $2ab$. We encorage the reader to come up with similar observations regarding the product $(a-b)(a-b)$. This regularity in multiplication of a binomial by itself leads us to the **perfect square formula**:

$$\boldsymbol{(a \pm b)^2 = a^2 \pm 2ab + b^2}$$

In the above notation, the " $\pm$ " sign is used to record two formulas at once, the perfect square of a sum and the perfect square of a difference. It tells us to either use a " $+$ " in both places, or a " $-$ " in both places. The $\boldsymbol{a}$ and $\boldsymbol{b}$ can actually represent any expression. For example, to expand $(2x-y^2)^2$, we can apply the perfect square formula by treating $2x$ as a and y^2 as b. Here is the calculation.

$$(2x-y^2)^2 = (2x)^2 - 2(2x)y^2 + (y^2)^2 = \boldsymbol{4x^2 - 4xy^2 + y^4}$$

Conjugate binomials have the **same first terms** and **opposite second terms**.

Another interesting pattern can be observed when multiplying two **conjugate** brackets, such as $(a+b)$ and $(a-b)$. Using the FOIL method,

$$\boldsymbol{(a+b)(a-b)} = a^2 + \cancel{ab} - \cancel{ab} - b^2 = \boldsymbol{a^2 - b^2},$$

we observe that the products of the inner and outer terms are opposite. So, they add to zero and the product of conjugate brackets becomes the difference of squares of the two terms. This regularity in multiplication of conjugate brackets leads us to the **difference of squares formula**.

$$\boldsymbol{(a+b)(a-b) = a^2 - b^2}$$

Again, $\boldsymbol{a}$ and $\boldsymbol{b}$ can represent any expression. For example, to find the product $(3x+0.1y^2)(3x-0.1y^2)$, we can apply the difference of squares formula by treating $3x$ as a and $0.1y^2$ as b. Here is the calculation.

$$\boldsymbol{(3x+0.1y^3)(3x-0.1y^3)} = (3x)^2 - (0.1y^3)^2 = \boldsymbol{9x^2 - 0.01y^6}$$

We encourage the use of the above formulas whenever applicable, as it allows for more efficient calculations and helps to observe patterns useful in future factoring.

Example 6 ► **Using Special Product Formulas in Polynomial Multiplication**

Find each product. Apply special products formulas, if applicable.

a. $(5x+3y)^2$ **b.** $(x+y-5)(x+y+5)$

Solution ► **a.** Applying the perfect square formula, we have

$$(5x+3y)^2 = (5x)^2 + 2(5x)3y + (3y)^2 = \boldsymbol{25x^2 + 30xy + 9y^2}$$

b. The product $(x+y-5)(x+y+5)$ can be found by multiplying each term of the first polynomial by each term of the second polynomial, using the distributive property. However, we can find the product $(x+y-5)(x+y+5)$ in a more efficient way by applying the difference of squares formula. Treating the expression $x+y$ as the first term a and the 5 as the second term b in the formula $(a+b)(a-b)=a^2-b^2$, we obtain

$$(x+y-5)(x+y+5)=(x+y)^2-5^2$$
$$=\underbrace{\boldsymbol{x^2+2xy+y^2}}_{\textit{here we apply the perfect square formula}}\ \boldsymbol{-\,25}$$

Caution: *The perfect square formula shows that* $\boldsymbol{(a+b)^2 \neq a^2+b^2}$.
The difference of squares formula shows that $\boldsymbol{(a-b)^2 \neq a^2-b^2}$.
More generally, $\boldsymbol{(a \pm b)^n \neq a^n \pm b^n}$ *for any natural* $n \neq 1$.

Product Functions

The operation of multiplication can be defined not only for polynomials but also for general functions.

Definition 2.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **product function**, denoted $\boldsymbol{f\cdot g}$ or $\boldsymbol{fg}$, is defined as

$$\boldsymbol{(f\cdot g)(x)=f(x)\cdot g(x)}.$$

The **domain** of the product function is the intersection $\boldsymbol{D_f \cap D_g}$ of the domains of the two functions.

Example 7 ► **Multiplying Polynomial Functions**

Suppose $P(x)=x^2-4x$ and $Q(x)=3x+2$. Find the following:

a. $(PQ)(x)$, $(PQ)(-2)$, and $P(-2)Q(-2)$
b. $(QQ)(x)$ and $(QQ)(1)$
c. $2(PQ)(k)$

Solution ► **a.** Using the definition of the product function, we have

$$(PQ)(x)=P(x)\cdot Q(x)=(x^2-4x)(3x+2)=3x^3+2x^2-12x^2-8x$$
$$=\boldsymbol{3x^3-10x^2-8x}$$

To find $(PQ)(-2)$, we substitute $x=-2$ to the above polynomial function. So,

$$(PQ)(-2)=3(-2)^3-10(-2)^2-8(-2)=3\cdot(-8)-10\cdot 4+16$$
$$=-24-40+16=\boldsymbol{-48}$$

To find $P(-2)Q(-2)$, we calculate

$$P(-2)Q(-2)=\left((-2)^2-4(-2)\right)(3(-2)+2)=(4+8)(-6+2)=12\cdot(-4)$$
$$=\boldsymbol{-48}$$

Observe that both expressions result in the same value. This was expected, as by the definition, $(PQ)(-2) = P(-2) \cdot Q(-2)$.

b. Using the definition of the product function as well as the perfect square formula, we have

$$(QQ)(x) = Q(x) \cdot Q(x) = [Q(x)]^2 = (3x+2)^2 = \mathbf{9x^2 + 12x + 4}$$

Therefore, $(QQ)(1) = 9 \cdot 1^2 + 12 \cdot 1 + 4 = 9 + 12 + 4 = \mathbf{25}$.

c. Since $(PQ)(x) = 3x^3 - 10x^2 - 8x$, as shown in the solution to *Example 7a*, then $(PQ)(k) = 3k^3 - 10k^2 - 8k$. Therefore,

$$2(PQ)(k) = 2[3k^3 - 10k^2 - 8k] = \mathbf{6k^3 - 20k^2 - 16k}$$

P.2 Exercises

1. Decide whether each expression has been simplified correctly. If not, correct it.

a. $x^2 \cdot x^4 = x^8$ **b.** $-2x^2 = 4x^2$ **c.** $(5x)^3 = 5^3x^3$

d. $-\left(\frac{x}{5}\right)^2 = -\frac{x^2}{25}$ **e.** $(a^2)^3 = a^5$ **f.** $4^5 \cdot 4^2 = 16^7$

g. $\frac{6^5}{3^2} = 2^3$ **h.** $xy^0 = 1$ **i.** $(-x^2y)^3 = -x^6y^3$

Simplify each expression.

2. $3x^2 \cdot 5x^3$ **3.** $-2y^3 \cdot 4y^5$ **4.** $3x^3(-5x^4)$

5. $2x^2y^5(7xy^3)$ **6.** $(6t^4s)(-3t^3s^5)$ **7.** $(-3x^2y)^3$

8. $\frac{12x^3y}{4xy^2}$ **9.** $\frac{15x^5y^2}{-3x^2y^4}$ **10.** $(-2x^5y^3)^2$

11. $\left(\frac{4a^2}{b}\right)^3$ **12.** $\left(\frac{-3m^4}{n^3}\right)^2$ **13.** $\left(\frac{-5p^2q}{pq^4}\right)^3$

14. $3a^2(-5a^5)(-2a)^0$ **15.** $-3a^3b(-4a^2b^4)(ab)^0$ **16.** $\frac{(-2p)^2pq^3}{6p^2q^4}$

17. $\frac{(-8xy)^2y^3}{4x^5y^4}$ **18.** $\left(\frac{-3x^4y^6}{18x^6y^7}\right)^3$ **19.** $((-2x^3y)^2)^3$

20. $((-a^2b^4)^3)^5$ **21.** x^nx^{n-1} **22.** $3a^{2n}a^{1-n}$

23. $(5^a)^{2b}$ **24.** $(-7^{3x})^{4y}$ **25.** $\frac{-12x^{a+1}}{6x^{a-1}}$

26. $\frac{25x^{a+b}}{-5x^{a-b}}$ **27.** $\left(x^{a+b}\right)^{a-b}$ **28.** $(x^2y)^n$

Find each product.

29. $8x^2y^3(-2x^5y)$

30. $5a^3b^5(-3a^2b^4)$

31. $2x(-3x+5)$

32. $4y(1-6y)$

33. $-3x^4y(4x-3y)$

34. $-6a^3b(2b+5a)$

35. $5k^2(3k^2-2k+4)$

36. $6p^3(2p^2+5p-3)$

37. $(x+6)(x-5)$

38. $(x-7)(x+3)$

39. $(2x+3)(3x-2)$

40. $3p(5p+1)(3p+2)$

41. $2u^2(u-3)(3u+5)$

42. $(2t+3)(t^2-4t-2)$

43. $(2x-3)(3x^2+x-5)$

44. $(a^2-2b^2)(a^2-3b^2)$

45. $(2m^2-n^2)(3m^2-5n^2)$

46. $(x+5)(x-5)$

47. $(a+2b)(a-2b)$

48. $(x+4)(x+4)$

49. $(a-2b)(a-2b)$

50. $(x-4)(x^2+4x+16)$

51. $(y+3)(y^2-3y+9)$

52. $(x^2+x-2)(x^2-2x+3)$

53. $(2x^2+y^2-2xy)(x^2-2y^2-xy)$

True or False? If it is false, show a counterexample by choosing values for a and b that would not satisfy the equation.

54. $(a+b)^2=a^2+b^2$

55. $a^2-b^2=(a-b)(a+b)$

56. $(a-b)^2=a^2+b^2$

57. $(a+b)^2=a^2+2ab+b^2$

58. $(a-b)^2=a^2+ab+b^2$

59. $(a-b)^3=a^3-b^3$

Find each product. Use the ***difference of squares*** *or the* ***perfect square*** *formula, if applicable.*

60. $(2p+3)(2p-3)$

61. $(5x-4)(5x+4)$

62. $\left(b-\frac{1}{3}\right)\left(b+\frac{1}{3}\right)$

63. $\left(\frac{1}{2}x-3y\right)\left(\frac{1}{2}x+3y\right)$

64. $(2xy+5y^3)(2xy-5y^3)$

65. $(x^2+7y^3)(x^2-7y^3)$

66. $(1.1x+0.5y)(1.1x-0.5y)$

67. $(0.8a+0.2b)(0.8a+0.2b)$

68. $(x+6)^2$

69. $(x-3)^2$

70. $(4x+3y)^2$

71. $(5x-6y)^2$

72. $\left(3a+\frac{1}{2}\right)^2$

73. $\left(2n-\frac{1}{3}\right)^2$

74. $(a^3b^2-1)^2$

75. $(x^4y^2+3)^2$

76. $(3a^2+4b^3)^2$

77. $(2x^2-3y^3)^2$

78. $3y(5xy^3+2)(5xy^3-2)$

79. $2a(2a^2+5ab)(2a^2+5ab)$

80. $3x(x^2y-xy^3)^2$

81. $(-xy+x^2)(xy+x^2)$

82. $(4p^2+3pq)(-3pq+4p^2)$

83. $(x+1)(x-1)(x^2+1)$

84. $(2x-y)(2x+y)(4x^2+y^2)$

85. $(a-b)(a+b)(a^2-b^2)$

86. $(a+b+1)(a+b-1)$

87. $(2x+3y-5)(2x+3y+5)$

88. $(3m+2n)(3m-2n)(9m^2-4n^2)$

89. $\left((2k-3)+h\right)^2$

90. $\left((4x+y)-5\right)^2$

91. $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}+y^{2b}\right)$

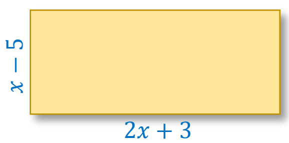
92. $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}-y^{2b}\right)$

Use the difference of squares formula, $(a+b)(a-b)=a^2-b^2$, to find each product.

93. $101 \cdot 99$ **94.** $198 \cdot 202$ **95.** $505 \cdot 495$

Find the area of each figure. Express it as a polynomial in descending powers of the variable x.

96.



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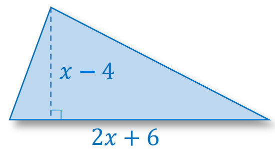

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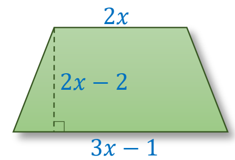


For each pair of functions, $\boldsymbol{f}$ and $\boldsymbol{g}$, find the ***product*** *function $(\boldsymbol{fg})(\boldsymbol{x})$.*

99. $f(x)=5x-6, \quad g(x)=-2+3x$

100. $f(x)=x^2+7x-2, \quad g(x)=6x+5$

101. $f(x)=3x^2-5x, \quad g(x)=9+x-x^2$

102. $f(x)=x^n-4, \quad g(x)=x^n+1$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x}$, and $\boldsymbol{R(x)=x-2}$. Find each of the following.

103. $(PR)(x)$ **104.** $(PQ)(x)$ **105.** $(PQ)(a)$

106. $(PR)(-1)$ **107.** $(PQ)(3)$ **108.** $(PR)(0)$

109. $(QR)(x)$ **110.** $(QR)\left(\frac{1}{2}\right)$ **111.** $(QR)(a+1)$

112. $P(a-1)$ **113.** $P(2a+3)$ **114.** $P(1+h)-P(1)$

Solve each problem.

115. Squares with x centimeters long sides are cut out from each corner of a rectangular piece of cardboard measuring 50 cm by 70 cm. Then the flaps of the remaining cardboard are folded up to construct a box. Find the volume $V(x)$ of the box in terms of the length x.

116. A rectangular flower-bed has a perimeter of 60 meters. If the rectangle is w meters wide, write a polynomial that can be used to determine the area $A(w)$ of the flower-bed in terms of w.

P3 Division of Polynomials

In this section we will discuss dividing polynomials. The result of division of polynomials is not always a polynomial. For example, $x+1$ divided by x becomes

$$\frac{x+1}{x}=\frac{x}{x}+\frac{1}{x}=1+\frac{1}{x},$$

which is not a polynomial. Thus, the set of polynomials is not closed under the operation of division. However, we can perform division with remainders, mirroring the algorithm of division of natural numbers. We begin with dividing a polynomial by a monomial and then by another polynomial.

Division of Polynomials by Monomials

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial, and then simplify each quotient. In other words, we use the reverse process of addition of fractions, as illustrated below.

$$\boldsymbol{\frac{a+b}{d}=\frac{a}{d}+\frac{b}{d}}$$

Example 1 ► **Dividing Polynomials by Monomials**

Divide and simplify.

a. $(6x^3+15x^2-2x)\div(3x)$ **b.** $\frac{xy^2-8x^2y+6x^3y^2}{-2xy^2}$

Solution ►

a. $(6x^3+15x^2-2x)\div(3x)=\frac{6x^3+15x^2-2x}{3x}=\frac{\overset{2}{\cancel{6}}\overset{2}{\cancel{x^3}}}{\cancel{3}\cancel{x}}+\frac{\overset{5}{\cancel{15}}\cancel{x^2}}{\cancel{3}\cancel{x}}-\frac{2\cancel{x}}{3\cancel{x}}=\boldsymbol{2x^2+5x-\frac{2}{3}}$

b. $\frac{xy^2-8x^2y+6x^3y^2}{-2xy^2}=-\frac{\cancel{x}\cancel{y^2}}{2\cancel{x}\cancel{y^2}}+\frac{\overset{4}{\cancel{8}}\cancel{x^2}\cancel{y}}{\cancel{2}\cancel{x}\cancel{y^2}}-\frac{\overset{3}{\cancel{6}}\overset{2}{\cancel{x^3}}\cancel{y^2}}{\cancel{2}\cancel{x}\cancel{y^2}}=\boldsymbol{-\frac{1}{2}+\frac{4x}{y}-3x^2}$

Division of Polynomials by Polynomials

To divide a polynomial by another polynomial, we follow an algorithm similar to the long division algorithm used in arithmetic. For example, observe the steps taken in the long division algorithm when dividing 158 by 13 and the corresponding steps when dividing x^2+5x+8 by $x+3$.

Step 1: Place the dividend under the long division symbol and the divisor in front of this symbol.

$13\,\overline{)\,158}$ $\underbrace{x+3}_{divisor}\,\overline{)\,\underbrace{x^2+5x+8}_{dividend}}$

Remember: Both polynomials should be written in **decreasing order of powers**. Also, any **missing terms** after the leading term should be displayed with a **zero coefficient**. This will ensure that the terms in each column are of the same degree.

Step 2: Divide the first term of the dividend by the first term of the divisor and record the quotient above the division symbol.

$$\begin{array}{rl} & \;\;1 \\ \underbrace{13} & \overline{)\,\underbrace{15}8} \end{array} \qquad\qquad \begin{array}{rl} & \overbrace{\;x\qquad\qquad}^{\text{quotient}} \\ \underbrace{x}+3 & \overline{)\,\underbrace{x^2}+5x+8} \end{array}$$

Step 3: Multiply the quotient from *Step 2* by the divisor and write the product under the dividend, lining up the columns with the same degree terms.

$$\begin{array}{rl} & \;\;1 \\ 13 & \overline{)\,158} \\ & \;13 \end{array} \qquad\qquad \begin{array}{rl} & \;x \\ x+3 & \overline{)\,x^2+5x+8} \\ & \;x^2+3x \end{array}$$

Step 4: Underline and subtract by adding opposite terms in each column. We suggest recording the new sign in a circle, so that it is clear what is being added.

$$\begin{array}{rl} & \;\;1 \\ 13 & \overline{)\,158} \\ & \underline{-13} \\ & \;\;2 \end{array} \qquad\qquad \begin{array}{rl} & \;x \\ x+3 & \overline{)\,x^2+5x+8} \\ & \underline{-(x^2 \overset{\ominus}{+} 3x)} \\ & \qquad 2x \end{array}$$

Step 5: Drop the next term (or digit) and repeat the algorithm until the degree of the remainder is lower than the degree of the divisor.

$$\begin{array}{rl} & \;\;12 \\ 13 & \overline{)\,158} \\ & \underline{-13\;} \\ & \;\;28 \\ & \underline{-26} \\ & \;\;\;2 \end{array} \qquad\qquad \begin{array}{rl} & \;x\;+\;2 \\ x+3 & \overline{)\,x^2+5x+8} \\ & \underline{-(x^2 \overset{\ominus}{+} 3x)} \\ & \qquad 2x+8 \\ & \qquad \underline{-(2x \overset{\ominus}{+} 6)} \\ & \qquad\qquad 2 \leftarrow \textit{remainder} \end{array}$$

In the example of long division of numbers, we have $158 = 13 \cdot 12 + 2$.

So, the quotient can be written as $\frac{158}{13} = \mathbf{12} + \frac{\mathbf{2}}{\mathbf{13}}$.

In the example of long division of polynomials, we have

$$x^2 + 5x + 8 = (x+3) \cdot (x \;+\; 2) + 2.$$

So, the quotient can be written as $\frac{x^2+5x+8}{x+3} = \boldsymbol{x+2} + \frac{\mathbf{2}}{\boldsymbol{x+3}}$.

Generally, if P, D, Q, and R are polynomials, such that $P(x) = D(x) \cdot Q(x) + R(x)$, then the ratio of polynomials P and D can be written as

$$\boxed{\frac{\boldsymbol{P(x)}}{\boldsymbol{D(x)}} = \boldsymbol{Q(x)} + \frac{\boldsymbol{R(x)}}{\boldsymbol{D(x)}},}$$

where $Q(x)$ is the quotient polynomial, and $R(x)$ is the remainder from the division of $P(x)$ by the divisor $D(x)$.

Observe: The degree of the remainder must be lower than the degree of the divisor, as otherwise, we could apply the division algorithm one more time.

Example 2 ► **Dividing Polynomials by Polynomials**

Divide.

a. $(3x^3 - 2x^2 + 5) \div (x^2 - 3)$ **b.** $\frac{2p^3+2p+3p^2}{5+2p}$

Solution ► **a.** When writing the polynomials in the long division format, we use a zero placeholder term in place of the missing linear terms in both the dividend and the divisor. So, we have

$$
\begin{array}{rr}
 & \mathbf{3x} \; - \; \mathbf{2} \\
x^2 + 0x - 3 & \overline{)\; 3x^3 - 2x^2 + 0x + 5} \\
 & -(3x^3 + 0x^2 \overset{\oplus}{-} 9x) \\
 & \overline{-2x^2 + 9x + 5} \\
 & -(-2x^2 - 0x \overset{\ominus}{+} 6) \\
 & \overline{\mathbf{9x - 1}}
\end{array}
$$

Thus, $(3x^3 - 2x^2 + 5) \div (x^2 - 3) = 3x - 2 + \frac{9x-1}{x^2-3}$.

b. To perform this division, we arrange both polynomials in decreasing order of powers, and replace the constant term in the dividend with a zero. So, we have

$$
\begin{array}{rr}
 & \boldsymbol{p^2 - p + \frac{7}{2}} \\
2p + 5 & \overline{)\; 2p^3 + 3p^2 + 2p + 0} \\
 & -(2p^3 \overset{\ominus}{+} 5p^2) \\
 & \overline{-2p^2 + 2p} \\
 & -(-2p^2 \overset{\oplus}{-} 5p) \\
 & \overline{7p + 0} \\
 & -\left(7p \overset{\ominus}{+} \frac{35}{2}\right) \\
 & \overline{\mathbf{-\frac{35}{2}}}
\end{array}
$$

Thus, $\frac{2p^3+2p+3p^2}{5+2p} = p^2 - p + \frac{7}{2} + \frac{-\frac{35}{2}}{2p+5} = \boldsymbol{p^2 - p + \frac{7}{2} - \frac{35}{4p+10}}$.

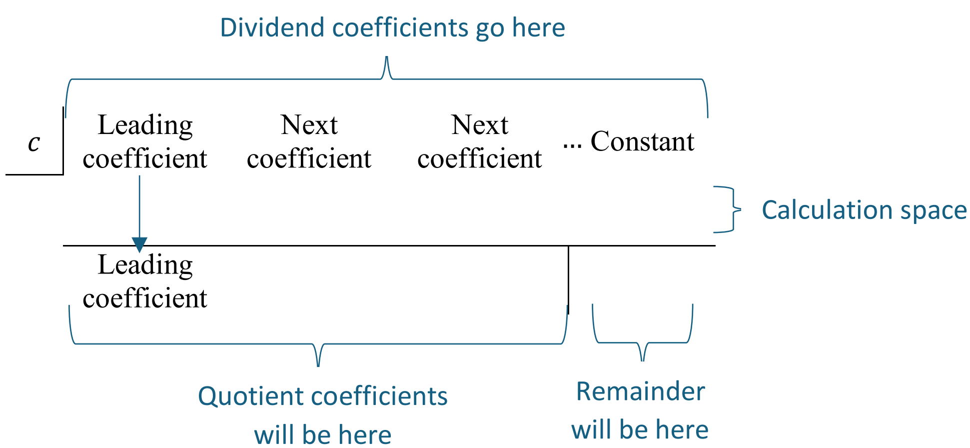
Observe in the above answer that $\frac{-\frac{35}{2}}{2p+5}$ is written in a simpler form, $-\frac{35}{4p+10}$. This is because $\frac{-\frac{35}{2}}{2p+5} = -\frac{35}{2} \cdot \frac{1}{2p+5} = -\frac{35}{4p+10}$.

Synthetic Division

Synthetic division is a shortcut for long division that works when the divisor is of the form $x - c$. It takes the constants and coefficients of long division and uses them to complete the calculation.

Step 1: Compare the divisor to $x - c$ to find c. For example, if the divisor is $x - 1$, then the value of c is 1. If the divisor is $x + 5$, then the value of c is 5.

Step 2: Set up the synthetic division grid as below:



Step 3: Multiply the last entry in the bottom row by the c value, write the result in the second row of the next column, then add down the column and write the sum in the bottom row. Repeat until you run out of columns.

Step 4: The bottom row contains the remainder and the coefficients of the quotient.

Example 3 ► **Using Synthetic Division to Divide Polynomials**

Use synthetic division to divide.

a. $(2x^3 + 3x^2 + x + 8) \div (x - 2)$ **b.** $\frac{x^5-16x^3-2x^2+6x-2}{x-4}$

c. $(4x^4 + 11x^3 + 5x - 8) \div (x + 3)$

Solution ► **a.** Comparing the divisor, $(x - 2)$, to the form $(x - c)$, we see that c is 2. Write the c value in the box at the top left and copy the coefficients of the dividend into the top row of the grid. Drop the leading coefficient into the bottom row of the grid.

$$\begin{array}{c|cccc}
2 & 2 & 3 & 1 & 8 \\
 & \downarrow & & & \\
\hline
 & 2 & & &
\end{array}$$

Multiply the 2 in the bottom row by the *c*-value in the box, then write the result in the second column, second row:

$$\begin{array}{c|cccc} 2 & 2 & 3 & 1 & 8 \\ & \downarrow & 4 & & \\ \hline & 2 & & & \end{array}$$

Add down the second column and write the result in the bottom row, second column:

$$\begin{array}{c|cccc} 2 & 2 & 3 & 1 & 8 \\ & \downarrow & 4 & & \\ \hline & 2 & 7 & & \end{array}$$

Multiply the 7 in the bottom row by the c-value in the box, then write the result in the third column, second row:

$$\begin{array}{c|cccc} 2 & 2 & 3 & 1 & 8 \\ & \downarrow & 4 & 14 & \\ \hline & 2 & 7 & & \end{array}$$

Add down the third column and write the result in the bottom row, third column:

$$\begin{array}{c|cccc} 2 & 2 & 3 & 1 & 8 \\ & \downarrow & 4 & 14 & \\ \hline & 2 & 7 & 15 & \end{array}$$

Multiply the 15 in the bottom row by the c-value in the box, then write the result in the fourth column, second row:

$$\begin{array}{c|cccc} 2 & 2 & 3 & 1 & 8 \\ & \downarrow & 4 & 14 & 30 \\ \hline & 2 & 7 & 15 & \end{array}$$

Add down the fourth column and write the result in the bottom row, fourth column:

$$\begin{array}{c|ccc|c} 2 & 2 & 3 & 1 & 8 \\ & \downarrow & 4 & 14 & 30 \\ \hline & 2 & 7 & 15 & 38 \end{array}$$

Quotient coefficients

Remainder

The quotient is $2x^2 + 7x + 15$ and the remainder is 38.

Tip: Work from right to left with the quotient coefficients – the rightmost number is the constant term, left of that is the linear coefficient, left of that is the quadratic coefficient, etc. The leading term of the quotient will be one degree less than the dividend.

b. The divisor is $x - 4$, so the c value will be 4. The dividend is missing a 4th degree term, so we add $0x^4$ as a placeholder and carry out the calculation as usual:

4	1	0	-16	-2	6	-2
	↓	4	16	0	-8	-8
	1	4	0	-2	-2	-10

The quotient is $x^4 + 4x^3 - 2x - 2$ and the remainder is -10.

c. Here the divisor is $x + 3$ so we rewrite it in the form $x - c$:

$$x + 3 = x - (-3)$$

and see that $c = -3$. We add $0x^2$ as a placeholder because the dividend is missing a 2nd degree term:

-3	4	11	0	5	-8
	↓	-12	3	-9	12
	4	-1	3	-4	4

The quotient is $4x^3 - x^2 + 3x - 4$ and the remainder is 4.

Quotient Functions

Similarly as in the case of polynomials, we can define quotients of functions.

Definition 3.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **quotient function**, denoted $\frac{f}{g}$, is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The **domain** of the quotient function is the intersection of the domains of the two functions, D_f and D_g, excluding the x-values for which $g(x) = 0$. So,

$$D_{\frac{f}{g}} = D_f \cap D_g \setminus \{x \mid g(x) = 0\}$$

Example 3 ► **Dividing Polynomial Functions**

Suppose $P(x) = 2x^2 - x - 6$ and $Q(x) = x - 2$. Find the following:

a. $\left(\frac{P}{Q}\right)(x)$ and $\left(\frac{P}{Q}\right)(-3)$,

b. $\left(\frac{P}{Q}\right)(2)$ and $\left(\frac{P}{Q}\right)(2a)$,

c. domain of $\frac{P}{Q}$.

Notice that this equation holds only for $x \neq 2$.

Solution ► **a.** By *Definition 3.1*, $\left(\frac{P}{Q}\right)(x) = \frac{P(x)}{Q(x)} = \frac{2x^2-x-6}{x-2} = \mathbf{2x+3}$

So, $\left(\frac{P}{Q}\right)(-3) = 2(-3) + 3 = \mathbf{-3}$. One can verify that the same value is found by evaluating $\frac{P(-3)}{Q(-3)}$.

b. Since the equation $\frac{(2x+3)(x-2)}{x-2} = 2x + 3$ is true only for $x \neq 2$, the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ cannot be used to evaluate $\left(\frac{P}{Q}\right)(2)$. However, by *Definition 3.1*, we have

$\left(\frac{P}{Q}\right)(2)$ is undefined, so 2 is not in the domain of $\frac{P}{Q}$

$$\left(\frac{P}{Q}\right)(2) = \frac{P(2)}{Q(2)} = \frac{2(2)^2 - (2) - 6}{(2) - 2} = \frac{8-2-6}{0} = \frac{0}{0} = \boldsymbol{undefined}$$

To evaluate $\left(\frac{P}{Q}\right)(2a)$, we first notice that if $a \neq 1$, then $2a \neq 2$. So, we can use the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ and evaluate $\left(\frac{P}{Q}\right)(2a) = 2(2a) + 3 = \mathbf{4a+3}$ **for all $\boldsymbol{a \neq 1}$.**

c. The domain of any polynomial is the set of all real numbers. So, the domain of $\frac{P}{Q}$ is the set of all real numbers except for the x-values for which the denominator $Q(x) = x - 2$ is equal to zero. Since the solution to the equation $x - 2 = 0$ is $x = 2$, then the value 2 must be excluded from the set of all real numbers. Therefore, $\boldsymbol{D_{\frac{P}{Q}}} = \mathbb{R} \setminus \{\mathbf{2}\}$.

P.3 Exercises

1. *True or False?* The quotient in a division of a six-degree polynomial by a second-degree polynomial is a third-degree polynomial. Justify your answer.

2. *True or False?* The remainder in a division of a polynomial by a second-degree polynomial is a first-degree polynomial. Justify your answer.

Divide.

3. $\frac{20x^3-15x^2+5x}{5x}$ **4.** $\frac{27y^4+18y^2-9y}{9y}$ **5.** $\frac{8x^2y^2-24xy}{4xy}$

6. $\frac{5c^3d+10c^2d^2-15cd^3}{5cd}$

7. $\frac{9a^5-15a^4+12a^3}{-3a^2}$

8. $\frac{20x^3y^2+44x^2y^3-24x^2y}{-4x^2y}$

9. $\frac{64x^3-72x^2+12x}{8x^3}$

10. $\frac{4m^2n^2-21mn^3+18mn^2}{14m^2n^3}$

11. $\frac{12ab^2c+10a^2bc+18abc^2}{6a^2bc}$

Choose whether it is best to use long division or synthetic division for each, then divide.

12. $(x^2+3x-18)\div(x+6)$

13. $(3y^2+17y+10)\div(3y+2)$

14. $(x^2-11x+16)\div(x+8)$

15. $(t^2-7t-9)\div(t-3)$

16. $\frac{6y^3-y^2-10}{3y+4}$

17. $\frac{4a^3+6a^2+14}{2a+4}$

18. $\frac{4x^3+5x^2-11x+3}{4x+1}$

19. $\frac{10z^3-26z^2+17z-13}{5z-3}$

20. $\frac{2x^3+4x^2-x+2}{x^2+2x-1}$

21. $\frac{3x^3-2x^2+5x-4}{x^2-x+3}$

22. $\frac{4k^4+6k^3+3k-1}{2k^2+1}$

23. $\frac{9k^4+12k^3-4k-1}{3k^2-1}$

24. $\frac{2p^3+7p^2+9p+3}{2p+2}$

25. $\frac{5t^2+19t+7}{4t+12}$

26. $\frac{x^4-4x^3+5x^2-3x+2}{x^2+3}$

27. $\frac{p^3-1}{p-1}$

28. $\frac{x^3+1}{x+1}$

29. $\frac{y^4+16}{y+2}$

30. $\frac{x^5-32}{x-2}$

For each pair of polynomials, $P(x)$ and $D(x)$, find such polynomials $Q(x)$ and $R(x)$ that $\boldsymbol{P(x)=Q(x)\cdot D(x)+R(x)}$.

31. $P(x)=4x^3-4x^2+13x-2$ and $D(x)=2x-1$

32. $P(x)=3x^3-2x^2+3x-5$ and $D(x)=3x-2$

For each pair of functions, $\boldsymbol{f}$ and $\boldsymbol{g}$, find the quotient function $\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right)(\boldsymbol{x})$ and state its ***domain***.

33. $f(x)=6x^2-4x,\ \ g(x)=2x$

34. $f(x)=6x^2+9x,\ \ g(x)=-3x$

35. $f(x)=x^2-36,\ \ g(x)=x+6$

36. $f(x)=x^2-25,\ \ g(x)=x-5$

37. $f(x)=2x^2-x-3,\ \ g(x)=2x-3$

38. $f(x)=3x^2+x-4,\ \ g(x)=3x+4$

39. $f(x)=8x^3+125,\ \ g(x)=2x+5$

40. $f(x)=64x^3-27,\ \ g(x)=4x-3$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x}$, *and* $\boldsymbol{R(x)=x-2}$. *Find each of the following. If the value can't be evaluated, say DNE (does not exist).*

41. $\left(\frac{R}{Q}\right)(x)$

42. $\left(\frac{P}{R}\right)(x)$

43. $\left(\frac{R}{P}\right)(x)$

44. $\left(\frac{R}{Q}\right)(2)$

45. $\left(\frac{R}{Q}\right)(0)$

46. $\left(\frac{P}{R}\right)(3)$

47. $\left(\frac{R}{P}\right)(-2)$

48. $\left(\frac{R}{P}\right)(2)$

49. $\left(\frac{P}{R}\right)(a)$, for $a\neq 2$

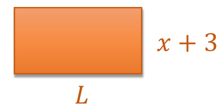
50. $\left(\frac{R}{Q}\right)\left(\frac{3}{2}\right)$

51. $\frac{1}{2}\left(\frac{Q}{R}\right)(x)$

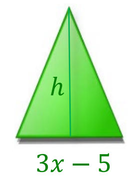
52. $\left(\frac{Q}{R}\right)(a-1)$

Solve each problem.

53. The area A of a rectangle is $3x^2 + 7x - 6$ and its width W is $x + 3$.
a. Find a polynomial that represents the length L of the rectangle.
b. Find the length of the rectangle if the width is 7 meters.



54. The area A of a triangle is $6x^2 - 13x + 5$. Find the height h of the triangle whose base is $3x - 5$. What is the height of such a triangle if its base is 7 centimeters?



P4 Graphs of Basic Polynomial Functions

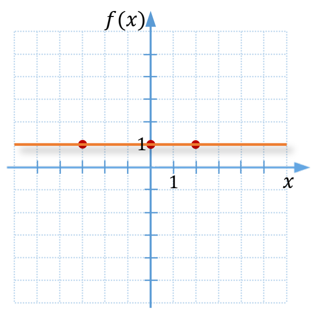
In this section, we will examine graphs of basic polynomial functions, such as constant, linear, quadratic, and cubic functions.

Graphs of Basic Polynomial Functions

Since polynomials are functions, they can be evaluated for different x-values and graphed in a system of coordinates. How do polynomial functions look like? Below, we graph several basic polynomial functions up to the third degree, and observe their shape, domain, and range.

CONSTANT

Let us start with a **constant function**, which is defined by a zero degree polynomial, such as $f(x) = 1$. In this example, for any real x-value, the corresponding y-value is constantly equal to 1. So, the graph of this function is a **horizontal line** with the y-intercept at 1.

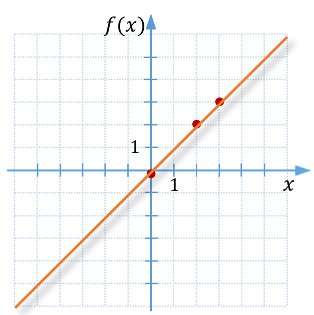


Domain: $\mathbb{R}$
Range: $\{1\}$

Generally, the graph of a **constant function**, $\boldsymbol{f(x) = c}$, is a horizontal line with the y-intercept at c. The domain of this function is $\mathbb{R}$ and the range is $\{c\}$.

LINEAR

The basic first degree polynomial function is the **identity function** given by the formula $\boldsymbol{f(x) = x}$. Since both coordinates of any point satisfying this equation are the same, the graph of the identity function is the diagonal line, as shown below.



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

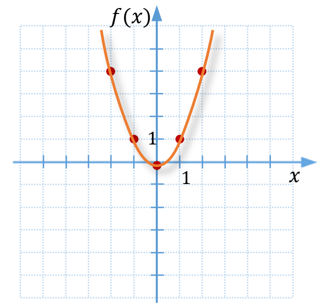
Generally, the graph of any first degree polynomial function, $\boldsymbol{f(x) = mx + b}$ with $m \neq 0$, is a slanted line. So, the domain and range of such function is $\mathbb{R}$.

QUADRATIC

The basic second degree polynomial function is the **squaring function** given by the formula $\boldsymbol{f(x) = x^2}$. The shape of the graph of this function is refered to as the **basic parabola**. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x) = x^2$	
-2	4	
-1	1	
0	0	*vertex*
1	1	
2	4	

symmetry about the y-axis



Domain: $\mathbb{R}$
Range: $[0, \infty)$

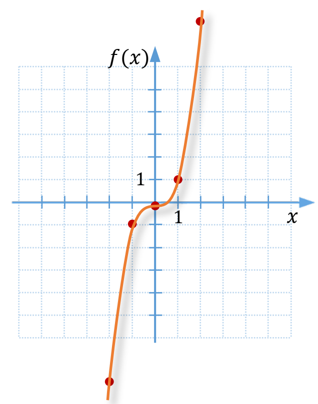
Generally, the graph of any second degree polynomial function, $\boldsymbol{f(x) = ax^2 + bx + c}$ with $a \neq 0$, is a **parabola**. The domain of such function is $\mathbb{R}$ and the range depends on how the parabola is directed, with arms up or down.

CUBIC

The basic third degree polynomial function is the **cubic function**, given by the formula $\boldsymbol{f(x) = x^3}$. The graph of this function has a shape of a 'snake'. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x) = x^3$	
-2	-8	
-1	-1	
0	0	*center*
1	1	
2	8	

symmetry about the origin



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of a third degree polynomial function, $\boldsymbol{f(x) = ax^3 + bx^2 + cx + d}$ with $a \neq 0$, has a shape of a 'snake' with different size waves in the middle. The domain and range of such function is $\mathbb{R}$.

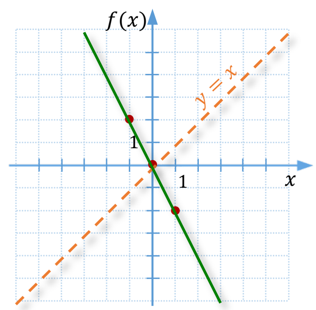
Example 1 ► Graphing Polynomial Functions

Graph each function using a table of values. Give the domain and range of each function by observing its graph. Then, on the same grid, graph the corresponding basic polynomial function. Observe and name the transformation(s) that can be applied to the basic shape in order to obtain the desired function.

a. $f(x) = -2x$ **b.** $f(x) = (x+2)^2$ **c.** $f(x) = x^3 - 2$

Solution ► **a.** The graph of $f(x) = -2x$ is a line passing through the origin and falling from left to right, as shown below in solid green.

x	$f(x) = -2x$
-1	2
0	0
1	-2



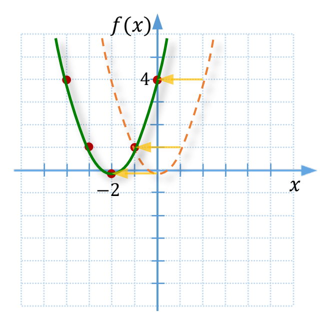
Domain of f: $\mathbb{R}$
Range of f: $\mathbb{R}$

Observe that to obtain the green line, we multiply y-coordinates of the orange line by a factor of -2. Such a transformation is called a **dilation** in the $\boldsymbol{y}$**-axis** by a factor of $\mathbf{-2}$. This dilation can also be achieved by applying a **symmetry in the** $\boldsymbol{x}$**-axis** first, and then **stretching** the resulting graph **in the** $\boldsymbol{y}$**-axis** by a factor of $\mathbf{2}$.

b. The graph of $f(x) = (x+2)^2$ is a parabola with a vertex at $(-2, 0)$, and its arms are directed upwards as shown below in solid green.

x	$f(x) = (x+2)^2$
-4	4
-3	1
-2	0 → *vertex*
-1	1
0	4

symmetry about the line $x = -2$



Domain: $\mathbb{R}$
Range: $[0, \infty)$

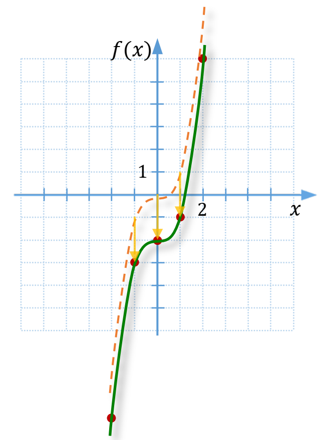
Observe that to obtain the solid green shape, it is enough to move the graph of the **basic parabola** by two units to the left. This transformation is called a **horizontal translation** by two units to the left. The translation to the left reflects the fact that the vertex of the parabola $f(x) = (x+2)^2$ is located at $x + 2 = 0$, which is equivalent to $x = -2$.

c. The graph of $f(x) = x^3 - 2$ has the shape of a basic cubic function with a center at $(0, -2)$.

x	$f(x) = x^3 - 2$
-2	-10
-1	-3
0	-2
1	-1
2	6

center

symmetry about $(0, -2)$



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Observe that the solid green graph can be obtained by shifting the graph of the **basic cubic function** by two units down. This transformation is called a **vertical translation** by two units down.

P.4 Exercises

1. *True or False?* The graph of $x^2 + 3$ is the same shape as a basic parabola with a vertex at $(3,0)$.

Graph *each function and state its* ***domain*** *and* ***range****.*

2. $f(x) = -2x + 3$ **3.** $f(x) = 3x - 4$ **4.** $f(x) = -x^2 + 4$

5. $f(x) = x^2 - 2$ **6.** $f(x) = \frac{1}{2}x^2$ **7.** $f(x) = -2x^2 + 1$

8. $f(x) = (x+1)^2 - 2$ **9.** $f(x) = -x^3 + 1$ **10.** $f(x) = (x-3)^3$

Guess the ***transformations*** *needed to apply to the graph of a basic parabola* $f(x) = x^2$ *to obtain the graph of the given function* $g(x)$*. Then* ***graph*** *both* $f(x)$ *and* $g(x)$ *on the same grid and confirm the original guess.*

11. $g(x) = -x^2$ **12.** $g(x) = x^2 - 3$ **13.** $g(x) = x^2 + 2$

14. $g(x) = (x+2)^2$ **15.** $g(x) = (x-3)^2$ **16.** $g(x) = (x+2)^2 - 1$

Attributions

p.**169** Roller Coaster in a Park by Priscilla Du Preez / Unsplash Licence