Quadratic Equations and Functions

In this chapter, we discuss various ways of solving quadratic equations, $ax^2 + bx + c = 0$, including equations quadratic in form, such as $x^{-2} + x^{-1} - 20 = 0$, and solving formulas for a variable that appears in the first and second power, such as k in $k^2 - 3k = 2N$. Frequently used strategies of solving quadratic equations include the **completing the square** procedure and its generalization in the form of the **quadratic formula**. Completing the square allows for rewriting quadratic functions in vertex form, $f(x) = a(x-h)^2 + k$, which is very useful for graphing as it provides information about the location, shape, and direction of the parabola.

In the second part of this chapter, we examine properties and graphs of quadratic functions, including basic transformations of these graphs.

Finally, these properties are used in solving application problems, particularly problems involving **optimization**.

Q1 Methods of Solving Quadratic Equations

As defined in *Section F4*, a quadratic equation is a second-degree polynomial equation in one variable that can be written in standard form as

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $\boldsymbol{a \neq 0}$. Such equations can be solved in many different ways, as presented below.

Solving by Graphing

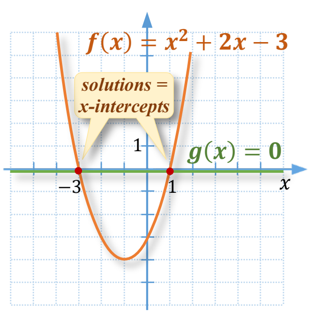


Figure 1.1

To solve a quadratic equation, for example $x^2 + 2x - 3 = 0$, we can consider its left side as a function $f(x) = x^2 + 2x - 3$ and the right side as a function $g(x) = 0$. To satisfy the original equation, both function values must be equal. After graphing both functions on the same grid, one can observe that this happens at points of intersection of the two graphs.

So the **solutions** to the original equation are the x-coordinates of the intersection points of the two graphs. In our example, these are the ***x*-intercepts** or the **roots** of the function $f(x) = x^2 + 2x - 3$, as indicated in *Figure 1.1*.

Thus, the solutions to $x^2 + 2x - 3 = 0$ are $\boldsymbol{x = -3}$ and $\boldsymbol{x = 1}$.

Note: Notice that the graphing method, although visually appealing, is not always reliable. For example, the solutions to the equation $49x^2 - 4 = 0$ are $x = \frac{2}{7}$ and $x = -\frac{2}{7}$. Such numbers would be very hard to read from the graph.

Thus, the graphing method is advisable to use when searching for integral solutions or estimations of solutions.

To find exact solutions, we can use one of the algebraic methods presented below.

Solving by Factoring

Many quadratic equations can be solved by factoring and employing the zero-product property, as in *Section F4*.

For example, the equation $x^2 + 2x - 3 = 0$ can be solved as follows:

$$(x + 3)(x - 1) = 0$$

so, by zero-product property,

$$x + 3 = 0 \text{ or } x - 1 = 0,$$

which gives us the solutions

$$\boldsymbol{x = -3} \text{ or } \boldsymbol{x = 1}.$$

Solving by Using the Square Root Property

Quadratic equations of the form $\boldsymbol{ax^2 + c = 0}$ can be solved by applying the **square root property**.

Square Root Property:

For any positive real number a, if $\boldsymbol{x^2 = a}$, then $\boldsymbol{x = \pm\sqrt{a}}$.
This is because $\sqrt{x^2} = |x|$. So, after applying the square root operator to both sides of the equation $x^2 = a$, we have

$$\sqrt{x^2} = \sqrt{a}$$
$$|x| = \sqrt{a}$$
$$x = \pm\sqrt{a}$$

The $\pm\sqrt{a}$ is a shorter recording of two solutions: $\sqrt{a}$ and $-\sqrt{a}$.

For example, the equation $49x^2 - 4 = 0$ can be solved as follows:

$$49x^2 - 4 = 0$$
$$49x^2 = 4$$
$$x^2 = \frac{4}{49}$$
$$\sqrt{x^2} = \sqrt{\frac{4}{49}}$$
$$x = \pm\sqrt{\frac{4}{49}}$$
$$\boldsymbol{x = \pm\frac{2}{7}}$$

apply square root to both sides of the equation

Here we use the **square root property**. Remember the $\pm$ sign!

Note: Using the square root property is a common solving strategy for quadratic equations where **one side is a perfect square** of an unknown quantity and the **other side is a constant** number.

Example 1 ► **Solve by the Square Root Property**

Solve each equation using the square root property.

a. $(x-3)^2 = 49$ **b.** $2(3x-6)^2 - 54 = 0$

Solution ►

a. Applying the square root property, we have

$$\sqrt{(x-3)^2} = \sqrt{49}$$
$$x - 3 = \pm 7$$
$$x = 3 \pm 7$$

so

$$\boldsymbol{x = 10} \text{ or } \boldsymbol{x = -4}$$

b. To solve $2(3x-6)^2 - 54 = 0$, we isolate the perfect square first and then apply the square root property. So,

$$2(3x-6)^2 - 54 = 0$$
$$(3x-6)^2 = \frac{54}{2}$$
$$\sqrt{(3x-6)^2} = \sqrt{27}$$
$$3x - 6 = \pm 3\sqrt{3}$$
$$3x = 6 \pm 3\sqrt{3}$$
$$x = \frac{6 \pm 3\sqrt{3}}{3}$$
$$x = \frac{\cancel{3}(2 \pm \sqrt{3})}{\cancel{3}}$$
$$x = 2 \pm \sqrt{3}$$

Thus, the solution set is $\{\boldsymbol{2 - \sqrt{3}},\ \boldsymbol{2 + \sqrt{3}}\}$.

Caution: To simplify expressions such as $\frac{6+3\sqrt{3}}{3}$, we **factor the numerator** first. The common errors to avoid are

incorrect order of operations ← $\frac{6+3\sqrt{3}}{3} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$

or

incorrect canceling ← $\frac{6+\cancel{3}\sqrt{3}}{\cancel{3}} = 6 + \sqrt{3}$

or

incorrect canceling ← $\frac{\overset{2}{\cancel{6}}+3\sqrt{3}}{\cancel{3}} = 2 + 3\sqrt{3}$

Solving by Completing the Square

So far, we have seen how to solve quadratic equations, $ax^2 + bx + c = 0$, if the expression $ax^2 + bx + c$ is factorable or if the coefficient b is equal to zero. To solve other quadratic equations, we may try to rewrite the variable terms in the form of a perfect square, so that the resulting equation can be solved by the square root property.

For example, to solve $x^2 + 6x - 3 = 0$, we observe that the variable terms $x^2 + 6x$ could be written in **perfect square** form if we add 9, as illustrated in *Figure 1.2*. This is because

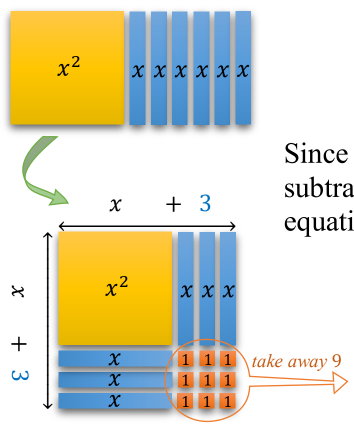


Figure 1.2

$$x^2 + 6x + 9 = (x+3)^2$$

observe that 3 comes from taking half of 6

Since the original equation can only be changed to an equivalent form, if we add 9, we must subtract 9 as well. *(Alternatively, we could add 9 to both sides of the equation.)* So, the equation can be transformed as follows:

$$x^2 + 6x - 3 = 0$$

Completing the Square Procedure

$$\underbrace{x^2 + 6x + 9}_{perfect\ square} - 9 - 3 = 0$$

$$(x+3)^2 = 12$$

$$\sqrt{(x+3)^2} = \sqrt{12}$$

square root property

$$x + 3 = \pm 2\sqrt{3}$$

$$x = \mathbf{-3 \pm 2\sqrt{3}}$$

Generally, to **complete the square** for the first two terms of the equation

$$\boldsymbol{x^2 + bx + c = 0},$$

we take half of the x-coefficient, which is $\frac{b}{2}$, and square it. Then, we add and subtract that number, $\left(\frac{b}{2}\right)^2$. *(Alternatively, we could add $\left(\frac{b}{2}\right)^2$ to both sides of the equation.)*
This way, we produce an equivalent equation

$$\boldsymbol{x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0},$$

and consequently,

$$\boldsymbol{\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0}.$$

We can write this equation directly, by following the rule:
Write the sum of x and **half of the middle coefficient**, **square the binomial**, and **subtract the perfect square of the constant** appearing in the bracket.

To **complete the square** for the first two terms of a quadratic equation with a leading coefficient of $a \neq 1$,

$$\boldsymbol{ax^2 + bx + c = 0},$$

we

- divide the equation by $\boldsymbol{a}$ *(alternatively, we could factor a out of the first two terms)* so that the leading coefficient is 1, and then
- complete the square as in the previous case, where $\boldsymbol{a = 1}$.

So, after division by $\boldsymbol{a}$, we obtain

$$\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}.$$

Since half of $\frac{b}{a}$ is $\frac{b}{2a}$, then we complete the square as follows:

$$\boldsymbol{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0}.$$

Remember to **subtract the perfect square of the constant** appearing in the bracket!

Example 2 ► **Solve by Completing the Square**

Solve each equation using the completing the square method.

a. $x^2 + 5x - 1 = 0$ **b.** $3x^2 - 12x - 5 = 0$

Solution ► **a.** First, we complete the square for $x^2 + 5x$ by adding and subtracting $\left(\frac{5}{2}\right)^2$ and then we apply the square root property. So, we have

$$\underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2}_{perfect\ square} - \left(\frac{5}{2}\right)^2 - 1 = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 1 \cdot \frac{4}{4} = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{29}{4} = 0$$

apply square root to both sides of the equation

$$\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$$

remember to use the $\pm$ sign!

$$x + \frac{5}{2} = \pm\sqrt{\frac{29}{4}}$$

$$x + \frac{5}{2} = \pm\frac{\sqrt{29}}{2}$$

$$x = \frac{-5 \pm \sqrt{29}}{2}$$

Thus, the solution set is $\left\{\frac{-5-\sqrt{29}}{2}, \frac{-5+\sqrt{29}}{2}\right\}$.

Note: Unless specified otherwise, we are expected to state the **exact solutions** rather than their calculator approximations. Sometimes, however, especially when solving application problems, we may need to use a calculator to approximate the solutions. The reader is encouraged to check that the two decimal **approximations** of the above solutions are

$$\frac{-5-\sqrt{29}}{2} \approx -5.19 \quad \text{and} \quad \frac{-5+\sqrt{29}}{2} \approx 0.19$$

b. In order to apply the strategy as in the previous example, we divide the equation by the leading coefficient, 3. So, we obtain

$$3x^2 - 12x - 5 = 0$$

$$x^2 - 4x - \frac{5}{3} = 0$$

Then, to complete the square for $x^2 - 4x$, we may add and subtract 4. This allows us to rewrite the equation equivalently, with the variable part in perfect square form.

$$(x-2)^2 - 4 - \frac{5}{3} = 0$$

$$(x-2)^2 = 4 \cdot \frac{3}{3} + \frac{5}{3}$$

$$(x-2)^2 = \frac{17}{3}$$

$$x - 2 = \pm\sqrt{\frac{17}{3}}$$

$$x = 2 \pm \frac{\sqrt{17}}{\sqrt{3}}$$

Note: The final answer could be written as a single fraction as shown below:

$$x = \frac{2\sqrt{3} \pm \sqrt{17}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6 \pm \sqrt{51}}{3}$$

Solving with Quadratic Formula

Applying the completing the square procedure to the quadratic equation

$$ax^2 + bx + c = 0,$$

with real coefficients $a \neq 0$, b, and c, allows us to develop a general formula for finding the solution(s) to any such equation.

Quadratic Formula ► The solution(s) to the equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are real coefficients, are given by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $x_{1,2}$ denotes the two solutions, $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Proof: ► First, since $a \neq 0$, we can divide the equation $ax^2 + bx + c = 0$ by a. So, the equation to solve is

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then, we complete the square for $x^2 + \frac{b}{a}x$ by adding and subtracting the perfect square of half of the middle coefficient, $\left(\frac{b}{2a}\right)^2$. So, we obtain

$$\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2}_{perfect\ square} - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and finally,

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which concludes the proof.

Example 3 ► **Solving Quadratic Equations with the Use of the Quadratic Formula**

Using the Quadratic Formula, solve each equation, if possible. Then visualize the solutions graphically.

a. $2x^2 + 3x - 20 = 0$ **b.** $3x^2 - 4 = 2x$ **c.** $x^2 - \sqrt{2}x + 3 = 0$

Solution ► **a.** To apply the quadratic formula, first, we identify the values of a, b, and c. Since the equation is in standard form, $a = 2$, $b = 3$, and $c = -20$. The solutions are equal to

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2(-20)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{9 + 160}}{4}$$

$$= \frac{-3 \pm 13}{4} = \begin{cases} \frac{-3 + 13}{4} = \frac{10}{4} = \mathbf{\frac{5}{2}} \\ \frac{-3 - 13}{4} = \frac{-16}{4} = \mathbf{-4} \end{cases}$$

Thus, the solution set is $\left\{\mathbf{-4}, \mathbf{\frac{5}{2}}\right\}$.

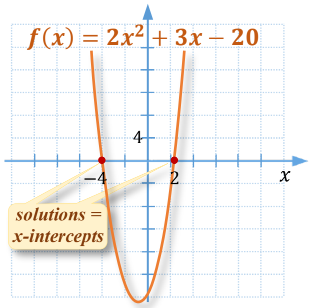


Figure 1.3

These solutions can be seen as x-intercepts of the function $f(x) = 2x^2 + 3x - 20$, as shown in *Figure 1.3.*

b. Before we identify the values of a, b, and c, we need to write the given equation $3x^2 - 4 = 2x$ in standard form. After subtracting $2x$ from both sides of the given equation, we obtain

$$3x^2 - 2x - 4 = 0$$

Since $a = 3$, $b = -2$, and $c = -4$, we evaluate the quadratic formula,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3(-4)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

$$= \frac{2 \pm \sqrt{4 \cdot 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{2(1 \pm \sqrt{13})}{6} = \frac{\mathbf{1 \pm \sqrt{13}}}{\mathbf{3}}$$

simplify by factoring

So, the solution set is $\left\{\mathbf{\frac{1-\sqrt{13}}{3}}, \mathbf{\frac{1+\sqrt{13}}{3}}\right\}$.

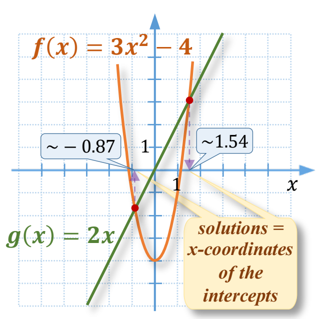


Figure 1.4

We may visualize solutions to the original equation, $3x^2 - 4 = 2x$, by graphing functions $f(x) = 3x^2 - 4$ and $g(x) = 2x$. The x-coordinates of the intersection points are the solutions to the equation $f(x) = g(x)$, and consequently to the original equation. As indicated in *Figure 1.4*, the approximations of these solutions are $\frac{1-\sqrt{13}}{3} \approx -0.87$ and $\frac{1+\sqrt{13}}{3} \approx 1.54$.

c. Substituting $a = 1$, $b = -\sqrt{2}$, and $c = 3$ into the Quadratic Formula, we obtain

$$x_{1,2} = \frac{\sqrt{2} \pm \sqrt{\left(-\sqrt{2}\right)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{\sqrt{2} \pm \sqrt{2 - 12}}{2} = \frac{\sqrt{2} \pm \sqrt{-10}}{2}$$

not a real number!

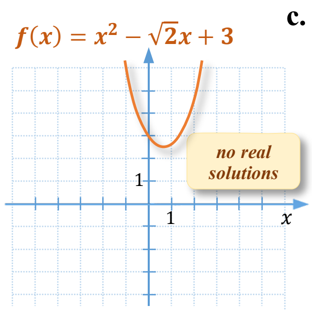


Figure 1.5

Since a square root of a negative number is not a real value, we have **no real solutions**. In a graphical representation, this means that the graph of the function $f(x) = x^2 - \sqrt{2}x + 3$ never equals 0 and therefore does not cross the x-axis. See *Figure 1.5.*

Although there are no real solutions to the equation, there are complex solutions that can be simplified as in *Section RD6*:

$$x_{1,2} = \frac{\sqrt{2} \pm \sqrt{10}\, i}{2}$$

Observation: Notice that we could find information about the solutions in *Example 3c* just by evaluating the radicand $b^2 - 4ac$. Since this radicand was negative, we concluded that there was no real solution to the given equation as a root of a negative number is not a real number. There was no need to evaluate the whole Quadratic Formula to determine the nature of the solutions.

So, the radicand in the Quadratic Formula carries important information about the number and nature of roots. Because of it, this radicand earned a special name, the discriminant.

Definition 1.1 ► The radicand $\boldsymbol{b^2 - 4ac}$ in the Quadratic Formula is called the **discriminant** and it is denoted by Δ.

Notice that in terms of Δ, the Quadratic Formula takes the form

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Observing the behaviour of the expression $\sqrt{\Delta}$ allows us to classify the number and type of solutions (roots) of a quadratic equation with rational coefficients.

Characteristics of Roots (Solutions) Depending on the Discriminant

Suppose $ax^2 + bx + c = 0$ has **rational** coefficients $a \neq 0$, b, c, and $\Delta = b^2 - 4ac$.

- If $\Delta < \mathbf{0}$, then the equation has **two complex conjugate solutions,** $\frac{-b-\sqrt{|\Delta|}\, i}{2a}$ and $\frac{-b+\sqrt{|\Delta|}\, i}{2a}$, as $\sqrt{negative}$ is an imaginary number.
- If $\Delta = \mathbf{0}$, then the equation has **one rational solution,** $\frac{-b}{2a}$.
- If $\Delta > \mathbf{0}$, then the equation has **two real solutions,** $\frac{-b-\sqrt{\Delta}}{2a}$ and $\frac{-b+\sqrt{\Delta}}{2a}$.
  These solutions are
  - **irrational**, if Δ is **not** a **perfect square number**
  - **rational**, if Δ is a **perfect square number** (as $\sqrt{perfect\ square} = integer$)

  In addition, if $\Delta \geq \mathbf{0}$ is a **perfect square number**, then the equation could be solved by **factoring**.

Example 4 ► **Determining the Number and Type of Solutions of a Quadratic Equation**

Using the discriminant, determine the number and type of solutions of each equation without solving the equation. If the equation can be solved by factoring, show the factored form of the trinomial.

a. $2x^2 + 7x - 15 = 0$

b. $4x^2 - 12x + 9 = 0$

c. $3x^2 - x + 1 = 0$

d. $2x^2 - 7x + 2 = 0$

Solution ►

a. $\Delta = 7^2 - 4 \cdot 2 \cdot (-15) = 49 + 120 = 169$

Since 169 is a perfect square number, the equation has **two rational solutions** and it can be solved by factoring. Indeed, $2x^2 + 7x - 15 = (2x - 3)(x + 5)$.

b. $\Delta = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$

$\Delta = 0$ indicates that the equation has **one rational solution** and it can be solved by factoring. Indeed, the expression $4x^2 - 12x + 9$ is a perfect square, $(2x - 3)^2$.

c. $\Delta = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11$

Since $\Delta < 0$, the equation has **two complex conjugate solutions** and therefore it cannot be solved by factoring.

d. $\Delta = (-7)^2 - 4 \cdot 2 \cdot 2 = 49 - 16 = 33$

Since $\Delta > 0$ but is not a perfect square number, the equation has **two real solutions** but it cannot be solved by factoring.

Example 5 ► **Solving Equations Equivalent to Quadratic**

Solve each equation.

a. $2 + \frac{7}{x} = \frac{5}{x^2}$

b. $1 - 2x^2 = (x + 2)(x - 1)$

Solution ►

a. This is a rational equation, with the set of $\mathbb{R} \setminus \{0\}$ as its domain. To solve it, we multiply the equation by the $LCD = x^2$. This brings us to a quadratic equation

$$2x^2 + 7x = 5$$

or equivalently

$$2x^2 + 7x - 5 = 0,$$

which can be solved by following the Quadratic Formula for $a = 2$, $b = 7$, and $c = -5$. So, we have

$$x_{1,2} = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 2(-5)}}{2 \cdot 2} = \frac{-7 \pm \sqrt{49 + 40}}{4} = \frac{\mathbf{-7 \pm \sqrt{89}}}{\mathbf{4}}$$

Since both solutions are in the domain, the solution set is $\left\{\frac{\mathbf{-7-\sqrt{89}}}{\mathbf{4}}, \frac{\mathbf{-7+\sqrt{89}}}{\mathbf{4}}\right\}$.

b. To solve $1 - 2x^2 = (x + 2)(x - 1)$, we simplify the equation first and rewrite it in standard form. So, we have

$$1 - 2x^2 = x^2 + x - 2$$
$$-3x^2 - x + 3 = 0$$
$$3x^2 + x - 3 = 0$$

Since the left side of this equation is not factorable, we may use the Quadratic Formula. So, the solutions are

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3(-3)}}{2 \cdot 3} = \frac{-1 \pm \sqrt{1 + 36}}{6} = \frac{\mathbf{-1 \pm \sqrt{37}}}{\mathbf{6}}.$$

Q.1 Exercises

True or False.

1. A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are any real numbers.

2. If the graph of $f(x) = ax^2 + bx + c$ intersects the x-axis twice, the equation $ax^2 + bx + c = 0$ has two solutions.

3. If the equation $ax^2 + bx + c = 0$ has no real solution, the graph of $f(x) = ax^2 + bx + c$ does not intersect the x-axis.

4. The Quadratic Formula cannot be used to solve the equation $x^2 - 5 = 0$ because the equation does not contain a linear term.

5. The solution set for the equation $x^2 = 16$ is $\{4\}$.

6. To complete the square for $x^2 + bx$, we add $\left(\frac{b}{2}\right)^2$.

7. If the discriminant is positive, the equation can be solved by factoring.

For each function f,

a) *graph $f(x)$ using a table of values;*

b) *find the x-intercepts of the graph;*

c) *solve the equation $f(x) = 0$ by factoring and compare these solutions to the x-intercepts of the graph.*

8. $f(x) = -x^2 - 4x - 3$

9. $f(x) = x^2 + 2x - 3$

10. $f(x) = 3x + x(x - 2)$

11. $f(x) = 2x - x(x - 3)$

12. $f(x) = 4x^2 - 4x - 3$

13. $f(x) = -\frac{1}{4}(2x^2 + 5x - 12)$

Solve each equation using the ***square root property****.*

14. $x^2 = 49$

15. $x^2 = 32$

16. $a^2 - 50 = 0$

17. $n^2 - 24 = 0$

18. $3x^2 - 72 = 0$

19. $5y^2 - 200 = 0$

20. $(x - 4)^2 = 64$

21. $(x + 3)^2 = 16$

22. $(3n - 1)^2 = 7$

23. $(5t + 2)^2 = 12$

24. $x^2 - 10x + 25 = 45$

25. $y^2 + 8y + 16 = 44$

26. $4a^2 + 12a + 9 = 32$

27. $25(y - 10)^2 = 36$

28. $16(x + 4)^2 = 81$

29. $(4x + 3)^2 = -25$

30. $(3n - 2)(3n + 2) = -5$

31. $2x - 1 = \frac{18}{2x-1}$

Solve each equation using the ***completing the square*** *procedure.*

32. $x^2 + 12x = 0$

33. $y^2 - 3y = 0$

34. $x^2 - 8x + 2 = 0$

35. $n^2 + 7n = 3n - 4$

36. $p^2 - 4p = 4p - 16$

37. $y^2 + 7y - 1 = 0$

38. $2x^2 - 8x = -4$

39. $3a^2 + 6a = -9$

40. $3y^2 - 9y + 15 = 0$

41. $5x^2 - 60x + 80 = 0$

42. $2t^2 + 6t - 10 = 0$

43. $3x^2 + 2x - 2 = 0$

44. $2x^2 - 16x + 25 = 0$

45. $9x^2 - 24x = -13$

46. $25n^2 - 20n = 1$

47. $x^2 - \frac{4}{3}x = -\frac{1}{9}$

48. $x^2 + \frac{5}{2}x = -1$

49. $x^2 - \frac{2}{5}x - 3 = 0$

In problems ***50-51****, find all values of x such that $f(x) = g(x)$ for the given functions f and g.*

50. $f(x) = x^2 - 9$ and $g(x) = 4x - 6$

51. $f(x) = 2x^2 - 5x$ and $g(x) = -x + 14$

52. Explain the errors in the following solutions of the equation $5x^2 - 8x + 2 = 0$:

a. $x = \frac{8 \pm \sqrt{-8^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 5} = \frac{8 \pm \sqrt{64 - 40}}{10} = \frac{8 \pm \sqrt{24}}{10} = \frac{8 \pm 2\sqrt{6}}{10} = \frac{4}{5} \pm 2\sqrt{6}$

b. $x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 5} = \frac{8 \pm \sqrt{64 - 40}}{10} = \frac{8 \pm \sqrt{24}}{10} = \frac{8 \pm 2\sqrt{6}}{10} = \begin{cases} \frac{10\sqrt{6}}{10} = \sqrt{6} \\ \frac{6\sqrt{6}}{10} = \frac{3\sqrt{6}}{5} \end{cases}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve each equation with the aid of the ***Quadratic Formula****, if possible. Illustrate your solutions graphically, using a table of values.*

53. $x^2 + 3x + 2 = 0$

54. $y^2 - 2 = y$

55. $x^2 + x = -3$

56. $2y^2 + 3y = -2$

57. $x^2 - 8x + 16 = 0$

58. $4n^2 + 1 = 4n$

Solve each equation with the aid of the ***Quadratic Formula****. Give the* ***exact*** *and* ***approximate*** *solutions up to two decimal places.*

59. $a^2 - 4 = 2a$

60. $2 - 2x = 3x^2$

61. $0.2x^2 + x + 0.7 = 0$

62. $2t^2 - 4t + 2 = 3$

63. $y^2 + \frac{y}{3} = \frac{1}{6}$

64. $\frac{x^2}{4} - \frac{x}{2} = 1$

65. $5x^2 = 17x - 2$

66. $15y = 2y^2 + 16$

67. $6x^2 - 8x = 2x - 3$

Use the discriminant to determine the ***number and type of solutions*** *for each equation. Also, without solving, decide whether the equation can be solved by* ***factoring*** *or whether the quadratic formula should be used.*

68. $3x^2 - 5x - 2 = 0$

69. $4x^2 = 4x + 3$

70. $x^2 + 3 = -2\sqrt{3}x$

71. $4y^2 - 28y + 49 = 0$

72. $3y^2 - 10y + 15 = 0$

73. $9x^2 + 6x = -1$

In problems 74-76, find all values of constant k, so that each equation will have ***exactly one*** *rational solution.*

74. $x^2 + kx + 49 = 0$

75. $9y^2 - 30y + k = 0$

76. $kx^2 + 8x + 1 = 0$

77. Suppose that one solution of a quadratic equation with integral coefficients is irrational. Assuming that the equation has two solutions, can the other solution be a rational number? Justify your answer.

Solve each equation using any algebraic method. State the solutions in their exact form.

78. $-2x(x + 2) = -3$

79. $(x + 2)(x - 4) = 1$

80. $(x + 2)(x + 6) = 8$

81. $(2x - 3)^2 = 8(x + 1)$

82. $(3x + 1)^2 = 2(1 - 3x)$

83. $2x^2 - (x + 2)(x - 3) = 12$

84. $(x - 2)^2 + (x + 1)^2 = 0$

85. $1 + \frac{2}{x} + \frac{5}{x^2} = 0$

86. $x = \frac{2(x+3)}{x+5}$

87. $2 + \frac{1}{x} = \frac{3}{x^2}$

88. $\frac{3}{x} + \frac{x}{3} = \frac{5}{2}$

89. $\frac{1}{x} + \frac{1}{x+4} = \frac{1}{7}$

Q2 Applications of Quadratic Equations

Some polynomial, rational or even radical equations are **quadratic in form.** As such, they can be solved using techniques described in the previous section. For instance, the rational equation $\frac{1}{x^2}+\frac{1}{x}-6=0$ is quadratic in form because if we replace $\frac{1}{x}$ with a single variable, say a, then the equation becomes quadratic, $a^2+a-6=0$. In this section, we explore applications of quadratic equations in solving equations quadratic in form as well as solving formulas containing variables in the second power.

We also revisit application problems that involve solving quadratic equations. Some of the application problems that are typically solved with the use of quadratic or polynomial equations were discussed in *Sections F4* and *RT6*. However, in the previous sections, the equations used to solve such problems were all possible to solve by factoring. In this section, we include problems that require the use of methods other than factoring.

Equations Quadratic in Form

Definition 2.1 ► A nonquadratic equation is referred to as **quadratic in form** or **reducible to quadratic** if it can be written in the form

$$a\boldsymbol{u}^2+b\boldsymbol{u}+c=0,$$

where $a \neq 0$ and $\boldsymbol{u}$ represents any *algebraic expression*.

Equations quadratic in form are usually easier to solve by using strategies for solving the related quadratic equation $a\boldsymbol{u}^2+b\boldsymbol{u}+c=0$ for the expression $\boldsymbol{u}$, and then solving for the original variable, as shown in the example below.

Example 1 ► **Solving Equations Quadratic in Form**

Solve each equation.

a. $(x^2-1)^2-(x^2-1)=2$

b. $x-3\sqrt{x}=10$

c. $\frac{1}{(a+2)^2}+\frac{1}{a+2}-6=0$

Solution ► **a.** First, observe that the expression x^2-1 appears in the given equation in the first and second power. So, it may be useful to replace x^2-1 with a new variable, for example u. After this substitution, the equation becomes quadratic,

$$u^2-u=2,$$

and can be solved via factoring

$$u^2-u-2=0$$

$$(u-2)(u+1)=0$$

$$u=2 \text{ or } u=-1$$

This can be any letter, as long as it is different than the original variable.

Since we need to solve the original equation for x, not for u, we replace u back with x^2-1. This gives us

$$x^2 - 1 = 2 \quad \text{or} \quad x^2 - 1 = -1$$

$$x^2 = 3 \quad \text{or} \quad x^2 = 0$$

$$x = \pm\sqrt{3} \quad \text{or} \quad x = 0$$

Thus, the solution set is $\{-\sqrt{3}, \mathbf{0}, \sqrt{3}\}$.

b. If we replace $\sqrt{x}$ with, for example, a, then $x = a^2$, and the equation becomes

$$a^2 - 3a = 10,$$

which can be solved by factoring

$$a^2 - 3a - 10 = 0$$

$$(a + 2)(a - 5) = 0$$

$$a = -2 \quad \text{or} \quad a = 5$$

After replacing a back with $\sqrt{x}$, we have

$$\sqrt{x} = -2 \quad \text{or} \quad \sqrt{x} = 5.$$

The first equation, $\sqrt{x} = -2$, does not give us any solution as the square root cannot be negative. After squaring both sides of the second equation, we obtain $x = 25$. So, the solution set is $\{\mathbf{25}\}$.

c. The equation $\frac{1}{(a+2)^2} + \frac{1}{a+2} - 6 = 0$ can be solved as any other rational equation, by clearing the denominators via multiplying by the $LCD = (a + 2)^2$. However, it can also be seen as a quadratic equation as soon as we replace $\frac{1}{a+2}$ with, for example, x. By doing so, we obtain

$$x^2 + x - 6 = 0,$$

which after factoring

$$(x + 3)(x - 2) = 0,$$

gives us

$$x = -3 \quad \text{or} \quad x = 2$$

Remember to use a different letter than the variable in the original equation.

Again, since we need to solve the original equation for a, we replace x back with $\frac{1}{a+2}$. This gives us

$$\frac{1}{a+2} = -3 \quad \text{or} \quad \frac{1}{a+2} = 2$$

take the reciprocal of both sides

$$a + 2 = \frac{1}{-3} \quad \text{or} \quad a + 2 = \frac{1}{2}$$

$$a = -\frac{7}{3} \quad \text{or} \quad a = -\frac{3}{2}$$

Since both values are in the domain of the original equation, which is $\mathbb{R} \setminus \{0\}$, then the solution set is $\left\{-\frac{7}{3}, -\frac{3}{2}\right\}$.

Solving Formulas

When solving formulas for a variable that appears in the second power, we use the same strategies as in solving quadratic equations. For example, we may use the square root property or the quadratic formula.

Example 2 ► **Solving Formulas for a Variable that Appears in the Second Power**

Solve each formula for the given variable.

a. $E = mc^2$, for c **b.** $N = \frac{k^2-3k}{2}$, for k

Solution ► **a.** To solve for c, first, we reverse the multiplication by m via the division by m. Then, we reverse the operation of squaring by taking the square root of both sides of the equation.

$$E = mc^2$$

$$\frac{E}{m} = c^2$$

Then, we reverse the operation of squaring by taking the square root of both sides of the equation. So, we have

$$\sqrt{\frac{E}{m}} = \sqrt{c^2},$$

and therefore

$$\boldsymbol{c = \pm\sqrt{\frac{E}{m}}}$$

Remember that $\sqrt{c^2} = |c|$, so we use the $\pm$ sign in place of $|\ |$.

b. Observe that the variable k appears in the formula $N = \frac{k^2-3k}{2}$ in two places. Once in the first and once in the second power of k. This means that we can treat this formula as a quadratic equation with respect to k and solve it with the aid of the quadratic formula. So, we have

$$N = \frac{k^2 - 3k}{2}$$

$$2N = k^2 - 3k$$

$$\boldsymbol{k}^2 - 3\boldsymbol{k} - 2N = 0$$

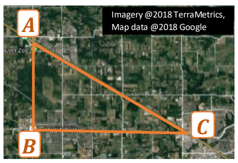
Substituting $a = 1$, $b = -3$, and $c = -2N$ to the quadratic formula, we obtain

$$\boldsymbol{k_{1,2}} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-2N)}}{2} = \boldsymbol{\frac{3 \pm \sqrt{9 + 8N}}{2}}$$

Application Problems

Many application problems require solving quadratic equations. Sometimes this can be achieved via factoring, but often it is helpful to use the quadratic formula.

Example 3 ► **Solving a Distance Problem with the Aid of the Quadratic Formula**



Three towns A, B, and C are positioned as shown in the accompanying figure. The roads at B form a right angle. The towns A and C are connected by a straight road as well. The distance from A to B is 7 kilometers less than the distance from B to C. The distance from A to C is 20 km. Approximate the remaining distances between the towns up to the tenth of a kilometer.

Solution ► Since the roads between towns form a right triangle, we can employ the Pythagorean equation

$$AC^2 = AB^2 + BC^2$$

Suppose that $BC = x$. Then $AB = x - 7$, and we have

$$20^2 = (x-7)^2 + x^2$$

$$400 = x^2 - 14x + 49 + x^2$$

$$2x^2 - 14x - 351 = 0$$

Applying the quadratic formula, we obtain

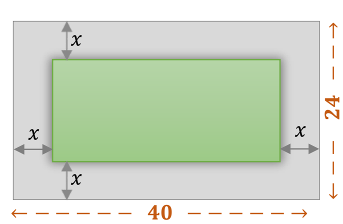
$$x_{1,2} = \frac{14 \pm \sqrt{14^2 + 4 \cdot 2 \cdot 351}}{4} = \frac{14 \pm \sqrt{196 + 2808}}{4} = \frac{14 \pm \sqrt{3004}}{4} \approx 17.2 \text{ } or \text{ } \cancel{-10.2}$$

Since x represents a distance, it must be positive. So, the only solution is $x \approx 17.2$ km. Thus, the distance $\boldsymbol{BC} \approx \mathbf{17.2}$ **km** and hence $\boldsymbol{AB} \approx 17.2 - 7 = \mathbf{10.2}$ **km**.

Example 4 ► **Solving a Geometry Problem with the Aid of the Quadratic Formula**

A city designated a 24 m by 40 m rectangular area for a playground with a sidewalk of uniform width around it. The playground itself is using $\frac{2}{3}$ of the original rectangular area. To the nearest centimeter, what is the width of the sidewalk?

Solution ► To visualize the situation, we may draw a diagram as below.



Suppose x represents the width of the sidewalk. Then, the area of the playground (the green area) can be expressed as $(40-2x)(24-2x)$. Since the green area is $\frac{2}{3}$ of the original rectangular area, we can form the equation

$$(40-2x)(24-2x)=\frac{2}{3}(40\cdot 24)$$

To solve it, we may want to lower the coefficients by dividing both sides of the equation by 4 first. This gives us

$$\frac{2(20-x)\cdot 2(12-x)}{4}=\frac{2}{3}\cdot\frac{(40\cdot 24)}{4}$$

$$(20-x)(12-x)=160$$

$$240-32x+x^2=160$$

$$x^2-32x+80=0,$$

which can be solved using the Quadratic Formula:

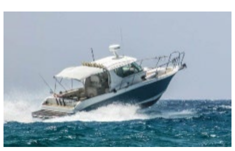
$$x_{1,2}=\frac{32\pm\sqrt{(-32)^2-4\cdot 80}}{2}=\frac{32\pm\sqrt{704}}{2}\approx\frac{32\pm 8\sqrt{11}}{2}$$

$$=16\pm 4\sqrt{11}\approx\begin{cases}29.27\\ 2.73\end{cases}$$

The width x must be smaller than 12, so this value is too large to be considered.

Thus, the sidewalk is approximately **2.73** meters wide.

Example 5 ► **Solving a Motion Problem That Requires the Use of the Quadratic Formula**



The Columbia River flows at a rate of 2 mph for the length of a popular boating route. In order for a boat to travel 3 miles upriver and return in a total of 4 hours, how fast must the boat be able to travel in still water?

Solution ► Suppose the rate of the boat moving in still water is r. Then, $r-2$ represents the rate of the boat moving upriver and $r+2$ represents the rate of the boat moving downriver. We can arrange these data in the table below.

	R ·	T	$= D$
upriver	$r-2$	$\frac{3}{r-2}$	3
downriver	$r+2$	$\frac{3}{r+2}$	3
total		4	

We fill the time-column by following the formula $T=\frac{D}{R}$.

By adding the times needed for traveling upriver and downriver, we form the rational equation

$$\frac{3}{r-2}+\frac{3}{r+2}=4,$$

which, after multiplying by the $LCD=r^2-4$, becomes a quadratic equation.

$$3(r+2)+3(r-2)=4(r^2-4)$$

$$3r + 6 + 3r - 6 = 4r^2 - 16$$
$$0 = 4r^2 - 6r - 16$$
$$0 = 2r^2 - 3r - 8$$

Using the Quadratic Formula, we have

$$r_{1,2} = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot (-8)}}{2 \cdot 2} = \frac{3 \pm \sqrt{9 + 64}}{4} = \frac{3 \pm \sqrt{73}}{4} \approx \begin{cases} 2.9 \\ \cancel{-1.4} \end{cases}$$

Since the rate cannot be negative, the boat moves in still water at approximately **2.9 mph**.

Example 6 ► **Solving a Work Problem That Requires the Use of the Quadratic Formula**

Krista and Joanna work in the same office. Krista can file the daily office documents in 1 hour less time than Joanna can. Working together, they can do the job in 1 hr 45 min. To the nearest minute, how long would it take each person working alone to file these documents?

Solution ► Suppose the time needed for Joanna to complete the job is t, in hours. Then, $t - 1$ represents the time needed for Krista to complete the same job. Since we keep time in hours, we need to convert 1 hr 45 min into $1\frac{3}{4}$ hr $= \frac{7}{4}$ hr. Now, we can arrange the given data in a table, as below.

	R ·	T	$= Job$
Joanna	$\frac{1}{t}$	t	1
Krista	$\frac{1}{t-1}$	$t - 1$	1
together	$\frac{4}{7}$	$\frac{7}{4}$	1

We fill the rate-column by following the formula $R = \frac{Job}{T}$.

By adding the rates of work for each person, we form the rational equation

$$\frac{1}{t} + \frac{1}{t-1} = \frac{4}{7},$$

which, after multiplying by the $LCD = 7t(t-1)$, becomes a quadratic equation.

$$7(t-1) + 7t = 4(t^2 - t)$$
$$7t - 7 + 7t = 4t^2 - 4t$$
$$0 = 4t^2 - 18t + 7$$

Using the Quadratic Formula, we have

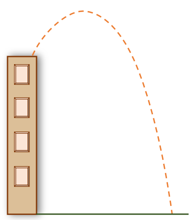
$$t_{1,2} = \frac{18 \pm \sqrt{(-18)^2 - 4 \cdot 4 \cdot 7}}{2 \cdot 4} = \frac{18 \pm \sqrt{212}}{8} = \frac{18 \pm 2\sqrt{53}}{8} = \frac{9 \pm \sqrt{53}}{4} \approx \begin{cases} 4.07 \\ \cancel{0.43} \end{cases}$$

Since the time needed for Joanna cannot be shorter than 1 hr, we reject the 0.43 possibility. So, working alone, **Joanna** requires approximately 4.07 hours ≈ **4 hours 4 minutes**, while **Krista** can do the same job in approximately 3.07 hours ≈ **3 hours 4 minutes**.

Example 7 ► **Solving a Projectile Problem Using a Quadratic Function**

A ball is projected upward from the top of a 96-ft building at 32 ft/sec. Its height above the ground, h, in feet, can be modelled by the function $h(t) = -16t^2 + 32t + 96$, where t is the time in seconds after the ball was projected. To the nearest tenth of a second, when does the ball hit the ground?

Solution ►



The ball hits the ground when its height h above the ground is equal to zero. So, we look for the solutions to the equation

$$h(t) = 0$$

which is equivalent to

$$-16t^2 + 32t + 96 = 0$$

Before applying the Quadratic Formula, we may want to lower the coefficients by dividing both sides of the equation by -16. So, we have

$$t^2 - 2t - 6 = 0$$

and

$$t_{1,2} = \frac{2 \pm \sqrt{(-2)^2 + 4 \cdot 6}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7} \approx \begin{cases} 3.6 \\ \cancel{-1.6} \end{cases}$$

Thus, the ball hits the ground in about **3.6 seconds**.

Q.2 Exercises

1. Discuss the validity of the following solution to the equation $\left(\frac{1}{x-2}\right)^2 - \frac{1}{x-2} - 2 = 0$:

Since this equation is quadratic in form, we solve the related equation $a^2 - a - 2 = 0$ by factoring

$$(a - 2)(a + 1) = 0.$$

The possible solutions are $a = 2$ and $a = -1$. Since 2 is not in the domain of the original equation, the solution set as $\{-1\}$.

Solve each equation by treating it as a quadratic in form.

2. $x^4 - 6x^2 + 9 = 0$

3. $x^8 - 29x^4 + 100 = 0$

4. $x - 10\sqrt{x} + 9 = 0$

5. $2x - 9\sqrt{x} + 4 = 0$

6. $y^{-2} - 5y^{-1} - 36 = 0$

7. $2a^{-2} + a^{-1} - 1 = 0$

8. $\left(1+\sqrt{t}\right)^2+\left(1+\sqrt{t}\right)-6=0$

9. $\left(2+\sqrt{x}\right)^2-3\left(2+\sqrt{x}\right)-10=0$

10. $(x^2+5x)^2+2(x^2+5x)-24=0$

11. $(t^2-2t)^2-4(t^2-2t)+3=0$

12. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}-5=0$

13. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}-8=0$

14. $1-\frac{1}{2p+1}-\frac{1}{(2p+1)^2}=0$

15. $\frac{2}{(u+2)^2}+\frac{1}{u+2}=3$

16. $\left(\frac{x+3}{x-3}\right)^2-\left(\frac{x+3}{x-3}\right)=6$

17. $\left(\frac{y^2-1}{y}\right)^2-4\left(\frac{y^2-1}{y}\right)-12=0$

In problems 23-40, solve each formula for the indicated variable.

18. $F=\frac{mv^2}{r}$, for v

19. $V=\pi r^2h$, for r

20. $A=4\pi r^2$, for r

21. $V=\frac{1}{3}s^2h$, for s

22. $F=\frac{Gm_1m_2}{r^2}$, for r

23. $N=\frac{kq_1q_2}{s^2}$, for s

24. $a^2+b^2=c^2$, for b

25. $I=\frac{703W}{H^2}$, for H

26. $A=\pi r^2+\pi rs$, for r

27. $A=2\pi r^2+2\pi rh$, for r

28. $s=v_0t+\frac{gt^2}{2}$, for t

29. $t=\frac{a}{\sqrt{a^2+b^2}}$, for a

30. $P=\frac{A}{(1+r)^2}$, for r

31. $P=EI-RI^2$, for I

32. $s(6s-t)=t^2$, for s

33. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for v, assuming that $c>0$ and $m>0$

34. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for c, assuming that $v>0$ and $m>0$

35. $p=\frac{E^2R}{(r+R)^2}$, for E, assuming that $(r+R)>0$

36. The "**golden**" proportions have been considered visually pleasing for the past 2900 years. A rectangle with the width w and length l has "golden" proportions if

$$\frac{\boldsymbol{w}}{\boldsymbol{l}}=\frac{\boldsymbol{l}}{\boldsymbol{w}+\boldsymbol{l}}.$$

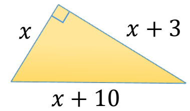
Solve this formula for $\boldsymbol{l}$. Then, find the value of the **golden ratio** $\frac{\boldsymbol{l}}{\boldsymbol{w}}$ up to three decimal places.

Answer each question.

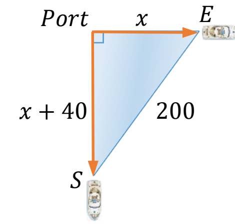
37. A boat moves r km/h in still water. If the rate of the current is c km/h,
- **a.** give an expression for the rate of the boat moving upstream;
- **b.** give an expression for the rate of the boat moving downstream.

38.
- **a.** Vivian marks a class test in n hours. Give an expression representing Vivian's rate of marking, in the number of marked tests per hour.
- **b.** How many tests will she have marked in h hours?

Solve each problem.



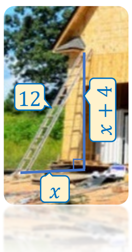
39. Find the exact length of each side of the triangle.



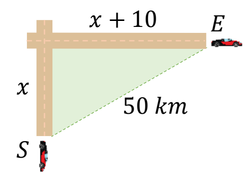
40. Two cruise ships leave a port at the same time, but they move at different rates. The faster ship is heading south, and the slower one is heading east. After a few hours, they are 200 km apart. If the faster ship went 40 km farther than the slower one, how far did each ship travelled?

41. The length of a rectangular area carpet is 2 ft more than twice the width. Diagonally, the carpet measures 13 ft. Find the dimensions of the carpet.

42. The legs of a right triangle with 26 cm long hypotenuse differ by 14 cm. Find the lengths of the legs.



43. A 12-ft ladder is tilting against a house. The top of the ladder is 4 ft further from the ground than the bottom of the ladder is from the house. To the nearest inch, how high does the ladder reach?



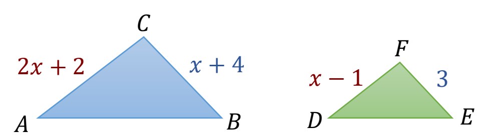
44. Two cars leave an intersection, one heading south and the other heading east. In one hour the cars are 50 kilometers apart. If the faster car went 10 kilometers farther than the slower one, how far did each car travelled?

45. The length and width of a computer screen differ by 4 inches. Find the dimensions of the screen, knowing that its area is 117 square inches.

46. The length of an American flag is 1 inch shorter than twice the width. If the area of this flag is 190 square inches, find the dimensions of the flag.

47. The length of a Canadian flag is twice the width. If the area of this flag is 100 square meters, find the exact dimensions of the flag.

48. ***Thales Theorem*** states that corresponding sides of similar triangles are proportional. The accompanying diagram shows two similar triangles, $\triangle ABC$ and $\triangle DEF$. Given the information in the diagram, find the length AC.

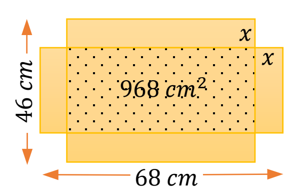


Solve each problem.

49. Sonia bought an area carpet for her 12 ft by 18 ft room. The carpet covers 135 ft^2, and when centered in the room, it leaves a strip of the bare floor of uniform width around the edges of the room. How wide is this strip?

50. Park management plans to create a rectangular 14 m by 20 m flower garden with a sidewalk of uniform width around the perimeter of the garden. There are enough funds to install 152 m^2 of a brick sidewalk. Find the width of the sidewalk.

51. Squares of equal area are cut from each corner of a 46 cm by 68 cm rectangular cardboard. Obtained this way flaps are folded up to create an open box with the area of the base equal to 968 cm^2. What is the height of the box?



52. The outside measurements of a picture frame are 22 cm and 28 cm. If the area of the exposed picture is 315 cm^2, find the width of the frame.

53. The length of a rectangle is one centimeter shorter than twice the width. The rectangle shares its longer side with a square of 169 cm^2 area. What are the dimensions of the rectangle?

54. A rectangular piece of cardboard is 15 centimeters longer than it is wide. 100 cm^2 squares are removed from each corner of the cardboard. Folding up the established flaps creates an open box of 13.5-litre volume. Find the dimensions of the original piece of cardboard. (*Hint:* 1 litre = 1000 cm^3)

55. Karin travelled 420 km by motorcycle to visit a friend. When planning the return trip by the same road, she calculated that her driving time could be 1 hour shorter if she increases her average speed by 10 km/h. On average, how fast was she driving to her friend?

56. An average, an Airbus A380 flies 80 km/h faster than a Boeing 787 Dreamliner. Suppose an Airbus A380 flew 2600 km in half an hour shorter time than it took a Boeing 787 Dreamliner to fly 2880 km. Determine the speed of each plane.

57. Two small planes, a Skyhawk and a Mooney Bravo, took off from the same place and at the same time. The Skyhawk flew 500 km. The Mooney Bravo flew 1050 km in one hour longer time and at a 100 km/h faster speed. If the planes fly faster than 200 km/h, find the average rate of each plane.

58. Gina drives 550 km to a conference. Due to heavier traffic, she returns at 10 km/h slower rate. If the round trip took her 10.5 hours, what was Gina's average rate of driving to the conference?

59. A barge travels 25 km upriver and then returns in a total of 5 hours. If the current in the river is 3 km/hr, approximately how fast would this barge move in still water?

60. A canoeist travels 3 kilometers down a river with a 3 km/h current. For the return trip upriver, the canoeist chose to use a longer branch of the river with a 2 km/hr current. If the return trip is 4 km long and the time needed for travelling both ways is 3 hours, approximate the speed of the canoe in still water.

61. Two planes take off from the same airport and at the same time. The first plane flies with an average speed r km/h and is heading North. The second plane flies faster by 40 km/h and is heading East. In thirty minutes the planes are 580 kilometers apart from each other. Determine the average speed of each plane.

62. Jack flew 650 km to visit his relatives in Alaska. On the way to Alaska, his plane encounter a 40 km/h headwind. On the returning trip, the plane flew with a 20 km/h tailwind. If the total flying time (both ways) was 5 hours 45 minutes, what was the average speed of the plane in still air?

63. Two janitors, an experienced and a newly hired one, need 4 hours to clean a school building. The newly hired worker would need 1.5 hour longer time than the experienced one to clean the school on its own. To the nearest minute, how much time is required for the experienced janitor to clean the school working alone?

64. Two workers can weed out a vegetable garden in 2 hr. On his own, one worker can do the same job in half an hour shorter time than the other. To the nearest minute, how long would it take the faster worker to weed out the garden by himself?

65. Helen and Monica are planting flowers in their garden. On her own, Helen would need an hour longer than Monica to plant all the flowers. Together, they can finish the job in 8 hr. To the nearest minute, how long would it take each person to plant all the flowers if working alone?

66. To prepare the required number of pizza crusts for a day, the owner of Ricardo's Pizza needs 40 minutes shorter time than his worker Sergio. Together, they can make these pizza crusts in 2 hours. To the nearest minute, how long would it take each of them to do this job alone?

67. A fish tank can be filled with water with the use of one of two pipes of different diameters. If only the larger-diameter pipe is used, the tank can be filled in an hour shorter time than if only the smaller-diameter pipe is used. If both pipes are open, the tank can be filled in 1 hr 12 min. How much time is needed for each pipe to fill the tank if working alone?

68. Two roofers, Garry and Larry, can install new asphalt roof shingles in 6 hours 40 min. On his own, Garry can do this job in 3 hours shorter time than Larry can. How much time each or the roofers need to install these shingles alone?

69. A ball is thrown down with the initial velocity of 6 m/sec from a balcony that is 100 m above the ground. Suppose that function $h(t) = -4.9t^2 - 6t + 100$ can be used to determine the height $h(t)$ of the ball t seconds after it was thrown down. Approximately in how many seconds the ball will be 5 meters above the ground?

70. A bakery's weekly profit, P (in dollars), for selling n poppyseed strudels can be modelled by the function $P(n) = -0.05n^2 + 7n - 200$. What is the minimum number of poppyseed strudels that must be sold to break even?

71. If P dollars is invested in an account that pays the annual interest rate r (in decimal form), then the amount A of money in the account after 2 years can be determined by the formula $A = P(1 + r)^2$. Suppose \$3000 invested in this account for 2 years grew to \$3257.29. What was the interest rate?

72. To determine the distance, d, of an object to the horizon we can use the equation $d = \sqrt{12800h + h^2}$, where h represents the distance of an object to the Earth's surface, and both, d and h, are in kilometers. To the nearest meter, how far above the Earth's surface is a plane if its distance to the horizon is 400 kilometers?