Advanced (Algebraic) Mathematics 1

For Math 072 – Camosun College Edition

Adapted from *Intermediate Algebra and Trigonometry – Camosun College Edition* by Crystal Lomas. For information about the adaptation, refer to the Copyright and Licensing statement.

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Advanced (Algebraic) Mathematics 1 and 2 are adapted from the Intermediate Algebra and Trigonometry – Camosun College Edition, which was adapted by **Crystal Lomas** from Anna Kuczynska's original open textbook.

These adapted editions were further reorganized by **Puja Gupta** to align with the learning outcomes for Math 072 and Math 073 at Camosun College.

- Minor changes and corrections were applied throughout the text.
- Section RD6 Complex Numbers was added.
- Content about secant, cosecant, and cotangent was added to Sections T2 and T3.
- Dependency on trigonometric identities was removed and replaced with alternate explanations where appropriate in Sections T2, T3, T5.
- Calculator instruction in Section T3 was changed to reference scientific calculators.

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Review of Operations on the Set of Real Numbers

Before we start our journey through algebra, let us review the structure of the real number system, properties of four operations, order of operations, the concept of absolute value, and set-builder and interval notation.

R1 Structure of the Set of Real Numbers

It is in human nature to group and classify objects with the same properties. For instance, items found in one's home can be classified as furniture, clothing, appliances, dinnerware, books, lighting, art pieces, plants, etc., depending on what each item is used for, what it is made of, how it works, etc. Furthermore, each of these groups could be subdivided into more specific categories (groups). For example, furniture includes tables, chairs, bookshelves, desks, etc. Sometimes an item can belong to more than one group. For example, a piece of furniture can also be a piece of art. Sometimes the groups do not have any common items (e.g. plants and appliances). Similarly to everyday life, we like to classify numbers with respect to their properties. For example, even or odd numbers, prime or composite numbers, common fractions, finite or infinite decimals, infinite repeating decimals, negative numbers, etc. In this section, we will take a closer look at commonly used groups (sets) of real numbers and the relations between those groups.

Set Notation and Frequently Used Sets of Numbers

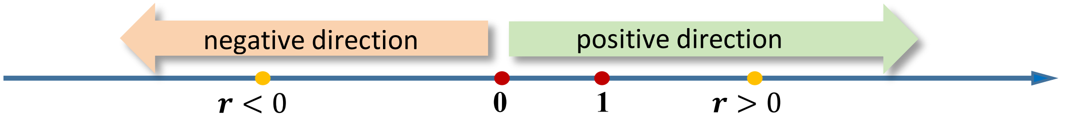
We start with terminology and notation related to sets.

Definition 1.1 ► A **set** is a collection of objects, called **elements** (or **members**) of this set.

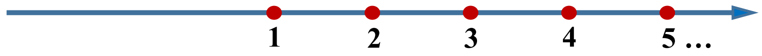
Roster Notation: A set can be given by listing its elements within the **set brackets** { } (braces).
The elements of the set are separated by commas.
To indicate that a pattern continues, we use three dots … .
Examples:
If set A consists of the numbers 1, 2, and 3, we write $A = \{1,2,3\}$.
If set B consists of consecutive counting numbers starting from 5, we write $B = \{5,6,7,\dots\}$.

More on Notation: To indicate that the number 2 **is an element** of set A, we write $\mathbf{2 \in A}$.
To indicate that the number 2 **is <u>not</u> an element** of set B, we write $\mathbf{2 \notin A}$.
A set with no elements, called **empty** set, is denoted by the symbol $\emptyset$ or { }.

In this course we will be working with the set of **real numbers**, denoted by $\mathbb{R}$. To visualise this set, we construct a line and choose two distinct points on it, 0 and 1, to establish direction and scale. This makes it a **number line**. Each real number r can be identified with exactly one point on such a number line by choosing the endpoint of the segment of length $|r|$ that starts from 0 and follows the line in the direction of 1, for positive r, or in the direction opposite to 1, for negative r.

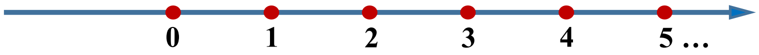


The set of real numbers contains several important subgroups (**subsets**) of numbers. The very first set of numbers that we began our mathematics education with is the set of counting numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, called **natural numbers** and denoted by ℕ.



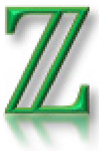
Natural numbers ℕ

The set of natural numbers together with the number **0** creates the set of **whole numbers** $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by 𝕎.

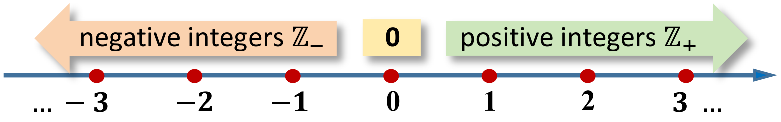


Whole numbers 𝕎

Notice that if we perform addition or multiplication of numbers from either of the above sets, ℕ and 𝕎, the result will still be an element of the same set. We say that the set of **natural numbers** ℕ and the set of **whole numbers** 𝕎 are both **closed** under **addition** and **multiplication**.



However, if we wish to perform subtraction of natural or whole numbers, the result may become a negative number. For example, $2 - 5 = -3 \notin \mathbb{W}$, so neither the set of whole numbers nor natural numbers is closed under subtraction. To be able to perform subtraction within the same set, it is convenient to extend the set of whole numbers to include negative counting numbers. This creates the set of **integers** $\{\dots, -\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by ℤ.

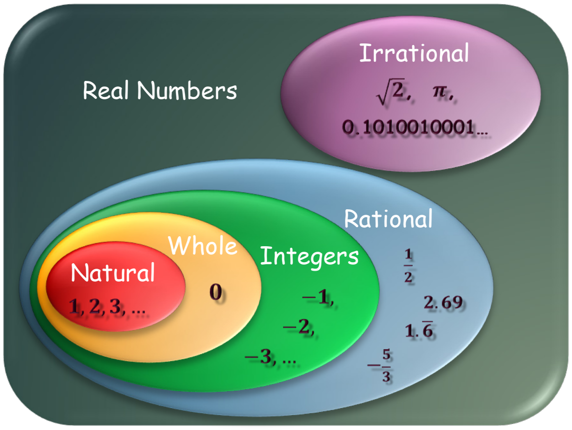


Integers ℤ

Alternatively, the set of integers can be recorded using the ± sign: $\{\mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \dots\}$. The ± sign represents two numbers at once, the positive and the negative.

So the set of **integers** ℤ is **closed** under **addition**, **subtraction** and **multiplication**. What about division? To create a set that would be closed under division, we extend the set of integers by including all quotients of integers (all common fractions). This new set is called the set of **rational numbers** and denoted by ℚ. Here are some examples of rational numbers: $\frac{3}{1} = \mathbf{3}$, $\frac{1}{2} = \mathbf{0.5}$, $-\frac{7}{4}$, or $\frac{4}{3} = \mathbf{1.\overline{3}}$.

Thus, the set of **rational numbers** ℚ is **closed** under **all four operations**. It is quite difficult to visualize this set on the number line as its elements are nearly everywhere. Between any two rational numbers, one can always find another rational number, simply by taking an average of the two. However, all the points corresponding to rational numbers still do not fulfill the whole number line. Actually, the number line contains a lot more unassigned points than points that are assigned to rational numbers. The remaining points correspond to numbers called **irrational** and are denoted by 𝕀ℚ. Here are some examples of irrational numbers: $\sqrt{\mathbf{2}}, \boldsymbol{\pi}, \boldsymbol{e}$, or $\mathbf{0.1010010001}\dots$.



By definition, the two sets, $\mathbb{Q}$ and $\mathbb{IQ}$ fill the entire number line, which represents the set of **real numbers**, $\mathbb{R}$.

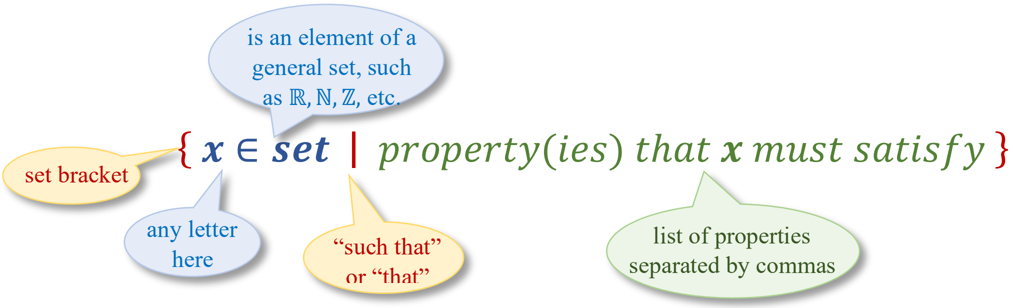
The sets $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{IQ}$, and $\mathbb{R}$ are related to each other as in the accompanying diagram. One can make the following observations:

$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where $\subset$ (*read* is a **subset**) represents the operator of **inclusion of sets**;

$\mathbb{Q}$ and $\mathbb{IQ}$ are **disjoint** (they have **no common element**);

$\mathbb{Q}$ together with $\mathbb{IQ}$ create $\mathbb{R}$.

So far, we introduced six double-stroke letter signs to denote the main sets of numbers. However, there are many more sets that one might be interested in describing. Sometimes it is enough to use a subindex with the existing letter-name. For instance, the set of all positive real numbers can be denoted as $\mathbb{R}_+$ while the set of negative integers can be denoted by $\mathbb{Z}_-$. But how would one represent, for example, the set of even or odd numbers or the set of numbers divisible by 3, 4, 5, and so on? To describe numbers with a particular property, we use the **set-builder notation**. Here is the structure of set-builder notation:



For example, to describe the set of even numbers, first, we think of a property that distinguishes even numbers from other integers. This is divisibility by 2. So each even number n can be expressed as $2k$, for some integer k. Therefore, the set of even numbers could be stated as $\{n \in \mathbb{Z} \mid n = 2k, k \in \mathbb{Z}\}$ (*read*: The set of all integers n such that each n is of the form $2k$, for some integral k.)

To describe the set of rational numbers, we use the fact that any rational number can be written as a common fraction. Therefore, the set of rational numbers $\mathbb{Q}$ can be described as $\left\{x \,\middle|\, x = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0\right\}$ (*read*: The set of all real numbers x that can be expressed as a fraction $\frac{p}{q}$, for integral p and q, with $q \neq 0$.)

Convention: *If the description of a set refers to the set of real numbers, there is no need to state $x \in \mathbb{R}$ in the first part of set-builder notation. For example, we can write $\{\boldsymbol{x} \in \mathbb{R} \mid \boldsymbol{x} > \boldsymbol{0}\}$ or $\{\boldsymbol{x} \mid \boldsymbol{x} > \boldsymbol{0}\}$. Both sets represent the set of all positive real numbers, which could also be recorded as simply $\mathbb{R}_+$. However, if we work with any other major set, this set must be stated. For example, to describe all positive integers $\mathbb{Z}_+$ using set-builder notation, we write $\{\boldsymbol{x} \in \mathbb{Z} \mid \boldsymbol{x} > \boldsymbol{0}\}$ and $\mathbb{Z}$ is essential there.*

Example 1 ► **Listing Elements of Sets Given in Set-builder Notation**

List the elements of each set.

a. $\{n \in \mathbb{Z} \mid -2 \le n < 5\}$ **b.** $\{n \in \mathbb{N} \mid n = 5k, k \in \mathbb{N}\}$

Solution ► **a.** This is the set of integers that are at least -2 but smaller than 5. So this is $\{-2, -1, 0, 1, 2, 3, 4\}$.

b. This is the set of natural numbers that are multiples of 5. Therefore, this is the infinite set $\{5, 10, 15, 20, \dots\}$.

Example 2 ► **Writing Sets with the Aid of Set-builder Notation**

Use set-builder notation to describe each set.

a. $\{1, 4, 9, 16, 25, \dots\}$ **b.** $\{-2, 0, 2, 4, 6\}$

Solution ► **a.** First, we observe that the given set is composed of consecutive perfect square numbers, starting from 1. Since all the elements are natural numbers, we can describe this set using the set-builder notation as follows: $\{n \in \mathbb{N} \mid n = k^2, \text{for } k \in \mathbb{N}\}$.

b. This time, the given set is finite and lists all even numbers starting from -2 up to 6. Since the general set we work with is the set of integers, the corresponding set in set-builder notation can be written as $\{n \in \mathbb{Z} \mid n \text{ is even}, -2 \le n \le 6\}$, or $\{n \in \mathbb{Z} \mid -2 \le n \le 6, n = 2k, \text{for } k \in \mathbb{Z}\}$.

Observations:

* *There are many equivalent ways to describe a set using the set-builder notation.*

* *The commas used between the conditions (properties) stated after the "such that" bar play the same role as the connecting word "and".*

Rational Decimals

How can we recognize if a number in decimal notation is rational or irrational?

A **terminating** decimal (with a **finite** number of nonzero digits after the decimal dot, like 1.25 or 0.1206) can be converted to a common fraction by replacing the decimal dot with the division by the corresponding power of 10 and then simplifying the resulting fraction. For example,

$$\underset{\substack{\text{2 places}\\ \to \text{2 zeros}}}{1.25} = \frac{125}{100} \underset{\substack{\text{simplify}\\ \text{by 25}}}{=} \frac{5}{4}, \quad \text{or} \quad \underset{\substack{\text{4 places}\\ \to \text{4 zeros}}}{0.1206} = \frac{1206}{10000} \underset{\substack{\text{simplify}\\ \text{by 2}}}{=} \frac{603}{5000}.$$

Therefore, any **terminating decimal is a rational number**.

One can also convert a **nonterminating** (**infinite**) decimal to a common fraction, as long as there is a **recurring** (**repeating**) sequence of digits in the decimal expansion. This can be done using the method shown in *Example 3a*. Hence, any **infinite repeating decimal is a rational number**.

Also, notice that any fraction $\frac{m}{n}$ can be converted to either a finite or infinite repeating decimal. This is because since there are only finitely many numbers occurring as remainders in the long division process when dividing by n, eventually, either a remainder becomes zero, or the sequence of remainders starts repeating.

So **a number is rational if and only if it can be represented by a finite or infinite repeating decimal**. Since the irrational numbers are defined as those that are not rational, we can conclude that **a number is irrational if and only if it can be represented as an infinite non-repeating decimal**.

Example 3

Proving that an Infinite Repeating Decimal is a Rational Number

Show that the given decimal is a rational number.

a. $0.333\ldots$ **b.** $2.3\overline{45}$

Solution

a. Let $a = 0.333\ldots$. After multiplying this equation by 10, we obtain $10a = 3.333\ldots$. Since in both equations, the number after the decimal dot is exactly the same, after subtracting the equations side by side, we obtain

$$-\begin{cases}10a = 3.\cancel{333\ldots}\\ \ \ a = 0.\cancel{333\ldots}\end{cases}$$
$$9a = 3$$

which solves to $a = \frac{3}{9} = \frac{1}{3}$. So $0.333\ldots = \frac{1}{3}$ is a rational number.

b. Let $a = 2.3\overline{45}$. The bar above 45 tells us that the sequence 45 repeats forever. To use the subtraction method as in solution to *Example 3a*, we need to create two equations involving the given number with the decimal dot moved after the repeating sequence and before the repeating sequence. This can be obtained by multiplying the equation $a = 2.3\overline{45}$ first by 1000 and then by 10, as below.

$$-\begin{cases}1000a = 2345.\cancel{\overline{45}}\\ \ \ \ 10a = 23.\cancel{\overline{45}}\end{cases}$$
$$990a = 2322$$

Therefore, $a = \frac{2322}{990} = \frac{129}{55} = 2\frac{19}{55}$, which proves that $2.3\overline{45}$ is rational.

Example 4

Identifying the Main Types of Numbers

List all numbers of the set

$\left\{-10,\ -5.34,\ 0,\ 1,\ \frac{12}{3},\ 3.\overline{16},\ \frac{4}{7},\ \sqrt{2},\ -\sqrt{36},\ \sqrt{-4},\ \pi,\ 9.010010001\ldots\right\}$ that are

a. natural **b.** whole **c.** integral **d.** rational **e.** irrational

Solution

a. The only natural numbers in the given set are 1 and $\frac{12}{3} = 4$.

b. The whole numbers include the natural numbers and the number 0, so we list $0, 1$ and $\frac{12}{3}$.

c. The integral numbers in the given set include the previously listed $0, 1, \frac{12}{3}$, and the negative integers -10 and $-\sqrt{36} = -6$.

d. The rational numbers in the given set include the previously listed integers $0, 1, \frac{12}{3}, -10,\ -\sqrt{36}$, the common fraction $\frac{4}{7}$, and the decimals -5.34 and $3.\overline{16}$.

e. The only irrational numbers in the given set are the constant π and the infinite decimal $9.010010001 \ldots$.

Note: $\sqrt{-4}$ is not a real number.

R.1 Exercises

True or False? If it is false, explain why.

1. Every natural number is an integer.

2. Some rational numbers are irrational.

3. Some real numbers are integers.

4. Every integer is a rational number.

5. Every infinite decimal is irrational.

6. Every square root of an odd number is irrational.

Use ***roster notation*** *to list all elements of each set.*

7. The set of all positive integers less than 9

8. The set of all odd whole numbers less than 11

9. The set of all even natural numbers

10. The set of all negative integers greater than -5

11. The set of natural numbers between 3 and 9

12. The set of whole numbers divisible by 4

Use ***set-builder notation*** *to describe each set.*

13. $\{0, 1, 2, 3, 4, 5\}$

14. $\{4, 6, 8, 10, 12, 14\}$

15. The set of all real numbers greater than -3

16. The set of all real numbers less than 21

17. The set of all multiples of 3

18. The set of perfect square numbers up to 100

Fill in each box with one of the signs $\in, \notin, \subset, \not\subset$ *or* $=$ *to make the statement true.*

19. -3 ☐ $\mathbb{Z}$

20. $\{0\}$ ☐ $\mathbb{W}$

21. $\mathbb{Q}$ ☐ $\mathbb{Z}$

22. $0.3555 \ldots$ ☐ $\mathbb{IQ}$

23. $\sqrt{3}$ ☐ $\mathbb{Q}$

24. $\mathbb{Z}_{-}$ ☐ $\mathbb{Z}$

25. $\pi \,\square\, \mathbb{R}$ **26.** $\mathbb{N} \,\square\, \mathbb{Q}$ **27.** $\mathbb{Z}_+ \,\square\, \mathbb{N}$

For the given set, state the subset of ***(a)*** *natural numbers,* ***(b)*** *whole numbers,* ***(c)*** *integers,* ***(d)*** *rational numbers,* ***(e)*** *irrational numbers,* ***(f)*** *real numbers.*

28. $\left\{-1,\ 2.16,\ -\sqrt{25},\ \frac{12}{2}, -\frac{12}{5},\ 3.\overline{25},\ \sqrt{5},\ \pi,\ 3.565665666\ldots\right\}$

29. $\left\{0.999\ldots,\ -5.001,\ 0,\ 5\frac{3}{4},\ 1.4\overline{05},\ \frac{7}{8},\ \sqrt{2},\ \sqrt{16},\ \sqrt{-9},\ 9.010010001\ldots\right\}$

Show that the given decimal is a rational number.

30. $0.555\ldots$ **31.** $1.\overline{02}$ **32.** $0.1\overline{34}$

33. $2.0\overline{125}$ **34.** $0.25\overline{7}$ **35.** $5.22\overline{54}$

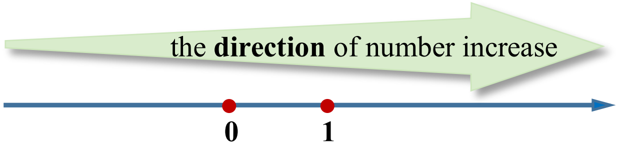
R2 Number Line and Interval Notation

As mentioned in the previous section, it is convenient to visualise the set of real numbers by identifying each number with a unique point on a number line.

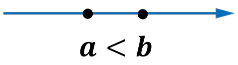
Order on the Number Line and Inequalities

Definition 2.1 ► A **number line** is a line with two distinct points chosen on it. One of these points is designated as **0** and the other point is designated as **1**.

The length of the segment from 0 to 1 represents one **unit** and provides the scale that allows to locate the rest of the numbers on the line. The **direction** from 0 to 1, marked by an **arrow** at the end of the line, indicates the **increasing order** on the number line. The numbers corresponding to the points on the line are called the **coordinates** of the points.



Note: *For simplicity, the coordinates of points on a number line are often identified with the points themselves.*



To compare numbers, we use **inequality signs** such as $<, \leq, >, \geq$, or $\neq$. For example, if a is **smaller than** b we write $a < b$. This tells us that the location of point a on the number line is to the left of point b. Equivalently, we could say that b is **larger than** a and write $b > a$. This means that the location of b is to the right of a.

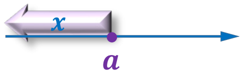
Example 1 ► **Identifying Numbers with Points on a Number Line**

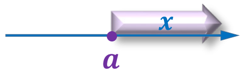
Match the numbers $-2,\ 3.5,\ \pi,\ -1.5,\ \frac{5}{2}$ with the letters on the number line:

Solution ► To match the given numbers with the letters shown on the number line, it is enough to order the numbers from the smallest to the largest. First, observe that negative numbers are smaller than positive numbers and $-2 < -1.5$. Then, observe that $\pi \approx 3.14$ is larger than $\frac{5}{2}$ but smaller than 3.5. Therefore, the numbers are ordered as follows:

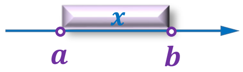
$$-2 < -1.5 < \frac{5}{2} < \pi < 3.5$$

Thus, $\boldsymbol{A = -2, B = -1.5, C = \frac{5}{2}, D = \pi}$, and $\boldsymbol{E = 3.5}$.

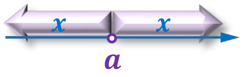




To indicate that a number x is **smaller or equal** a, we write $x \leq a$. This tells us that the location of point x on the number line is to the left of point a or exactly at point a. Similarly, if x is **larger or equal** a, we write $x \geq a$, and we locate x to the right of point a or exactly at point a.



To indicate that a number x is **between** a and b, we write $a < x < b$. This means that the location of point x on the number line is somewhere on the segment joining points a and b, but not at a nor at b. Such stream of two inequalities is referred to as a **three-part inequality**.



Finally, to state that a number x is **different than** a, we write $x \neq a$. This means that the point x can lie anywhere on the entire number line, except at the point a.

Here is a list of some English phrases that indicate the use of particular inequality signs.

English Phrases	Inequality Sign(s)
is less than; smaller than	$<$
is less or equal; smaller or equal; at most; is no more than	$\leq$
is more than; larger than; greater than;	$>$
is more or equal; larger or equal; greater or equal; at least; no less than	$\geq$
is different than	$\neq$
is between	$<\quad<$

Example 2 ► Using Inequality Symbols

Write each statement as a single or a three-part inequality.

a. -7 **is less than** 5
b. $2x$ **is greater or equal** 6
c. $3x + 1$ **is between** -1 and 7
d. x **is between** 1 and 8, **including** 1 and **excluding** 8
e. $5x - 2$ **is different than** 0
f. x **is negative**

Solution ►

a. Write $-7 < 5$. *Notice:* The inequality "points" to the smaller number. This is an example of a **strong** inequality. One side is "strongly" smaller than the other side.

b. Write $2x \geq 6$. This is an example of a **weak** inequality, as it allows for equation.

c. Enclose $3x + 1$ within two strong inequalities to obtain $-1 < 3x + 1 < 7$. *Notice:* The word "between" indicates that the endpoints are not included.

d. Since 1 is included, the statement is $1 \leq x < 8$.

e. Write $5x - 2 \neq 0$.

f. Negative x means that x is smaller than zero, so the statement is $x < 0$.

Example 3 ► **Graphing Solutions to Inequalities in One Variable**

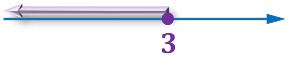
Using a number line, graph all x-values that satisfy (are **solutions** of) the given inequality or inequalities:

a. $x > -2$ **b.** $x \leq 3$ **c.** $1 \leq x < 4$

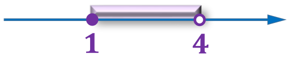
Solution ►

a. The x-values that satisfy the inequality $x > -2$ are larger than -2, so we shade the part of the number line that corresponds to numbers greater than -2. Those are all points to the right of -2, but not including -2. To indicate that the -2 is not a solution to the given inequality, we draw a hollow circle at -2.

b. The x-values that satisfy the inequality $x \leq 3$ are smaller than or equal to 3, so we shade the part of the number line that corresponds to the number 3 or numbers smaller than 3. Those are all points to the left of 3, including the point 3. To indicate that the 3 is a solution to the given inequality, we draw a filled-in circle at 3.

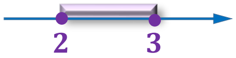


c. The x-values that satisfy the inequalities $1 \leq x < 4$ are larger than or equal to 1 and at the same time smaller than 4. Thus, we shade the part of the number line that corresponds to numbers between 1 and 4, including the 1 but excluding the 4. Those are all the points that lie between 1 and 4, including the point 1 but excluding the point 4. So, we draw a segment connecting 1 with 4, with a filled-in circle at 1 and a hollow circle at 4.

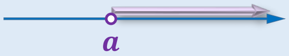


Interval Notation

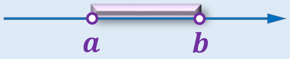
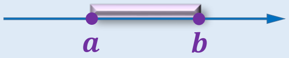
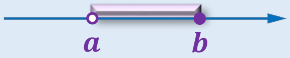
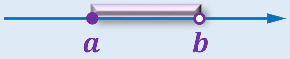
As shown in the solution to *Example 3*, the graphical solutions of inequalities in one variable result in a segment of a number line (if we extend the definition of a segment to include the endpoint at infinity). To record such a solution segment algebraically, it is convenient to write it by stating its left endpoint (corresponding to the lower number) and then the right endpoint (corresponding to the higher number), using appropriate brackets that would indicate the inclusion or exclusion of the endpoint. For example, to record algebraically the segment that starts from 2 and ends on 3, including both endpoints, we write $[2, 3]$. Such notation very closely depicts the graphical representation of the segment, , and is called **interval notation**.



Interval Notation: A set of numbers satisfying a single inequality of the type $<, \leq, >$, or $\geq$ can be recorded in interval notation, as stated in the table below.

inequality	**set-builder notation**	**graph**	**interval notation**	**comments**
$x > a$	$\{x \mid x > a\}$		$(\boldsymbol{a}, \infty)$	- list the endvalues from **left to right** - to exclude the endpoint use a **round bracket (** or **)**
$x \geq a$	$\{x \mid x \geq a\}$		$[\boldsymbol{a}, \infty)$	- **infinity** sign is used with a **round bracket**, as there is no last point to include - to include the endpoint use a **square bracket [** or **]**
$x < a$	$\{x \mid x < a\}$		$(-\infty, \boldsymbol{a})$	- to indicate **negative infinity**, use the **negative sign** in front of ∞ - to indicate **positive infinity**, there is no need to write a positive sign in front of the infinity sign
$x \leq a$	$\{x \mid x \leq a\}$		$(-\infty, \boldsymbol{a}]$	- remember to list the endvalues from **left to right**; this also refers to infinity signs

Similarly, a set of numbers satisfying two inequalities resulting in a segment of solutions can be recorded in interval notation, as stated below.

inequality	**set-builder notation**	**graph**	**interval notation**	**comments**
$a < x < b$	$\{x \mid a < x < b\}$		$(\boldsymbol{a}, \boldsymbol{b})$	- we read: an **open interval** from a to b
$a \leq x \leq b$	$\{x \mid a \leq x \leq b\}$		$[\boldsymbol{a}, \boldsymbol{b}]$	- we read: a **closed interval** from a to b
$a < x \leq b$	$\{x \mid a < x \leq b\}$		$(\boldsymbol{a}, \boldsymbol{b}]$	- we read: an interval from a to b, without a but with b This is called **half-open** or **half-closed** interval.
$a \leq x < b$	$\{x \mid a \leq x < b\}$		$[\boldsymbol{a}, \boldsymbol{b})$	- we read: an interval from a to b, with a but without b This is called **half-open** or **half-closed** interval.

In addition, the set of all real numbers $\mathbb{R}$ is represented in the interval notation as $(-\infty, \infty)$.

Example 4 ► **Writing Solutions to One Variable Inequalities in Interval Notation**

Write solutions to the inequalities from *Example 3* in set-builder and interval notation.

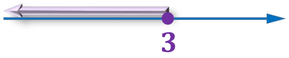
a. $x > -2$ **b.** $x \leq 3$ **c.** $1 \leq x < 4$

Solution ►

a. The solutions to the inequality $x > -2$ can be stated in set-builder notation as $\boldsymbol{\{x|x > -2\}}$. Reading the graph of this set

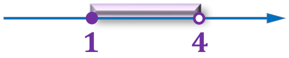
from **left to right**, we start from -2, without -2, and go towards infinity. So, the interval of solutions is written as $(\mathbf{-2}, \infty)$. We use the round bracket to indicate that the endpoint is not included. The infinity sign is always written with the round bracket, as infinity is a concept, not a number. So, there is no last number to include.

b. The solutions to the inequality $x \leq 3$ can be stated in set-builder notation as $\boldsymbol{\{x|x \leq 3\}}$. Again, reading the graph of this set



from **left to right**, we start from $-\infty$ and go up to 3, including 3. So, the interval of solutions is written as $(-\infty, \mathbf{3}]$. We use the square bracket to indicate that the endpoint is included. As before, the infinity sign takes the round bracket. Also, we use "−" in front of the infinity sign to indicate negative infinity.

c. The solutions to the three-part inequality $1 \leq x < 4$ can be stated in set-builder notation as $\boldsymbol{\{x|1 \leq x < 4\}}$. Reading the graph of this set



from **left to right**, we start from 1, including 1, and go up to 4, excluding 4. So, the interval of solutions is written as $[\mathbf{1}, \mathbf{4})$. We use the square bracket to indicate 1 and the round bracket, to exclude 4.

Absolute Value, and Distance

The **absolute value** of a number x, denoted $|x|$, can be thought of as the distance from x to 0 on a number line. Based on this interpretation, we have $|\boldsymbol{x}| = |\boldsymbol{-x}|$. This is because both numbers x and $-x$ are at the same distance from 0. For example, since both 3 and -3 are exactly three units apart from the number 0, then $|3| = |-3| = 3$.

Since distance can not be negative, we have $|\boldsymbol{x}| \geq \mathbf{0}$.

Here is a formal definition of the absolute value operator.

Definition 2.2 ► For any real number x,

$$|x| \overset{def}{=} \begin{cases} \boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} \geq \mathbf{0} \\ \boldsymbol{-x}, & \boldsymbol{if} \quad \boldsymbol{x} < \mathbf{0} \end{cases}.$$

The above definition of absolute value indicates that for $x \geq 0$ we use the equation $|x| = x$, and for $x < 0$ we use the equation $|x| = -x$ (the absolute value of x is the opposite of x, which is a positive number).

Example 5 ► **Evaluating Absolute Value Expressions**

Evaluate.

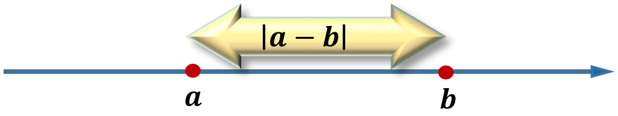
a. $-|-4|$ **b.** $|-5| - |2|$ **c.** $|-5-(-2)|$

Solution ► **a.** Since $|-4| = 4$ then $-|-4| = \mathbf{-4}$.

b. Since $|-5| = 5$ and $|2| = 2$ then $|-5| - |2| = 5 - 2 = \mathbf{3}$.

c. Before applying the absolute value operator, we first simplify the expression inside the absolute value sign. So we have $|-5-(-2)| = |-5+2| = |-3| = \mathbf{3}$.

On a number line, the **distance** between two points with coordinates a and b is calculated by taking the difference between the two coordinates. So, if $b > a$, the distance is $b - a$. However, if $a > b$, the distance is $a - b$. What if we don't know which value is larger, a or b? Since the distance must be positive, we can choose to calculate any of the differences and apply the absolute value on the result.



Definition 2.3 ► The **distance** $d(a, b)$ between points a and b on a number line is given by the expression $|\boldsymbol{a} - \boldsymbol{b}|$, or equivalently $|\boldsymbol{b} - \boldsymbol{a}|$.

Notice that $d(x, 0) = |x - 0| = |x|$, which is consistent with the intuitive definition of absolute value of x as the distance from x to 0 on the number line.

Example 6 ► **Finding Distance Between Two Points on a Number Line**

Find the distance between the two given points on the number line.

a. -3 and 5 **b.** x and 2

Solution ► **a.** Using the distance formula for two points on a number line, we have $d(-3,5) = |-3-5| = |-8| = \mathbf{8}$. Notice that we could also calculate $|5-(-3)| = |8| = \mathbf{8}$.

b. Following the formula, we obtain $d(x, 2) = |x - 2|$. Since a is unknown, the distance between a and 2 is stated as an expression $|\boldsymbol{x} - \mathbf{2}|$ rather than a specific number.

R.2 Exercises

Write each statement with the use of an ***inequality*** *symbol.*

1. -6 is less than -3

2. 0 is more than -1

3. 17 is greater or equal to x

4. x is smaller or equal to 8

5. $2x + 3$ is different than zero

6. $2 - 5x$ is negative

7. x is between 2 and 5

8. $3x$ is between -5 and 7

9. $2x$ is between -2 and 6, including -2 and excluding 6

10. $x + 1$ is between -5 and 11, excluding -5 and including 11

Graph each set of numbers on a number line and write it in ***interval notation***.

11. $\{x \mid x \geq -4\}$

12. $\{x \mid x \leq -3\}$

13. $\left\{x \middle| x < \frac{5}{2}\right\}$

14. $\left\{x \middle| x > -\frac{2}{5}\right\}$

15. $\{x \mid 0 < x < 6\}$

16. $\{x \mid -1 \leq x \leq 4\}$

17. $\{x \mid -5 \leq x < 16\}$

18. $\{x \mid -12 < x \leq 4.5\}$

Evaluate.

19. $-|-7|$

20. $|5| - |-13|$

21. $|11 - 19|$

22. $|-5 - (-9)|$

23. $-|9| - |-3|$

24. $-|-13 + 7|$

Replace each $\square$ *with one of the signs* $<, >, \leq, \geq, =$ *to make the statement true.*

25. $-7 \ \square \ -5$

26. $|-16| \ \square \ -|16|$

27. $-3 \ \square \ -|3|$

28. $x^2 \ \square \ 0$

29. $x \ \square \ |x|$

30. $|x| \ \square \ |-x|$

Find the distance between the given points.

31. $-7, -32$

32. $46, -13$

33. $-\frac{2}{3}, \frac{5}{6}$

34. $x, 0$

35. $5, y$

36. x, y

Find numbers that are 5 units apart from the given point.

37. 0

38. 3

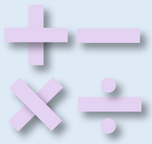
39. a

R3 Properties and Order of Operations on Real Numbers

In algebra, we are often in need of changing an expression to a different but equivalent form. This can be observed when simplifying expressions or solving equations. To change an expression equivalently from one form to another, we use appropriate properties of operations and follow the order of operations.

Properties of Operations on Real Numbers

The four basic operations performed on real numbers are addition $(+)$, subtraction $(-)$, multiplication $(\cdot)$, and division $(\div)$. Here are the main properties of these operations:

Closure:	The result of an operation on real numbers is also a real number. We can say that the **set of real numbers** is **closed** under **addition**, **subtraction** and **multiplication**. We cannot say this about division, as **division by zero is not allowed**.
Neutral Element: 	A real number that leaves other real numbers unchanged under a particular operation. For example, **zero** is the **neutral element** (also called **additive identity**) of **addition**, since $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$, and $\mathbf{0} + \boldsymbol{a} = \boldsymbol{a}$, for any real number a. Similarly, **one** is the **neutral element** (also called **multiplicative identity**) of **multiplication**, since $\boldsymbol{a} \cdot \mathbf{1} = \boldsymbol{a}$, and $\mathbf{1} \cdot \boldsymbol{a} = \boldsymbol{a}$, for any real number a.
Inverse Operations: 	Operations that reverse the effect of each other. For example, **addition and subtraction** are **inverse operations**, as $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{a}$, and $\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{b} = \boldsymbol{a}$, for any real a and b. Similarly, **multiplication and division** are **inverse operations**, as $\boldsymbol{a} \cdot \boldsymbol{b} \div \boldsymbol{b} = \boldsymbol{a}$, and $\boldsymbol{a} \cdot \boldsymbol{b} \div \boldsymbol{b} = \boldsymbol{a}$ for any real a and $b \neq 0$.
Opposites:	Two quantities are **opposite** to each other if they **add to zero**. Particularly, $\boldsymbol{a}$ and $-\boldsymbol{a}$ are **opposites** (also referred to as **additive inverses**), as $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$. For example, the opposite of 3 is -3, the opposite of $-\frac{3}{4}$ is $\frac{3}{4}$, the opposite of $x+1$ is $-(x+1) = -x-1$.
Reciprocals:	Two quantities are **reciprocals** of each other if they **multiply to one**. Particularly, $\boldsymbol{a}$ and $\frac{\mathbf{1}}{\boldsymbol{a}}$ are **reciprocals** (also referred to as **multiplicative inverses**), since $\boldsymbol{a} \cdot \frac{\mathbf{1}}{\boldsymbol{a}} = \mathbf{1}$. For example, the reciprocal of 3 is $\frac{1}{3}$, the reciprocal of $-\frac{3}{4}$ is $-\frac{4}{3}$, the reciprocal of $x+1$ is $\frac{1}{x+1}$.
Multiplication by 0:	Any real quantity **multiplied by zero** becomes **zero**. Particularly, $\boldsymbol{a} \cdot \mathbf{0} = \mathbf{0}$, for any real number a.
Zero Product:	If a product of two real numbers is zero, then at least one of these numbers must be zero. Particularly, for any real $\boldsymbol{a}$ and $\boldsymbol{b}$, if $\boldsymbol{a} \cdot \boldsymbol{b} = \mathbf{0}$, then $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{b} = \mathbf{0}$. For example, if $x(x-1) = 0$, then either $x = 0$ or $x - 1 = 0$.

Commutativity: **The order of numbers does not change the value** of a particular operation. In particular, **addition** and **multiplication** is **commutative**, since

$$a + b = b + a \text{ and } a \cdot b = b \cdot a,$$

for any real a and b. For example, $5 + 3 = 3 + 5$ and $5 \cdot 3 = 3 \cdot 5$.

Note: Neither subtraction nor division is commutative. See a counterexample: $5 - 3 = 2$ but $3 - 5 = -2$, so $5 - 3 \neq 3 - 5$. Similarly, $5 \div 3 \neq 3 \div 5$.

Associativity: **Association (grouping) of numbers does not change the value** of an expression involving only one type of operation. In particular, **addition** and **multiplication** is **associative**, since

$$(a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c),$$

for any real a and b. For example, $(5 + 3) + 2 = 5 + (3 + 2)$ and $(5 \cdot 3) \cdot 2 = 5 \cdot (3 \cdot 2)$.

Note: Neither subtraction nor division is associative. See a counterexample:
$(8 - 4) - 2 = 2$ but $8 - (4 - 2) = 6$, so $(8 - 4) - 2 \neq 8 - (4 - 2)$.
Similarly, $(8 \div 4) \div 2 = 1$ but $8 \div (4 \div 2) = 4$, so $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$.

Distributivity: Multiplication can be **distributed** over addition or subtraction by following the rule:

$$a(b \pm c) = ab \pm ac,$$

for any real a, b and c. For example, $2(3 \pm 5) = 2 \cdot 3 \pm 2 \cdot 5$, or $2(x \pm y) = 2x \pm 2y$.

Note: The reverse process of distribution is known as **factoring a common factor** out. For example, $2ax + 2ay = 2a(x + y)$.

Example 1 ► Showing Properties of Operations on Real Numbers

Complete each statement to illustrate the indicated property.

a. $mn =$ _______ (commutativity of multiplication)
b. $5x + (7x + 8) =$ ___________________ (associativity of addition)
c. $5x(2 - x) =$ ___________________ (distributivity of multiplication)
d. $-y +$ _____ $= 0$ (additive inverse)
e. $-6 \cdot$ _____ $= 1$ (multiplicative inverse)
f. If $7x = 0$, then ____ $= 0$ (zero product)

Solution ►

a. To show that multiplication is commutative, we change the order of letters, so $mn = nm$.

b. To show that addition is associative, we change the position of the bracket, so $5x + (7x + 8) = (5x + 7x) + 8$.

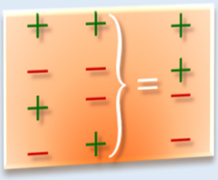
c. To show the distribution of multiplication over subtraction, we multiply $5x$ by each term of the bracket. So we have $5x(2 - x) = 5x \cdot 2 - 5x \cdot x$.

d. Additive inverse to $-y$ is its opposite, which equals to $-(-y) = y$.
So we write $-y + y = 0$.

e. Multiplicative inverse of -6 is its reciprocal, which equals to $-\frac{1}{6}$.
So we write $-6 \cdot \left(-\frac{1}{6}\right) = 1$.

f. By the zero product property, one of the factors, 7 or x, must equal to zero.
Since $7 \neq 0$, then x must equal to zero. So, we write: If $7x = 0$, then $x = 0$.

Sign Rule:



When multiplying or dividing two numbers of the **same sign**, the result is **positive**.

When multiplying or dividing two numbers of **different signs**, the result is **negative**.

This rule also applies to double signs. If the two **signs** are the **same**, they can be replaced by a **positive** sign. For example $+(+3) = 3$ and $-(-3) = 3$.
If the two **signs** are **different**, they can be replaced by a **negative** sign. For example $-(+3) = -3$ and $+(-3) = -3$.

Observation:

Since a double negative $(--)$ can be replaced by a positive sign $(+)$, the ***opposite of an opposite*** *leaves the original quantity unchanged. For example, $-(-2) = 2$, and generally $-(-a) = a$.*
Similarly, taking the ***reciprocal of a reciprocal*** *leaves the original quantity unchanged. For example, $\frac{1}{\frac{1}{2}} = 1 \cdot \frac{2}{1} = 2$, and generally $\frac{1}{\frac{1}{a}} = 1 \cdot \frac{a}{1} = a$.*

Caution:

Be careful not to eliminate the $+$ operator when simplifying subtraction of a negative.
For example, $5 - (-5) = 5 + 5$ with the $+$ operator being essential.

Example 2 ► Using Properties of Operations on Real Numbers

Use applicable properties of real numbers to simplify each expression.

a. $-\frac{-2}{-3}$

b. $3 + (-2) - (-7) - 11$

c. $2x(-3y)$

d. $3a - 2 - 5a + 4$

e. $-(2x - 5)$

f. $2(x^2 + 1) - 2(x - 3x^2)$

g. $-\frac{100ab}{25a}$

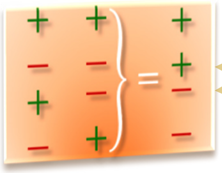
h. $\frac{2x-6}{2}$

Solution ►

a. The quotient of two negative numbers is positive, so $-\frac{-2}{-3} = -\frac{2}{3}$.

Note: *To determine the overall sign of an expression involving only multiplication and division of signed numbers, it is enough to count how many of the negative signs appear in the expression. An* ***even number of negatives*** *results in a* ***positive*** *value; an* ***odd number of negatives*** *leaves the answer* ***negative****.*

b. First, according to the sign rule, replace each double sign by a single sign. Therefore,



$$3+(-2)-(-7)-11=3-2+7-11.$$

It is convenient to treat this expression as a **sum** of signed numbers. So, it really means

$$\boxed{3}+\boxed{-2}+\boxed{7}+\boxed{-11}$$

but, for shorter notation, we tend not to write the plus signs.

Then, using the commutative property of addition, we collect all positive numbers, and all negative numbers to obtain

$$3\underbrace{-2+7}_{\substack{\text{switch}\\ \text{addends}}}-11=\underbrace{3+7}_{\substack{\text{collect}\\ \text{positive}}}\underbrace{-2-11}_{\substack{\text{collect}\\ \text{negative}}}=\underbrace{10-13}_{\text{subtract}}=\mathbf{-3}.$$

c. Since associativity of multiplication tells us that the order of performing multiplication does not change the outcome, there is no need to use any brackets in expressions involving only multiplication. So, the expression $2x(-3y)$ can be written as $2\cdot x\cdot(-3)\cdot y$. Here, the bracket is used only to isolate the negative number, not to prioritize any of the multiplications. Then, applying commutativity of multiplication to the middle two factors, we have

$$2\cdot\underbrace{x\cdot(-3)}_{\substack{\text{switch}\\ \text{factors}}}\cdot y=\underbrace{2\cdot(-3)}_{\substack{\text{perform}\\ \text{multiplication}}}\cdot x\cdot y=\boldsymbol{-6xy}$$

d. First, use commutativity of addition to switch the two middle addends, then factor out the a, and finally perform additions where possible.

$$3a\underbrace{-2-5a}_{\substack{\text{switch}\\ \text{addends}}}+4=\underbrace{3a-5a}_{\text{factor } \boldsymbol{a} \text{ out}}-2+4=\underbrace{(3-5)}_{\text{combine}}a\underbrace{-2+4}_{\text{combine}}=\boldsymbol{-2a+2}$$

***Note:** In practice, to combine terms with the same variable, add their coefficients.*

e. The expression $-(2x-5)$ represents the **opposite** to $2x-5$, which is $\boldsymbol{-2x+5}$. This expression is indeed the opposite because

$$-2x+5+2x-5=-2x\underbrace{+2x+5}_{\substack{\text{commutativity}\\ \text{of addition}}}-5=\underbrace{2x-2x}_{\text{opposites}}\underbrace{-5+5}_{\text{opposites}}=0+0=0.$$

Notice that the negative sign in front of the bracket in the expression $-(2x-5)$ can be treated as multiplication by -1. Indeed, using the distributive property of multiplication over subtraction and the sign rule, we achieve the same result

$$-1(2x-5)=-1\cdot 2x+(-1)(-5)=\boldsymbol{-2x+5}.$$

Note: *In practice, to release a bracket with a negative sign (or a negative factor) in front of it, change all the addends into opposites. For example*

$$-(2x-y+1) = -2x+y-1$$
$$\text{and} \quad -3(2x-y+1) = -6x+3y-3$$

f. To simplify $2(x^2+1)-2(x-3x^2)$, first, we apply the distributive property of multiplication and the sign rule.

$$2(x^2+1)-2(x-3x^2) = 2x^2+2-2x+6x^2$$

Then, using the commutative property of addition, we group the terms with the same powers of x. So, the equivalent expression is

$$2x^2+6x^2-2x+2$$

Finally, by factoring x^2 out of the first two terms, we can add them to obtain

$$(2+6)x^2-2x+2 = \boldsymbol{8x^2-2x+2}.$$

Note: *In practice, to combine terms with the same powers of a variable (or variables), add their coefficients. For example*

$$\underline{\underline{2x^2}}\ \underline{\underline{-5x^2}}\ \underline{+3xy}\ \underline{-xy}\ -3+2 = \underline{\underline{-3x^2}}\ \underline{+2xy}-1.$$

g. To simplify $-\frac{100ab}{25a}$, we reduce the common factors of the numerator and denominator by following the property of the neutral element of multiplication, which is one. So,

$$-\frac{100ab}{25a} = -\frac{25\cdot 4ab}{25a} = -\frac{25a\cdot 4b}{25a\cdot 1} = -\boxed{\frac{25a}{25a}}\cdot\frac{4b}{1} = -1\cdot\frac{4b}{1} = \boldsymbol{-4b}.$$

This process is called **canceling** and can be recorded in short as

$$-\frac{\overset{4}{\cancel{100}}\cancel{a}b}{\cancel{25}\cancel{a}} = \boldsymbol{-4b}.$$

h. To simplify $\frac{2x-6}{2}$, factor the numerator and then remove from the fraction the factor of one by canceling the common factor of 2 in the numerator and the denominator. So, we have

$$\frac{2x-6}{2} = \frac{\cancel{2}\cdot(x-3)}{\cancel{2}} = \boldsymbol{x-3}.$$

In the solution to *Example 2d* and *2f*, we used an intuitive understanding of what a "term" is. We have also shown how to combine terms with a common variable part (like terms). Here is a more formal definition of a term and of like terms.

Definition 3.1 ► A **term** is a **product** of constants (numbers), variables, or expressions. Here are examples of single terms:

$$1,\ x,\ \tfrac{1}{2}x^2,\ -3xy^2,\ 2(x+1),\ \tfrac{x+2}{x(x+1)},\ \pi\sqrt{x}.$$

Observe that the expression $2x+2$ consists of two terms connected by addition, while the equivalent expression $2(x+1)$ represents just one term, as it is a product of the number 2 and the expression $(x+1)$.

Like terms are the terms that have exactly the same variable part (the same variables or expressions raised to the same exponents). Like terms can be **combined** by adding their **coefficients** (numerical part of the term).

For example, $5x^2$ and $-2x^2$ are like, so they can be combined (added) to $3x^2$,
$(x+1)$ and $3(x+1)$ are like, so they can be combined to $4(x+1)$,
but $5x$ and $2y$ are unlike, so they cannot be combined.

Example 3 ► Combining Like Terms

Simplify each expression by combining like terms.

a. $-x^2+3y^2+x-6+2y^2-x+1$

b. $\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$

Solution ► **a.** Before adding like terms, it is convenient to underline the groups of like terms by the same type of underlining. So, we have

add to zero

$$\underline{-x^2}\underline{+3y^2}\ \underline{\underline{+x}}\ \underline{-6}\ \underline{+2y^2}\ \underline{\underline{-x}}\ \underline{+1}=\underline{\boldsymbol{-x^2}}\underline{\boldsymbol{+5y^2}}\ \underline{\boldsymbol{-5}}$$

b. Notice that the numerical coefficients of the first two like terms in the expression

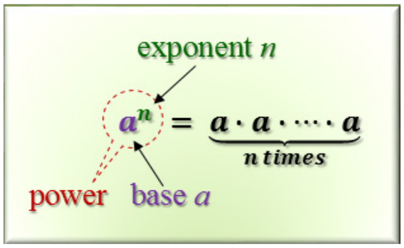
$$\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$$

are 2 and -5, and of the last two like terms are 1 and $-\frac{1}{2}$. So, by adding these coefficients, we obtain

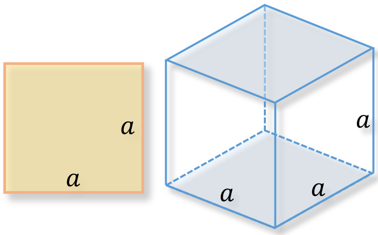
$$\boldsymbol{-\frac{3}{x+1}+\frac{1}{2}\sqrt{x}}$$

Observe that $\frac{1}{2}\sqrt{x}$ can also be written as $\frac{\sqrt{x}}{2}$. Similarly, $-\frac{3}{x+1}$, $\frac{-3}{x+1}$, or $-3\cdot\frac{1}{x+1}$ are equivalent forms of the same expression.

Exponents and Roots



Exponents are used as a shorter way of recording repeated multiplication by the same quantity. For example, to record the product $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, we write 2^5. The **exponent** 5 tells us how many times to multiply the **base** 2 by itself to evaluate the product, which is 32. The expression 2^5 is referred to as the 5th **power** of 2, or "2 to the 5th". In the case of exponents 2 or 3, terms "squared" or "cubed" are often used. This is because of the connection to geometric figures, a square and a cube.



The area of a square with sides of length a is expressed by a^2 (*read:* "a squared" or "the square of a") while the volume of a cube with sides of length a is expressed by a^3 (*read:* "a cubed" or "the cube of a").

If a negative number is raised to a certain exponent, a bracket must be used around the base number. For example, if we wish to multiply -3 by itself two times, we write $(-3)^2$, which equals $(-3)(-3) = 9$. The notation -3^2 would indicate that only 3 is squared, so $-3^2 = -3 \cdot 3 = -9$. This is because an **exponent refers only to the number immediately below the exponent**. Unless we use a bracket, a negative sign in front of a number is not under the influence of the exponent.

Example 4 ► **Evaluating Exponential Expressions**

Evaluate each exponential expression.

a. -3^4 **b.** $(-2)^6$

c. $(-2)^5$ **d.** $-(-2)^3$

e. $\left(-\frac{2}{3}\right)^2$ **f.** $-\left(-\frac{2}{3}\right)^5$

Solution ► **a.** $-3^4 = (-1) \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \mathbf{-81}$

b. $(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \mathbf{64}$

c. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = \mathbf{-32}$

> ***Observe:*** *Negative sign in front of a power works like multiplication by* -1.
>
> *A **negative** base raised to an **even exponent** results in a **positive** value.*
> *A **negative** base raised to an **odd exponent** results in a **negative** value.*

d. $-(-2x)^3 = -(-2x)(-2x)(-2x)$

$$= -(-2)(-2)(-2)xxx = -(-2)^3x^3 = -(-8)x^3 = \mathbf{8x^3}$$

e. $\left(-\frac{2}{3}\right)^2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = \frac{(-2)^2}{3^2} = \mathbf{\frac{4}{9}}$

f. $-\left(-\frac{2}{3}\right)^5 = -\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = -\frac{(-2)^5}{3^5} = -\frac{-32}{243} = \boldsymbol{\frac{32}{243}}$

Observe: *Exponents apply to every factor of the numerator and denominator of the base. This exponential property can be stated as*

$$\boxed{(ab)^n = a^n b^n} \quad \textit{and} \quad \boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

To reverse the process of squaring, we apply a **square root**, denoted by the **radical sign** $\sqrt{}$. For example, since $5 \cdot 5 = 25$, then $\sqrt{25} = 5$. Notice that $(-5)(-5) = 25$ as well, so we could also claim that $\sqrt{25} = -5$. However, we wish to define the operation of taking square root in a unique way. We choose to take the **positive** number (called **principal square root**) as the value of the square root. Therefore $\sqrt{25} = 5$, and generally

$$\boxed{\sqrt{x^2} = |x|.}$$

Since the square of any nonzero real number is positive, the square root of a negative number is not a real number. For example, we can say that $\sqrt{-16}$ **does not exist** (in the set of real numbers), as there is no real number a that would satisfy the equation $a^2 = -16$.

Example 5 ► Evaluating Radical Expressions

Evaluate each radical expression.

a. $\sqrt{0}$ **b.** $\sqrt{64}$

c. $-\sqrt{121}$ **d.** $\sqrt{-100}$

e. $\sqrt{\frac{1}{9}}$ **f.** $\sqrt{0.49}$

Solution ►

a. $\sqrt{0} = \mathbf{0}$, as $0 \cdot 0 = 0$

b. $\sqrt{64} = \mathbf{8}$, as $8 \cdot 8 = 64$

c. $-\sqrt{121} = \mathbf{-11}$, as we copy the negative sign and $11 \cdot 11 = 121$

d. $\sqrt{-100} = \boldsymbol{DNE}$ (*read:* doesn't exist), as no real number squared equals -100

e. $\sqrt{\frac{1}{9}} = \boldsymbol{\frac{1}{3}}$, as $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Notice that $\frac{\sqrt{1}}{\sqrt{9}}$ also results in $\frac{1}{3}$. So, $\sqrt{\frac{1}{9}} = \frac{\sqrt{1}}{\sqrt{9}}$ and generally $\boxed{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$ for any nonnegative real numbers a and $b \neq 0$.

f. $\sqrt{0.49} = \mathbf{0.7}$, as $0.7 \cdot 0.7 = 0.49$

Order of Operations

In algebra, similarly as in arithmetic, we like to perform various operations on numbers or on variables. To record in what order these operations should be performed, we use grouping signs, mostly brackets, but also division bars, absolute value symbols, radical symbols, etc. In an expression with many grouping signs, we perform operations in the **innermost grouping sign first**. For example, the innermost grouping sign in the expression

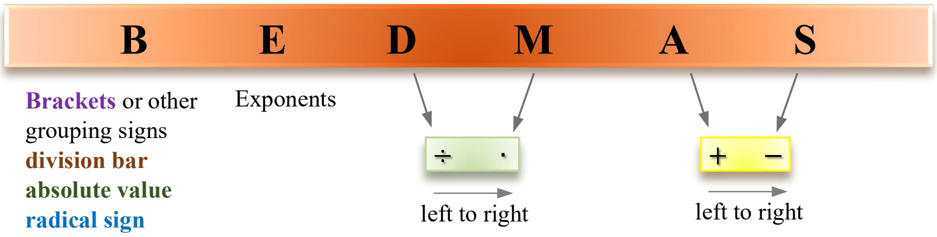
$$[4 + (3 \cdot |2 - 4|)] \div 2$$

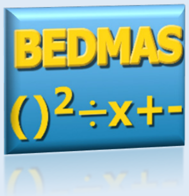
is the absolute value sign, then the round bracket, and finally, the square bracket. So first, perform subtraction, then apply the absolute value, then multiplication, addition, and finally the division. Here are the calculations:

$$\begin{aligned} &[4 + (3 \cdot |2 - 4|)] \div 2 \\ &= [4 + (3 \cdot |-2|)] \div 2 \\ &= [4 + (3 \cdot 2)] \div 2 \\ &= [4 + 6] \div 2 \\ &= 10 \div 2 \\ &= \mathbf{5} \end{aligned}$$

Observe that the more operations there are to perform, the more grouping signs would need to be used. To simplify the notation, additional rules of order of operations have been created. These rules, known as BEDMAS, allow for omitting some of the grouping signs, especially brackets. For example, knowing that multiplication is performed before addition, the expression $[4 + (3 \cdot |2 - 4|)] \div 2$ can be written as $[4 + 3 \cdot |2 - 4|] \div 2$ or $\frac{4+3\cdot|2-4|}{2}$.

Let's review the BEDMAS rule.





BEDMAS Rule:

1. Perform operations in the innermost **Bracket** (or other grouping sign) first.
2. Then work out **E**xponents.
3. Then perform **D**ivision and **M**ultiplication in order of their occurrence (left to right). *Notice that there is no priority between division and multiplication. However, both division and multiplication have priority before any addition or subtraction.*
4. Finally, perform **A**ddition and **S**ubtraction in order of their occurrence (left to right). *Again, there is no priority between addition and subtraction.*

Example 6 ► **Simplifying Arithmetic Expressions According to the Order of Operations**

Use the order of operations to simplify each expression.

a. $3 + 2 \cdot 6$

b. $4 \cdot 6 \div 3 - 2$

c. $12 \div 4 + 2|3 - 4|$

d. $2 \cdot 3^2 - 3(-2 + 6)$

e. $\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$

f. $\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot 2}$

Solution ► **a.** Out of the two operations, + and · , multiplication is performed first. So, we have

$3 + 2 \cdot 6$
$= 3 + 12$
$= \mathbf{15}$

b. There is no priority between multiplication and division, so we perform these operations in the order in which they appear, from left to right. Then we subtract. Therefore,

$4 \cdot 6 \div 3 - 2$
$= 24 \div 3 - 2$
$= 8 - 2$
$= \mathbf{6}$

c. In this expression, we have a grouping sign (the absolute value bars), so we perform the subtraction inside the absolute value first. Then, we apply the absolute value and work out the division and multiplication before the final addition. So, we obtain

$12 \div 4 + 2|3 - 4|$
$= 12 \div 4 + 2|-1|$
$= 12 \div 4 + 2 \cdot 1$
$= 3 + 2$
$= \mathbf{5}$

d. In this expression, work out the bracket first, then perform the exponent, then both multiplications, and finally the subtraction. Thus,

$2 \cdot 3^2 - 3(-2 + 6)$
$= 2 \cdot 3^2 - 3(4)$
$= 2 \cdot 9 - 3(4)$
$= 18 - 12$
$= \mathbf{6}$

e. The expression $\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$ contains two grouping signs, the bracket and the radical sign. Since these grouping signs are located at separate places (they are not nested), they can be worked out simultaneously. As usual, out of the operations inside the bracket, multiplication is done before addition. So, we calculate

$\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$
$= \sqrt{25} - 2\big(3 + (-8)\big)^2$
$= 5 - 2(-5)^2$

Work out the power first, then multiply, and finally subtract.

$$\begin{aligned} &= 5 - 2 \cdot 25 \\ &= 5 - 50 \\ &= \mathbf{-45} \end{aligned}$$

f. To simplify the expression $\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot2}$, work on the numerator and the denominator before performing the division. Therefore,

$$\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot2}$$

$-3^2 = -9$

$$= \frac{3-(-18)}{3\cdot2-6\cdot2}$$

$$= \frac{3+18}{6-12}$$

$$= \frac{21}{-6}$$

reduce the common factor of 3

$$= -\frac{\mathbf{7}}{\mathbf{2}}$$

Example 7 ► Simplifying Expressions with Nested Brackets

Simplify the expression $2\{1-5[3x+2(4x-1)]\}$.

Solution ► The expression $2\{1-5[3x+2(4x-1)]\}$ contains three types of brackets: the innermost parenthesis ( ), the middle brackets [ ], and the outermost braces { }. We start with working out the innermost parenthesis first, and then after collecting like terms, we proceed with working out consecutive brackets. So, we simplify

$$\begin{aligned} &2\{1-5[3x+2(4x-1)]\} && \text{distribute 2 over the ( ) bracket} \\ &= 2\{1-5[3x+8x-2]\} && \text{collect like terms before working out the [ ] bracket} \\ &= 2\{1-5[11x-2]\} && \text{distribute } -5 \text{ over the [ ] bracket} \\ &= 2\{1-55x+10\} && \text{collect like terms before working out the \{ \} bracket} \\ &= 2\{-55x+11\} && \text{distribute 2 over the \{ \} bracket} \\ &= \mathbf{-110x+22} \end{aligned}$$

Evaluation of Algebraic Expressions

An **algebraic expression** consists of letters, numbers, operation signs, and grouping symbols. Here are some examples of algebraic expressions:

$$6ab, \quad x^2-y^2, \quad 3(2a+5b), \quad \frac{x-3}{3-x}, \quad 2\pi r, \quad \frac{d}{t}, \quad Prt, \quad \sqrt{x^2+y^2}$$

When a letter is used to stand for various numerical values, it is called a **variable**. For example, if t represents the number of hours needed to drive between particular towns, then t changes depending on the average speed used during the trip. So, t is a variable. Notice however, that the distance d between the two towns represents a constant number. So, even though letters in algebraic expressions usually represent variables, sometimes they may represent a **constant** value. One such constant is the letter π, which represents approximately 3.14.

Notice that algebraic expressions do not contain any comparison signs (equality or inequality, such as $=$, $\neq$, $<$, $\leq$, $>$, $\geq$), therefore, they are **not to be solved** for any variable. Algebraic expressions can only be **simplified** by implementing properties of operations (see *Example 2* and *3*) or **evaluated** for particular values of the variables. The evaluation process involves substituting given values for the variables and evaluating the resulting arithmetic expression by following the order of operations.

Advice: *To evaluate an algebraic expression for given variables, first rewrite the expression replacing each variable with* ***empty brackets*** *and then write appropriate values inside these brackets. This will help to avoid possible errors of using incorrect signs or operations.*

Example 8 ► **Evaluating Algebraic Expressions**

Evaluate each expression for $a = -2$, $b = 3$, and $c = 6$.

a. $b^2 - 4ac$ **b.** $2c \div 3a$ **c.** $\frac{|a^2-b^2|}{-a^2+\sqrt{b+c}}$

Solution ►

a. First, we replace each letter in the expression $b^2 - 4ac$ with an empty bracket. So, we write

$$(\ \ )^2 - 4(\ \ )(\ \ ).$$

Now, we fill in the brackets with the corresponding values and evaluate the resulting expression. So, we have

$$(3)^2 - 4(-2)(6) = 9 - (-48) = 9 + 48 = \mathbf{57}.$$

b. As above, we replace the letters with their corresponding values to obtain

$$2c \div 3a = 2(6) \div 3(-2).$$

Since we work only with multiplication and division here, they are to be performed in order from left to right. Therefore,

$$2(6) \div 3(-2) = 12 \div 3(-2) = 4(-2) = \mathbf{-8}.$$

c. As above, we replace the letters with their corresponding values to obtain

$$\frac{|a^2 - b^2|}{-a^2 + \sqrt{b + c}} = \frac{|(-2)^2 - (3)^2|}{-(-2)^2 + \sqrt{(3) + (6)}} = \frac{|4 - 9|}{-4 + \sqrt{9}} = \frac{|-5|}{-4 + 3} = \frac{5}{-1} = \mathbf{-5}.$$

Equivalent Expressions

Algebraic expressions that produce the same value for all allowable values of the variables are referred to as **equivalent expressions**. Notice that properties of operations allow us to rewrite algebraic expressions in a simpler but equivalent form. For example,

$$\frac{\boldsymbol{x-3}}{\boldsymbol{3-x}} = \frac{\cancel{x-3}}{-(\cancel{x-3})} = \boldsymbol{-1}$$

or

$$(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) = (x+y)x-(x+y)y = x^2+\cancel{yx}-\cancel{xy}-y^2 = \boldsymbol{x^2}-\boldsymbol{y^2}.$$

To show that two expressions are **<u>not</u> equivalent**, it is enough to find a particular set of variable values for which the two expressions evaluate to a different value. For example,

$$\sqrt{x^2+y^2} \neq x+y$$

because if $x = 1$ and $y = 1$ then $\sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$ while $x+y = 1+1 = 2$. Since $\sqrt{2} \neq 2$ the two expressions $\sqrt{x^2+y^2}$ and $x+y$ are not equivalent.

Example 9 ► **Determining Whether a Pair of Expressions is Equivalent**

Determine whether the given expressions are equivalent.

a. $(a+b)^2$ and a^2+b^2

b. $\frac{x^8}{x^4}$ and x^4

Solution ►

a. Suppose $a = 1$ and $b = 1$. Then

$$(a+b)^2 = (1+1)^2 = 2^2 = 4$$

but

$$a^2+b^2 = 1^2+1^2 = 2.$$

So the expressions $(a+b)^2$ and a^2+b^2 are not equivalent.

Using the distributive property and commutativity of multiplication, check on your own that

$$(a+b)^2 = a^2+2ab+b^2.$$

b. Using properties of exponents and then removing a factor of one, we show that

$$\frac{x^8}{x^4} = \frac{x^4 \cdot x^4}{x^4} = x^4.$$

So the two expressions are indeed equivalent.

Review of Operations on Fractions

A large part of algebra deals with performing operations on algebraic expressions by generalising the ways that these operations are performed on real numbers, particularly, on common fractions. Since operations on fractions are considered to be one of the most challenging topics in arithmetic, it is a good idea to review the rules to follow when performing these operations before we move on to other topics of algebra.

Operations on Fractions:

Simplifying

To simplify a fraction to its lowest terms, **remove** the **greatest common factor** (**GCF**) of the numerator and denominator. For example, $\frac{48}{64} = \frac{3\cdot\cancel{16}}{4\cdot\cancel{16}} = \frac{3}{4}$, and generally $\frac{\boldsymbol{a}\cancel{\boldsymbol{k}}}{\boldsymbol{b}\cancel{\boldsymbol{k}}} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.

This process is called ***reducing*** or ***canceling***.

Note that the reduction can be performed several times, if needed. In the above example, if we didn't notice that 16 is the greatest common factor for 48 and 64, we could reduce the fraction by dividing the numerator and denominator by any common factor (2, or 4, or 8) first, and then repeat the reduction process until there is no common factors (other than 1) anymore. For example,

$$\frac{\cancel{48}}{\cancel{64}} \underset{\div\text{ by }2}{=} \frac{\cancel{24}}{\cancel{32}} \underset{\div\text{ by }4}{=} \frac{\cancel{6}}{\cancel{8}} \underset{\div\text{ by }2}{=} \frac{\mathbf{3}}{\mathbf{4}}.$$

Multiplying

To multiply fractions, we multiply their numerators and denominators. So generally,

$$\frac{\boldsymbol{a}}{\boldsymbol{b}}\cdot\frac{\boldsymbol{c}}{\boldsymbol{d}} = \frac{\boldsymbol{ac}}{\boldsymbol{bd}}.$$

However, before performing multiplication of numerators and denominators, it is a good idea to reduce first. This way, we work with smaller numbers, which makes the calculations easier. For example,

$$\frac{18}{15}\cdot\frac{25}{14} = \frac{18\cdot\cancel{25}}{\cancel{15}\cdot 14} \underset{\div\text{ by }5}{=} \frac{\cancel{18}\cdot 5}{\cancel{3}\cdot 14} \underset{\div\text{ by }3}{=} \frac{\cancel{6}\cdot 5}{1\cdot\cancel{14}} \underset{\div\text{ by }2}{=} \frac{3\cdot 5}{1\cdot 7} = \frac{\mathbf{15}}{\mathbf{7}}.$$

Dividing

To divide fractions, we **multiply** the dividend (the first fraction) **by the reciprocal** of the **divisor** (the second fraction). So generally,

$$\frac{\boldsymbol{a}}{\boldsymbol{b}} \div \frac{\boldsymbol{c}}{\boldsymbol{d}} \underset{\cdot\text{ by reciprocal}}{=} \frac{\boldsymbol{a}}{\boldsymbol{b}}\cdot\frac{\boldsymbol{d}}{\boldsymbol{c}} = \frac{\boldsymbol{ad}}{\boldsymbol{bc}}.$$

For example,

$$\frac{8}{15}\div\frac{4}{5} = \frac{\cancel{8}}{\cancel{15}}\cdot\frac{\cancel{5}}{\cancel{4}} \overset{\div\text{ by }4}{\underset{\div\text{ by }5}{=}} \frac{2\cdot 1}{3\cdot 1} = \frac{\mathbf{2}}{\mathbf{3}}.$$

Adding or Subtracting

To add or subtract fractions, follow the steps:

1. Find the **Lowest Common Denominator** (**LCD**).
2. Extend each fraction to higher terms to obtain the desired common denominator.
3. Add or subtract the numerators, keeping the common denominator.
4. Simplify the resulting fraction, if possible.

For example, to evaluate $\frac{5}{6}+\frac{3}{4}-\frac{4}{15}$, first we find the LCD for denominators 6, 4, and 15. We can either guess that 60 is the least common multiple of 6, 4, and 15, or we can use the following method of finding LCD:

$$\begin{array}{c|ccc} 2 & 6 & 4 & 15 \\ \cdot\, 3 & 3 & 2 & 15 \\ \hline \cdot & 1 & \cdot\, 2 & \cdot\, 5 \end{array} = 60$$

- divide by a common factor of at least two numbers; for example, by 2
- write the quotients in the line below; 15 is not divisible by 2, so just copy it down
- keep dividing by common factors until all numbers become relatively prime
- the LCD is the product of all numbers listed in the letter L, so it is 60

Then, we extend the fractions so that they share the same denominator of 60, and finally perform the operations in the numerator. Therefore,

$$\frac{5}{6}+\frac{3}{4}-\frac{4}{5}=\underbrace{\frac{5\cdot 10}{6\cdot 10}+\frac{3\cdot 15}{4\cdot 15}-\frac{4\cdot 12}{5\cdot 12}}_{\text{in practice, this step doesn't have to be written}}=\frac{5\cdot 10+3\cdot 15-4\cdot 12}{60}=\frac{50+45-48}{60}=\mathbf{\frac{47}{60}}.$$

Example 10 ► Evaluating Fractional Expressions

Simplify each expression.

a. $-\frac{2}{3}-\left(-\frac{5}{12}\right)$

b. $-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$

Solution ►

a. After replacing the double negative by a positive sign, we add the two fractions, using 12 as the lowest common denominator. So, we obtain

$$-\frac{2}{3}-\left(-\frac{5}{12}\right)=-\frac{2}{3}+\frac{5}{12}=\frac{-2\cdot 4+5}{12}=\frac{-\cancel{3}}{\cancel{12}}=-\mathbf{\frac{1}{4}}.$$

b. Following the order of operations, we calculate

$$-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$$

First, perform the division in the bracket by converting it to a multiplication by the reciprocal. The quotient becomes negative.

$$=-3\left[\frac{3}{2}-\frac{5}{\cancel{6}_{3}}\cdot\frac{\cancel{8}^{4}}{3}\right]$$

Reduce, before multiplying.

$$=-3\left(\frac{3}{2}-\frac{20}{9}\right)$$

Extend both fractions to higher terms using the common denominator of 18.

$$=-3\left(\frac{27-40}{18}\right)$$

Perform subtraction.

$$=-\cancel{3}\left(\frac{-13}{\cancel{18}}\right)$$

Reduce before multiplying. The product becomes positive.

$$=\mathbf{\frac{13}{6}}\text{ or equivalently }\mathbf{2\frac{1}{6}}.$$

R.3 Exercises

True or False?

1. The set of integers is closed under multiplication.

2. The set of natural numbers is closed under subtraction.

3. The set of real numbers different than zero is closed under division.

4. According to the BEDMAS rule, division should be performed before multiplication.

5. For any real number $\sqrt{x^2} = x$.

6. Square root of a negative number is not a real number.

7. If the value of a square root exists, it is positive.

8. $-x^3 = (-x)^3$

9. $-x^2 = (-x)^2$

Complete each statement to illustrate the indicated property.

10. $x + (-y) =$ _________ , commutative property of addition

11. $(7 \cdot 5) \cdot 2 =$ __________ , associative property of multiplication

12. $(3 + 8x) \cdot 2 =$ _____________ , distributive property of multiplication over addition

13. $a +$ ____ $= 0$, additive inverse

14. $-\frac{a}{b} \cdot$ _____ $= 1$, multiplicative inverse

15. $\frac{3x}{4y} \cdot$ _____ $= \frac{3x}{4y}$, multiplicative identity

16. _____ $+ (-a) = -a$, additive identity

17. $(2x - 7) \cdot$ ____ $= 0$, multiplication by zero

18. If $(x + 5)(x - 1) = 0$, then ________ $= 0$ or ________ $= 0$, zero product property

Perform operations.

19. $-\frac{2}{5} + \frac{3}{4}$

20. $\frac{5}{6} - \frac{2}{9}$

21. $\frac{5}{8} \cdot \left(-\frac{2}{3}\right) \cdot \frac{18}{15}$

22. $-3\left(-\frac{5}{9}\right)$

23. $-\frac{3}{4}(8x)$

24. $\frac{15}{16} \div \left(-\frac{9}{12}\right)$

Use order of operations to evaluate each expression.

25. $64 \div (-4) \div 2$

26. $3 + 3 \cdot 5$

27. $8 - 6(5 - 2)$

28. $20 + 4^3 \div (-8)$

29. $6\left(9 - 3\sqrt{9 - 5}\right)$

30. $-2^5 - 8 \div 4 - (-2)$

31. $-\frac{5}{6}+\left(-\frac{7}{4}\right)\div 2$

32. $\left(-\frac{3}{2}\right)\cdot\frac{1}{6}-\frac{2}{5}$

33. $-\frac{3}{2}\div\left(-\frac{4}{9}\right)-\frac{5}{4}\cdot\frac{2}{3}$

34. $-3\left(-\frac{4}{9}\right)-\frac{1}{4}\div\frac{3}{5}$

35. $2-3|3-4\cdot 6|$

36. $\frac{3|5-7|-6\cdot 4}{5\cdot 6-2|4-1|}$

Simplify each expression.

37. $-(x-y)$

38. $-2(3a-5b)$

39. $\frac{2}{3}(24x+12y-15)$

40. $\frac{3}{4}(16a-28b+12)$

41. $5x-8x+2x$

42. $3a+4b-5a+7b$

43. $5x-4x^2+7x-9x^2$

44. $8\sqrt{2}-5\sqrt{2}+\frac{1}{x}+\frac{3}{x}$

45. $2+3\sqrt{x}-6-\sqrt{x}$

46. $\frac{a-b}{b-a}$

47. $\frac{2(x-3)}{3-x}$

48. $-\frac{100ab}{75a}$

49. $-(5x)^2$

50. $\left(-\frac{2}{3}a\right)^2$

51. $5a-(4a-7)$

52. $6x+4-3(9-2x)$

53. $5x-4(2x-3)-7$

54. $8x-(-4y+7)+(9x-1)$

55. $6a-[4-3(9a-2)]$

56. $5\{x+3[4-5(2x-3)-7]\}$

57. $-2\{2+3[4x-3(5x+1)]\}$

58. $4\{[5(x-3)+5^2]-3[2(x+5)-7^2]\}$

59. $3\{[6(x+4)-3^3]-2[5(x-8)-8^2]\}$

Evaluate each algebraic expression for $a=-2$, $b=3$, and $c=2$.

60. b^2-a^2

61. $6c\div 3a$

62. $\frac{c-a}{c-b}$

63. $b^2-3(a-b)$

64. $\frac{-b+\sqrt{b^2-4ac}}{2a}$

65. $c\left(\frac{a}{b}\right)^{|a|}$

Determine whether each pair of expressions is equivalent.

66. $x^3\cdot x^2$ and x^5

67. a^2-b^2 and $(a-b)^2$

68. $\sqrt{x^2}$ and x

69. $(x^3)^2$ and x^5

Use the distributive property to calculate each value mentally.

70. $96\cdot 18+4\cdot 18$

71. $29\cdot 70+29\cdot 30$

72. $57\cdot\frac{3}{5}-7\cdot\frac{3}{5}$

73. $\frac{8}{5}\cdot 17+\frac{8}{5}\cdot 13$

Insert one pair of parentheses to make the statement true.

74. $2\cdot 3+6\div 5-3=9$

75. $9\cdot 5+2-8\cdot 3+1=22$

Attributions

p.**31** Question Mark by 3dman_eu, CC0; Search by 3dman_eu, CC0.

Linear Equations and Inequalities

One of the main concepts in Algebra is solving equations or inequalities. This is because solutions to most application problems involve setting up and solving equations or inequalities that describe the situation presented in the problem. In this unit, we will study techniques of solving linear equations and inequalities in one variable, linear forms of absolute value equations and inequalities, and applications of these techniques in word problems.

L1 Linear Equations in One Variable

When two algebraic expressions are compared by an equal sign $(=)$, an **equation** is formed. An equation can be interpreted as a scale that balances two quantities. It can also be seen as a mathematical sentence with the verb "equals" or the verb phrase "is equal to". For example, the equation $3x - 1 = 5$ corresponds to the sentence:

One less than three times an unknown number equals five.

Unless we know the value of the unknown number (the variable x), we are unable to determine whether or not the above sentence is a true or false statement. For example, if $x = 1$, the equation $3x - 1 = 5$ becomes a false statement, as $3 \cdot 1 - 1 \neq 5$ (the "scale" is not in balance); while, if $x = 2$, the equation $3x - 1 = 5$ becomes a true statement, as $3 \cdot 2 - 1 = 5$ (the "scale" is in balance). For this reason, such sentences (equations) are called **open sentences**. Each variable value that satisfies an equation (i.e., makes it a true statement) is a **solution** (i.e., a ***root***, or a ***zero***) of the equation. An equation is **solved** by finding its **solution set**, the set of all solutions.

Attention:

* ***Equations*** *can be* ***solved*** *by finding the variable value(s) satisfying the equation.*

Example: $$\overbrace{\mathbf{2x + x - 1}}^{left\ side} \underset{\substack{\uparrow \\ equal\ sign}}{=} \overbrace{\mathbf{5}}^{right\ side}$$ *can be solved for x*

* ***Expressions*** *can only be* ***simplified*** *or* ***evaluated***

Example: $\mathbf{2x + x - 1}$ *can be simplified to* $3x - 1$
or evaluated for a particular x-value

Example 1 ► Distinguishing Between Expressions and Equations

Decide whether each of the following is an expression or an equation.

a. $4x - 16$ **b.** $4x - 16 = 0$

Solution ► **a.** $4x - 16$ is an **expression** as it does not contain any symbol of equality.

This expression can be ***evaluated*** *(for instance, if* $x = 4$*, the expression assumes the value 0), or it can be written in a different form. For example, we could* ***factor*** *it. So, we could write*

$$4x - 16 = 4(x - 4).$$

Notice that the equal symbol (=) in the above line does not indicate an equation, but rather an equivalency between the two expressions, $4x - 16$ *and* $4(x - 4)$.

b. $4x - 16 = 0$ is an **equation** as it contains an equal symbol $(=)$ that connects two sides of the equation.

To solve this equation we could factor the left-hand side expression,

$$4(x-4) = 0,$$

and then from the Zero Product Property (see Section F4), we have

$$x - 4 = 0,$$

which leads us to the solution

$$x = \mathbf{4}.$$

Attention: *Even though the two algebraic forms,* $4x - 16$ *and* $4x - 16 = 0$ *are related to each other, it is important that we neither* ***voluntarily add*** *the "$= 0$" part when we want to change the form of the expression, nor* ***voluntarily drop*** *the "$= 0$" part when we solve the equation.*

Example 2 ► Determining if a Given Number is a Solution to an Equation

Determine whether the number -2 is a solution to the given equation.

a. $4x = 10 + x$

b. $x^2 - 4 = 0$

Solution ►

a. To determine whether -2 is a solution to the equation $4x = 10 + x$, we substitute -2 in place of the variable x and find out whether the resulting equation is a true statement. This gives us

$$4(-2) \overset{?}{=} 10 + (-2)$$

$$-8 \overset{?}{=} 8$$

Since the resulting equation is not a true statement, the number $\mathbf{-2}$ **is not a solution** to the given equation.

b. After substituting -2 for x in the equation $x^2 - 4 = 0$, we obtain

$$(-2)^2 - 4 \overset{?}{=} 0$$

$$4 - 4 \overset{?}{=} 0,$$

which becomes $0 = 0$, a true statement. Therefore, the number $\mathbf{-2}$ **is a solution** to the given equation.

Equations can be classified with respect to the number of solutions. There are **identities**, **conditional** equations, and **contradictions** (or ***inconsistent*** equations).

Definition 1.1 ► An **identity** is an equation that is satisfied by every real number for which the expressions on both sides of the equation are defined. Some examples of identities are

$$2x + 5x = 7x, \quad x^2 - 4 = (x+2)(x-2), \quad \text{or} \quad \frac{x}{x} = 1.$$

The solution set of the first two identities is the set of all real numbers, $\mathbb{R}$*. However, since the expression* $\frac{x}{x}$ *is undefined for* $x = 0$*, the solution set of the equation* $\frac{x}{x} = 1$ *is the set of all nonzero real numbers,* $\{\boldsymbol{x} | \boldsymbol{x} \neq \boldsymbol{0}\}$.

A **conditional** equation is an equation that is satisfied by at least one real number, but is not an identity. This is the most commonly encountered type of equation. Here are some examples of conditional equations:

$$3x - 1 = 5, \quad x^2 - 4 = 0, \quad \text{or} \quad \sqrt{x} = 2.$$

The solution set of the first equation is $\{2\}$*; of the second equation is* $\{-2,2\}$*; and of the last equation is* $\{4\}$.

A **contradiction** (an ***inconsistent*** equation) is an equation that has **no solution**. Here are some examples of contradictions:

$$5 = 1, \quad 3x - 3x = 8, \quad \text{or} \quad 0x = 1.$$

The solution set of any contradiction is the empty set, $\emptyset$.

Example 3 ► **Recognizing Conditional Equations, Identities, and Contradictions**

Determine whether the given equation is *conditional*, an *identity*, or a *contradiction*.

a. $x = x$ **b.** $x^2 = 0$ **c.** $\frac{1}{x} = 0$

Solution ►

a. This equation is satisfied by any real number. Therefore, it is an **identity**.

b. This equation is satisfied by $x = 0$, as $0^2 = 0$. However, any nonzero real number when squared becomes a positive number. So, the left side of the equation $x^2 = 0$ does not equal to zero for a nonzero x. That means that a nonzero number does not satisfy the equation. Therefore, the equation $x^2 = 0$ has exactly one solution, $x = 0$. So, the equation is **conditional**.

c. A fraction equals zero only when its numerator equals to zero. Since the numerator of $\frac{1}{x}$ does not equal to zero, then no matter what the value of x would be, the left side of the equation will never equal zero. This means that there is no x-value that would satisfy the equation $\frac{1}{x} = 0$. Therefore, the equation has no solution, which means it is a **contradiction**.

Attention: *Do not confuse the solution* $x = 0$ *to the equation in Example 3b with an empty set* $\emptyset$*. An empty set means that there is no solution.* $x = 0$ *means that there is one solution equal to zero.*

Solving Linear Equations in One Variable

In this section, we will focus on solving linear (up to the first degree) equations in one variable. Before introducing a formal definition of a linear equation, let us recall the definition of a term, a constant term, and a linear term.

Definition 1.2 ► A **term** is a **<u>product</u>** of numbers, letters, and possibly other algebraic expressions.

Examples of terms: $2,\ -3x,\ \frac{2}{3}(x+1),\ 5x^2y,\ -5\sqrt{x}$

A **constant term** is a number or a product of numbers.

Examples of constant terms: $2,\ -3,\ \frac{2}{3},\ 0,\ -5\pi$

A **linear term** is a product of numbers and the first power of a single variable.

Examples of linear terms: $-3x,\ \frac{2}{3}x,\ x,\ -5\pi x$

Definition 1.3 ► A **linear** equation is an equation with only **constant** or **linear terms**. A linear equation in one variable can be written in the form $\boldsymbol{Ax+B=0}$, for some real numbers A and B, and a variable x.

Here are some examples of *linear* equations: $2x+1=0,\quad 2=5,\quad 3x-7=6+2x$

Here are some examples of *nonlinear* equations: $x^2=16,\quad x+\sqrt{x}=-1,\quad 1+\frac{1}{x}=\frac{1}{x+1}$

So far, we have been finding solutions to equations mostly by guessing a value that would make the equation true. To find a methodical way of solving equations, observe the relations between equations with the same solution set. For example, equations

$$3x-1=5,\ \ 3x=6,\ \ and\ \ x=2$$

all have the same solution set $\{2\}$. While the solution to the last equation, $x=2$, is easily "seen" to be 2, the solution to the first equation, $3x-1=5$, is not readily apparent. Notice that the second equation is obtained by adding 1 to both sides of the first equation. Similarly, the last equation is obtained by dividing the second equation by 3. This suggests that to solve a linear equation, it is enough to write a sequence of simpler and simpler equations that preserve the solution set, and eventually result in an equation of the form:

$$x = constant \quad \text{or} \quad 0 = constant.$$

If the resulting equation is of the form $x = constant$, the solution is this constant.
If the resulting equation is $\mathbf{0}=\mathbf{0}$, then the original equation is an **identity**, as it is true for **all real values $\boldsymbol{x}$**.
If the resulting equation is $\mathbf{0} = \boldsymbol{constant\ other\ than\ zero}$, then the original equation is a **contradiction**, as there is **no real values $\boldsymbol{x}$** that would make it true.

Definition 1.4 ► **Equivalent equations** are equations with the same solution set.

How can we transform an equation to obtain a simpler but equivalent one?

We can certainly simplify expressions on both sides of the equation, following properties of operations listed in *section R3*. Also, recall that an equation works like a scale in balance.

Therefore, adding (or subtracting) the same quantity to (from) both sides of the equation will preserve this balance. Similarly, multiplying (or dividing) both sides of the equation by a nonzero quantity will preserve the balance.

Suppose we work with an equation $A = B$, where A and B represent some algebraic expressions. In addition, suppose that C is a real number (or another expression).
Here is a summary of the basic equality operations that can be performed to produce equivalent equations:

Equality Operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) = 1+3$ can be written as $2x-6=4$
Addition	$A + C = B + C$	$2x - 6 + 6 = 4 + 6$
Subtraction	$A - C = B - C$	$2x - 6 - 4 = 4 - 4$
Multiplication	$CA = CB, \quad if\ C \neq 0$	$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 10$
Division	$\frac{A}{C} = \frac{B}{C}, \quad if\ C \neq 0$	$\frac{2x}{2} = \frac{10}{2}$

Example 4 ► Using Equality Operations to Solve Linear Equations in One Variable

Solve each equation.

a. $4x - 12 + 3x = 3 + 5x - 2x$ **b.** $2[3(x-6) - x] = 3x - 2(5 - x)$

Solution ► **a.** First, simplify each side of the equation and then isolate the linear terms (terms containing x) on one side of the equation. Here is a sequence of equivalent equations that leads to the solution:

$4x - 12 + 3x = 3 + 6x - 2x$	collect like terms	(1)
$7x - 12 = 3 + 4x$	move the x-terms to the left side by subtracting $4x$ and the constant terms to the right side by adding 12	(2)
$7x - \cancel{12} - 4x + \cancel{12} = 3 + \cancel{4x} - \cancel{4x} + 12$		(3)
$7x - 4x = 3 + 12$	collect like terms	(4)
$3x = 15$	isolate x by dividing both sides by 3	(5)
$\frac{\cancel{3}x}{\cancel{3}} = \frac{15}{3}$	simplify each side	(6)
$x = \mathbf{5}$		(7)

Each equation is written underneath the previous one, with the **"=" symbol aligned in a column**

There is only **one "=" symbol** in each line

Let us analyze the relation between line (2) and (4).

$$7x - \mathbf{12} = 3 + \mathbf{4x} \qquad (2)$$

$$7x - \mathbf{4x} = 3 + \mathbf{12} \qquad (4)$$

By subtracting $4x$ from both sides of the equation (2), we actually 'moved' the term $+4x$ to the left side of equation (4) as $-4x$. Similarly, the addition of 12 to both sides of equation (2) caused the term -12 to 'move' to the other side as $+12$, in equation (4). This shows that the addition and subtraction property of equality allows us to change the position of a term from one side of an equation to another, by simply changing its sign. Although line (3) is helpful when explaining why we can move particular terms to another side by changing their signs, it is often cumbersome, especially when working with longer equations. So, in practice, we will avoid writing lines such as (3). Here is how we could record the solution to equation (1) in a concise way.

$$4x - 12 + 3x = 3 + 6x - 2x$$

$$7x - 12 = 3 + 4x$$

$$7x - 4x = 3 + 12$$

$$3x = 15$$

$$x = \mathbf{5}$$

b. First, release all the brackets, starting from the inner-most brackets. If applicable, remember to collect like terms after releasing each bracket. Finally, isolate x by applying appropriate equality operations. Here is our solution:

$$2[3(x-6) - x] = 3x - 2(5 - x) \qquad \text{release round red brackets}$$

$$2[3x - 18 - x] = 3x - 10 + 2x \qquad \text{collect like terms}$$

$$2[2x - 18] = 5x - 10 \qquad \text{release the square blue brackets}$$

$$4x - 36 = 5x - 10 \qquad \text{add } -5x \text{ and } 36 \text{ to both sides}$$

$$-x = 26 \qquad \text{divide both sides by } -1$$

$$x = \mathbf{-26}$$

multiplication by -1 works as well

Note: *Notice that we could choose to collect x-terms on the right side of the equation as well. This would shorten the solution by one line and save us the division by -1. Here is the alternative ending of the above solution.*

$$4x - 36 = 5x - 10 \qquad \text{add } -4x \text{ and } 10 \text{ to both sides}$$

$$-26 = x$$

Example 5 ► Solving Linear Equations Involving Fractions

Solve

$$\frac{x-4}{4}+\frac{2x+1}{6}=5.$$

Solution ► First, clear the fractions and then solve the resulting equation as in *Example 4*. To clear fractions, multiply both sides of the equation by the LCD of 4 and 6, which is 12.

$$\frac{x-4}{4}+\frac{2x+1}{6}=5 \qquad \text{multiply both sides by 12} \tag{1}$$

$$\overset{3}{\cancel{12}}\left(\frac{x-4}{\cancel{4}}\right)+\overset{2}{\cancel{12}}\left(\frac{2x+1}{\cancel{6}}\right)=12\cdot 5 \tag{2}$$

$$3(x-4)+2(2x+1)=60 \tag{3}$$

$$3x-12+4x+2=60 \tag{4}$$

$$7x-10=60 \tag{5}$$

$$7x=70 \tag{6}$$

$$x=\frac{70}{7}=\mathbf{10} \tag{7}$$

When multiplying each term by the $LCD = 12$, **simplify** it with the **denominator** before **multiplying** the result by the **numerator**.

So the solution to the given equation is $x = \mathbf{10}$.

Note: *Notice, that if the division of 12 by 4 and then by 6 can be performed fluently in our minds, writing equation* (2) *is not necessary. One could write equation* (3) *directly after the original equation* (1). *One could think: 12 divided by 4 is 3 so I multiply the resulting 3 by the numerator* $(x-4)$. *Similarly, 12 divided by 6 is 2 so I multiply the resulting 2 by the numerator* $(2x+1)$. *It is important though that each term, including the free term 5, gets multiplied by 12.*

Also, notice that the reason we multiply equations involving fractions by LCD's is to clear the denominators of those fractions. That means that if the multiplication by an appropriate LCD is performed correctly, the resulting equation should not involve any denominators!

Example 6 ► Solving Linear Equations Involving Decimals

Solve $0.07x - 0.03(15-x) = 0.05(14)$.

Solution ► To solve this equation, it is convenient (although not necessary) to clear the decimals first. This is done by multiplying the given equation by 100.

Each **term** (product of numbers and variable expressions) needs to be multiplied by 100.

$$0.07x-0.03(15-x)=0.05(14) \qquad \text{multiply both sides by 100}$$

$$7x-3(15-x)=5(14)$$

$$7x-45+3x=70$$

$$10x = 70 + 45$$

$$x = \frac{115}{10} = \mathbf{11.5}$$

So the solution to the given equation is $x = \mathbf{11.5}$.

Note: *In general, if* n *is the highest number of decimal places to clear in an equation, we multiply it by* 10^n.

Attention: *To multiply a product AB by a number C, we multiply just one factor of this product, either A or B, but not both! For example,*

$$10 \cdot 0.3(0.5 - x) = (10 \cdot 0.3)(0.5 - x) = 3(0.5 - x) \quad \checkmark$$

or

$$10 \cdot 0.3(0.5 - x) = 0.3 \cdot [10(0.5 - x)] = 0.3(5 - 10x) \quad \checkmark$$

but

$$10 \cdot 0.3(0.5 - x) \neq (10 \cdot 0.3)[10(0.5 - x)] = 3(5 - 10x) \quad ✗$$

Summary of Solving a Linear Equation in One Variable

- **Clear fractions or decimals**. Eliminate fractions by multiplying each side by the least common denominator (LCD). Eliminate decimals by multiplying by a power of 10.
- **Clear brackets** (starting from the inner-most ones) by applying the distributive property of multiplication. **Simplify** each side of the equation by **combining like terms**, as needed.
- **Collect** and **combine variable terms** on one side and free terms on the other side of the equation. Use the addition property of equality to collect all variable terms on one side of the equation and all free terms (numbers) on the other side.
- **Isolate the variable** by dividing the equation by the linear coefficient (coefficient of the variable term).

L.1 Exercises

True or False? Justify your answer.

1. The equation $5x - 1 = 9$ is equivalent to $5x - 5 = 5$.

2. The equation $x + \sqrt{x} = -1 + \sqrt{x}$ is equivalent to $x = -1$.

3. The solution set to $12x = 0$ is $\emptyset$.

4. The equation $x - 0.3x = 0.97x$ is an identity.

5. To solve $-\frac{2}{3}x = \frac{3}{5}$, we could multiply each side by the reciprocal of $-\frac{2}{3}$.

6. If a and b are real numbers, then $ax + b = 0$ has a solution.

Decide whether each of the following is an ***equation to solve*** *or an* ***expression to simplify****.*

7. $3x + 2(x - 6) - 1$

8. $3x + 2(x - 6) = 1$

9. $-5x + 19 = 3x - 5$

10. $-5x + 19 - 3x + 5$

Determine whether or not the given equation is linear.

11. $4x + 2 = x - 3$

12. $12 = x^2 + x$

13. $x + \frac{1}{x} = 1$

14. $2 = 5$

15. $\sqrt{16} = x$

16. $\sqrt{x} = 9$

Determine whether the given value is a solution of the equation.

17. $2,\ 3x - 4 = 2$

18. $-2,\ \frac{1}{x} - \frac{1}{2} = -1$

19. $6,\ \sqrt{2x + 4} = -4$

20. $-4,\ (x - 1)^2 = 25$

Solve each equation. If applicable, tell whether the equation is an ***identity*** *or a* ***contradiction****.*

21. $6x - 5 = 0$

22. $-2x + 5 = 0$

23. $-3x + 6 = 12$

24. $5x - 3 = -13$

25. $3y - 5 = 4 + 12y$

26. $9y - 4 = 14 + 15y$

27. $2(2a - 3) - 7 = 4a - 13$

28. $3(4 - 2b) = 4 - (6b - 8)$

29. $-3t + 5 = 4 - 3t$

30. $5p - 3 = 11 + 4p + p$

31. $13 - 9(2n + 3) = 4(6n + 1) - 15n$

32. $5(5n - 7) + 40 = 2n - 3(8n + 5)$

33. $3[1 - (4x - 5)] = 5(6 - 2x)$

34. $-4(3x + 7) = 2[9 - (7x + 10)]$

35. $3[5 - 3(4 - t)] - 2 = 5[3(5t - 4) + 8] - 16$

36. $6[7 - 4(8 - t)] - 13 = -5[3(5t - 4) + 8]$

37. $\frac{2}{3}(9n - 6) - 5 = \frac{2}{5}(30n - 25) - 7n$

38. $\frac{1}{2}(18 - 6n) + 5n = 10 - \frac{1}{4}(16n + 20)$

39. $\frac{8x}{3} - \frac{5x}{4} = -17$

40. $\frac{7x}{2} - \frac{5x}{10} = 5$

41. $\frac{3x-1}{4} + \frac{x+3}{6} = 3$

42. $\frac{3x+2}{7} - \frac{x+4}{5} = 2$

43. $\frac{2}{3}\left(\frac{7}{8} + 4x\right) - \frac{5}{8} = \frac{3}{8}$

44. $\frac{3}{4}\left(3x - \frac{1}{2}\right) - \frac{2}{3} = \frac{1}{3}$

45. $x - 2.3 = 0.08x + 3.5$

46. $x + 1.6 = 0.02x - 3.6$

47. $0.05x + 0.03(5000 - x) = 0.04 \cdot 5000$

48. $0.02x + 0.04 \cdot 3000 = 0.03(x + 3000)$

L2 Formulas and Applications

In the previous section, we studied how to solve linear equations. Those skills are often helpful in problem solving. However, the process of solving an application problem has many components. One of them is the ability to construct a mathematical model of the problem. This is usually done by observing the relationship between the variable quantities in the problem and writing an equation that describes this relationship.

Definition 2.1 ► An equation that represents or models a relationship between two or more quantities is called a **formula**.

To model real situations, we often use well-known formulas, such as $R \cdot T = D$, or $a^2 + b^2 = c^2$. However, sometimes we need to construct our own models.

Data Modelling

Example 1 ► **Constructing a Formula to Model a Set of Data Following a Linear Pattern**

In Santa Barbara, CA, a passenger taking a taxicab for a d-mile-long ride pays the fare of F dollars as per the table below.

distance $\boldsymbol{d}$ (in miles)	1	2	3	4	5
fare $\boldsymbol{F}$ (in dollars)	5.50	8.50	11.50	14.50	17.50

a. Write a formula that calculates fare F, in dollars, when distance driven d, in miles, is known.
b. Find the fare for a 16-mile ride by this taxi.
c. How long was the ride of a passenger who paid the fare of \$29.50?

Solution ► **a.** Observe that the increase in fare when driving each additional mile after the first is constantly \$3.00. This is because

$$17.5 - 14.5 = 14.5 - 11.5 = 11.5 - 8.5 = 8.5 - 5.5 = 3$$

If d represents the number of miles driven, then the number of miles after the first can be represented by $(d - 1)$. The fare for driving n miles is the cost of driving the first mile plus the cost of driving the additional miles, after the first one. So, we can write

$$\boldsymbol{fare\ F} = \begin{pmatrix} \text{cost of the} \\ \textbf{first mile} \end{pmatrix} + \begin{pmatrix} \textbf{cost increase} \\ \text{per mile} \end{pmatrix} \cdot \begin{pmatrix} \textbf{number of} \\ \text{additional } \textbf{miles} \end{pmatrix}$$

or symbolically,

$$F = 5.5 + 3(d - 1)$$

The above equation can be simplified to

$$F = 5.5 + 3d - 3 = 3d + 2.5.$$

Therefore, $\boldsymbol{F = 3d + 2.5}$ is the formula that models the given data.

b. Since the number of driven miles is $d = 16$, we evaluate

$$F = 3 \cdot 16 + 2.5 = 50.5$$

Therefore, the fare for a 16-mile ride is **\$50.50**.

c. This time, we are given the fare $F = 29.50$, and we are looking for the corresponding number of miles d. To find d, we substitute 31.7 for F in our formula $F = 3d + 2.5$ and then solve the resulting equation for d. We obtain

$$\begin{aligned} 29.5 &= 3d + 2.5 && \text{add } -2.5 \text{ to both sides} \\ 27 &= 3d && \text{divide both sides by 3} \\ d &= 9 \end{aligned}$$

So, the ride was **9 miles** long.

Notice that in the solution to *Example 1c*, we could first solve the equation $F = 3d + 2.5$ for d:

$$\begin{aligned} F &= 3d + 2.5 \\ F - 2.5 &= 3d \\ \boldsymbol{d} &= \boldsymbol{\frac{F - 2.5}{3}}, \end{aligned}$$

and then use the resulting formula to evaluate d at $F = 29.50$.

$$d = \frac{29.5 - 2.5}{3} = \frac{27}{3} = 9.$$

The advantage of solving the formula $F = 3d + 2.5$ for the variable d first is such that the resulting formula $d = \frac{F-2.5}{3}$ makes evaluations of d for various values of F easier. For example, to find the number of miles d driven for the fare of \$35.5, we could evaluate directly using $d = \frac{35.5-2.5}{3} = \frac{33}{3} = 11$ rather than solving the equation $35.5 = 3d + 2.5$ again.

Solving Formulas for a Variable

If a formula is going to be used for repeated evaluation of a specific variable, it is convenient to rearrange this formula in such a way that the desired variable is **isolated on one side** of the equation and it does not appear on the other side. Such a formula may also be called a function.

Definition 2.2 ► A **function** is a rule for determining the value of one variable from the values of one or more other variables, in a unique way. We say that the first variable **is a function** of the other variable(s).

For example, consider the uniform motion (constant speed) relation between distance, rate, and time.

To evaluate rate when distance and time is given, we use the formula

$$\boldsymbol{R = \frac{D}{T}}.$$

This formula describes **rate as a function of distance and time**, as rate can be uniquely calculated for any possible input of distance and time.

To evaluate time when distance and rate is given, we use the formula

$$\boldsymbol{T = \frac{D}{R}}.$$

This formula describes **time as a function of distance and rate**, as time can be uniquely calculated for any possible input of distance and rate.

Finally, to evaluate distance when rate and time is given, we use the formula

$$\boldsymbol{D = R \cdot T}.$$

Here, the **distance is** presented as **a function of rate and time**, as it can be uniquely calculated for any possible input of rate and time.

To **solve a formula for a given variable** means to rearrange the formula so that the desired **variable equals to an expression that contains only other variables** but not the one that we solve for. This can be done the same way as when solving equations.

Here are some hints and guidelines to keep in mind when solving formulas:

- **Highlight** the variable of interest and solve the equation as if the other variables were just numbers (think of easy numbers), without actually performing the given operations.

  Example: To solve $mx + b = c$ for m,

  we pretend to solve, for example:

  $$\begin{aligned} m \cdot 2 + 3 &= 1 && \text{add -3 to both sides} \\ m \cdot 2 &= 1 - 3 && \text{divide both sides by 2} \\ m &= \frac{1-3}{2} \end{aligned}$$

  so we write:

  $$\begin{aligned} mx + b &= c && \text{add } -b \text{ to both sides} \\ mx &= c - b && \text{divide both sides by } x \\ \boldsymbol{m} &= \boldsymbol{\frac{c-b}{x}} \end{aligned}$$

- **Reverse** (undo) **operations** to isolate the desired variable.

  Example: To solve $2L + 2W = P$ for W, first, observe the operations applied to W:

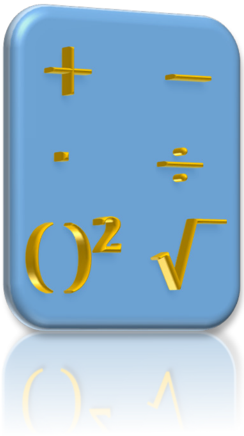
  $$W \xrightarrow{\cdot 2} 2W \xrightarrow{+2L} 2L + 2W$$

  Then, reverse these operations, starting from the last one first.

  $$W \xleftarrow[\div 2]{} 2W \xleftarrow[-2L]{} 2L + 2W$$

  So, we solve the formula as follows:

  $$\begin{aligned} 2L + 2W &= P && \text{add } -2L \text{ to both sides} \\ 2W &= P - 2L && \text{divide both sides by 2} \end{aligned}$$



$$\boldsymbol{W} = \frac{\boldsymbol{P} - \boldsymbol{2L}}{\boldsymbol{2}}$$

Notice that the last equation can also be written in the equivalent form $\boldsymbol{W} = \frac{\boldsymbol{P}}{\boldsymbol{2}} - \boldsymbol{L}$.

- **Keep the** desired **variable in the numerator**.

Example: To solve $R = \frac{D}{T}$ for T, we could take the reciprocal of each side of the equation to keep T in the numerator,

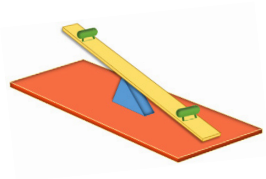
$$\frac{T}{D} = \frac{1}{R},$$

and then multiply by D to "undo" the division. Therefore, $\boldsymbol{T} = \frac{\boldsymbol{D}}{\boldsymbol{R}}$.

Observation: *Another way of solving $R = \frac{D}{T}$ for T is by multiplying both sides by T and dividing by R.*

$$\frac{T}{\cancel{R}} \cdot \cancel{R} = \frac{D}{\cancel{T}} \cdot \frac{\cancel{T}}{R}$$

This would also result in $\boldsymbol{T} = \frac{\boldsymbol{D}}{\boldsymbol{R}}$. Observe, that no matter how we solve this formula for T, the result differs from the original formula by interchanging (swapping) the variables T and R.



Note: *When working only with multiplication and division, by applying inverse operations, any factor of the numerator can be moved to the other side into the denominator, and likewise, any factor of the denominator can be moved to the other side into the numerator. Sometimes it helps to think of this movement of variables as the movement of a "teeter-totter".*

For example, the formula $\frac{bh}{2} = A$ can be solved for h by dividing by b and multiplying by 2. So, we can write directly $\boldsymbol{h} = \frac{\boldsymbol{2A}}{\boldsymbol{b}}$.

$\boldsymbol{2}$ was down so now goes up
and
$\boldsymbol{b}$ was up so now goes down

- **Keep the** desired **variable in one place.**

Example: To solve $A = P + Prt$ for P, we can factor P out,

$$A = P(1 + rt)$$

and then divide by the bracket. Thus,

$$\boldsymbol{P} = \frac{\boldsymbol{A}}{\boldsymbol{1 + rt}}.$$

Example 2 ► Solving Formulas for a Variable

Solve each formula for the indicated variable.

a. $a_n = a_1 + (n-1)d$ for n

b. $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T

Solution ► **a.** To solve $a_n = a_1 + (n-1)d$ for n we use the reverse operations strategy, starting with reversing the addition, then multiplication, and finally the subtraction.

$$a_n = a_1 + (n-1)d$$
$$a_n - a_1 = (n-1)d$$
$$\frac{a_n - a_1}{d} = n - 1$$
$$\boldsymbol{n = \frac{a_n - a_1}{d} + 1}$$

The last equation can also be written in the equivalent form $\boldsymbol{n} = \frac{\boldsymbol{a_n - a_1 + d}}{\boldsymbol{d}}$.

b. To solve $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T, first, we can take the reciprocal of each side of the equation to keep T in the numerator,

$$\frac{T}{PV} = \frac{T_0}{P_0V_0},$$

and then multiply by PV to "undo" the division. So,

$$\boldsymbol{T = \frac{T_0PV}{P_0V_0}}.$$

Attention: *Taking reciprocal of each side of an equation is a good strategy only if both sides are in the form of a* ***single fraction****. For example, to use the reciprocal property when solving* $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ *for a, first, we perform the addition to create a single fraction,* $\frac{1}{a} = \frac{c+b}{bc}$. *Then, taking reciprocals of both sides will give us an instant result of* $a = \frac{bc}{c+b}$.

Warning! *The reciprocal of* $\frac{1}{b} + \frac{1}{c}$ *is* ***not*** *equal to* $b + c$.

Example 3 ► **Using Formulas in Application Problems**

To determine a healthy weight for a person's height, we can use the body mass index I given by the formula $\boldsymbol{I} = \frac{\boldsymbol{W}}{\boldsymbol{H^2}}$, where W represents weight, in kilograms, and H represents height, in meters. Weight is considered healthy when the index is in the range 18.5–24.9.

a. Adam is 182 cm tall and weighs 89 kg. What is his body index?

b. Barb has a body mass index of 24.5 and a height of 1.7 meters. What is her weight?

Solution ► **a.** Since the formula calls for height H is in meters, first, we convert 182 cm to 1.82 meters, and then we substitute $H = 1.82$ and $W = 89$ into the formula. So,

$$\boldsymbol{I} = \frac{89}{1.82^2} \approx \boldsymbol{26.9}$$

When rounded to one decimal place. Thus Adam is overweighted.

b. To find Barb's weight, we may want to solve the given formula for W first, and then plug in the given data. So, Barb's weight is

$$\boldsymbol{W} = I\,H^2 = 24.5 \cdot 1.7^2 \approx \mathbf{70.8}\,\boldsymbol{kg}$$

Direct and Joined Variation

When two quantities vary proportionally, we say that there is a **direct variation** between them. For example, such a situation can be observed in the relation between time T and distance D covered by a car moving at a constant speed R. In particular, if $R = 60$ kph, we have

$$D = 60T$$

This relation tells us that the distance is 60 times larger than the time. Observe though that when the time doubles, the distance doubles as well. When the time triples, the distance also triples. So, the distance increases proportionally to the increase of the time. Such a **linear relation** between the two quantities is called a **direct variation**.

Definition 2.3 ► Two quantities, $\boldsymbol{x}$ and $\boldsymbol{y}$, are **directly proportional** to each other (there is a **direct variation** between them) iff there is a real constant $\boldsymbol{k \neq 0}$, such that

$$\boldsymbol{y = kx}.$$

We say that $\boldsymbol{y}$ **varies directly** as $\boldsymbol{x}$ with the **variation constant** $\boldsymbol{k}$.
(or equivalently: $\boldsymbol{y}$ is **directly proportional to** $\boldsymbol{x}$ with the **proportionality constant** $\boldsymbol{k}$.)

Example 4 ► **Solving Direct Variation Problems**

In scale drawing, the actual distance between two objects is directly proportional to the distance between the drawn objects. Suppose a kitchen room that is 4.6 meters long appears on a drawing as 2 cm long.

a. Find the direct variation equation that relates the actual distance D and the corresponding distance S on the drawing.

b. Find the actual dimensions of a 2.6 cm by 3.7 cm room on this drawing?

Solution ►

a. To find the direct variation equation that relates D and S, we need to find the variation constant k first. This can be done by substituting $D = 4.6$ and $S = 2$ into the equation $D = kS$. So, we obtain

$$4.6 = k \cdot 2$$
$$k = \frac{4.6}{2} = 2.3.$$

Therefore, the direct variation equation is $\boldsymbol{D = 2.3S}$.

b. To find the actual dimensions of a room that is drawn as 2.6 cm by 3.7 cm rectangle, we substitute these S-values into the above equation. This gives us

$$D = 2.3 \cdot 2.6 = 5.98 \quad \text{and} \quad D = 2.3 \cdot 3.7 = 8.51.$$

So, the actual dimensions of the room are $\mathbf{5.98}$ by $\mathbf{8.51}$ meters.

Sometimes a quantity varies directly as the n-th power of another quantity. For example, the formula $A = \pi r^2$ describes the direct variation between the area A of a circle and the square of the radius r of this circle. Here, the proportionality constant is π, while $n = 2$.

Extension: Generally, the fact that $\boldsymbol{y}$ **varies directly** as the $\boldsymbol{n}$**-th power** of $\boldsymbol{x}$ tells us that

$$\boldsymbol{y = kx^n},$$

for some nonzero constant $\boldsymbol{k}$.

Example 5 ► Solving a Direct Variation Problem Involving the Square of a Variable

Disregarding air resistance, the distance a body falls from rest is directly proportional to the square of the elapsed time. If a skydiver falls 24 meters in the first 2 seconds, how far will he fall in 5 seconds?

Solution ► Let d represents the distance the skydiver falls and t the time elapsed during this fall. Since d varies directly as t^2, we set the equation

$$d = kt^2$$

After substituting the data given in the problem, we find the value of k:

$$24 = k \cdot 2^2$$

$$k = \frac{24}{4} = 6$$

So, the direct variation equation is $d = 6t^2$. Hence, during 5 seconds the skydiver falls the distance $d = 6 \cdot 5^2 = 6 \cdot 25 = \mathbf{150}$ **meters**.

If one variable varies directly as the product of several other variables (possibly raised to some powers), we say that the first variable varies **jointly** as the other variables. For example, a joint variation can be observed in the formula for the area of a triangle, $A = \frac{1}{2}bh$, where the area A varies directly as the base b and directly as the height h of this triangle. We say that A varies jointly as b and h.

Definition 2.4 ► Variable $\boldsymbol{z}$ is **jointly proportional** to a set of variables (possibly raised to some powers) iff $\boldsymbol{z}$ is **directly proportional** to each of these variables (or equivalently: $\boldsymbol{z}$ is **directly proportional** to the product of these variables including their powers.)

Example:
z varies jointly as $\boldsymbol{x}$ and a **cube** of $\boldsymbol{y}$ iff $\boldsymbol{y = kxy^3}$, for some real constant $\boldsymbol{k \neq 0}$.

Example 6 ► Solving Joint Variation Problems

a. Kinetic energy is jointly proportional to the mass and the square of the velocity. Suppose a mass of 5 kilograms moving at a velocity of 4 meters per second has a kinetic energy of 40 joules.

b. Find the kinetic energy of a 3-kilogram ball moving at 6 meters per second.

c. Find the mass of an object that has 50 joules of kinetic energy when moving at 5 meters per second.

Solution ► **a.** Let E, m, and v represent respectively kinetic energy, mass, and velocity. Since E is jointly proportional to m and v^2, we set the equation

$$E = kmv^2.$$

Substituting the data given in the problem, we have

$$40 = k \cdot 5 \cdot 4^2,$$

which gives us

$$k = \frac{40}{80} = \frac{1}{2}.$$

So, the joint variation equation is $E = \frac{1}{2}mv^2$. Hence, the kinetic energy of the 3-kilogram ball moving at 6 meters per second is $E = \frac{1}{2} \cdot 3 \cdot 6^2 = 3 \cdot 18 =$ **54 joules**.

b. First, we may want to solve the equation $E = \frac{1}{2}mv^2$ for m and then evaluate it using substitutions $E = 50$, and $v = 5$. So, we have

$$E = \frac{1}{2}mv^2$$

$$2E = mv^2$$

$$m = \frac{2E}{v^2} = \frac{2 \cdot \overset{2}{\cancel{50}}}{\cancel{5^2}} = 4$$

reduce by 25

The mass of an object with the required parameters is **4 kilograms**.

L.2 Exercises

1. *When solving a formula for a particular variable, the answer can often be stated in various forms. Which of the following formulas are correct answers when solving* $A = \frac{a+b}{2}h$ *for* b*?*

A. $b = \frac{2A-a}{h}$ **B.** $b = \frac{2A}{h} - a$ **C.** $b = \frac{2A-ah}{h}$ **D.** $b = \frac{A-ah}{\frac{1}{2}h}$

2. *Which of the following formulas are* ***not*** *correct answers when solving* $A = P + Prt$ *for* P*? Justify your answer.*

A. $P = \frac{A}{rt}$ **B.** $P = \frac{A}{1+rt}$ **C.** $P = A - Prt$ **D.** $P = \frac{A-P}{rt}$

Solve each formula for the specified variable.

3. $I = Prt$ for r (*simple interest*)

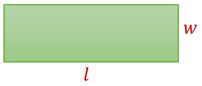
4. $C = 2\pi r$ for r (*circumference of a circle*)

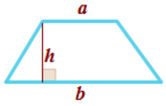
5. $E = mc^2$ for m (*mass-energy relation*)

6. $F = \frac{mv^2}{r}$ for m (*force in a circular motion*)

7. $A = \frac{(a+b)}{2}$ for b (*average*)

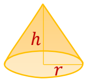
8. $Ax + By = c$ for y (*equation of a line*)

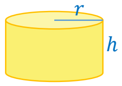




9. $P = 2l + 2w$ for l (*perimeter of a rectangle*)

10. $A = \frac{h}{2}(a + b)$ for a (*area of a trapezoid*)





11. $S = \pi rs + \pi r^2$ for π (*surface area of a cone*)

12. $S = 2\pi rh + 2\pi r^2$ for h (*surface area of a cylinder*)

13. $F = \frac{9}{5}C + 32$ for C (*Celsius to Fahrenheit*)

14. $C = \frac{9}{5}(F - 32)$ for F (*Fahrenheit to Celsius*)

15. $Q = \frac{p-q}{2}$ for p

16. $Q = \frac{p-q}{2}$ for q

17. $T = B + Bqt$ for q

18. $d = R - Rst$ for t

19. $d = R - Rst$ for R

20. $T = B + Bqt$ for B

Solve each problem.

21. On average, a passenger who drives n-kilometers in a taxicab in Abbotsford, BC, is charged C dollars as per the table below.

distance $\boldsymbol{n}$ (in kilometers)	1	2	3	4	5
cost $\boldsymbol{C}$ (in dollars)	5.10	7.00	8.90	10.80	12.70

a. Write a formula that calculates the cost C, in dollars, of driving a distance of n kilometers.
b. Find the cost for a 10-kilometer ride by this taxi.
c. If a passenger paid \$31.70, how far did he drive by this taxi?

22. On average, a passenger who drives n-kilometers in a taxicab in Vancouver, BC, is charged C dollars as per the table below.

distance $\boldsymbol{n}$ (in kilometers)	1	2	3	4	5
cost $\boldsymbol{C}$ (in dollars)	5.35	7.20	9.05	10.90	12.75

a. Write a formula that calculates the cost C, in dollars, of driving a distance of n kilometers.
b. Find the cost for a 20-kilometer ride by this taxi.
c. If a passenger paid \$22.00, how far did he drive by this taxi?

23. 

Assume that the amount of a medicine dosage for a child can be determined by the formula

$$c = \frac{ad}{a + 12},$$

where a represents the child's age, in years, and d represents the usual adult dosage, in milliliters.

a. If the adult dosage of a certain medication is 25 ml, what is the corresponding dosage for a three-year-old child?

b. Solve the formula for d.

c. Find the corresponding adult dosage, if a six-year-old child uses 5 ml of a certain medication.

24. The number of "full-time-equivalent" students, F, is often determined by the formula

$$F = \frac{n}{15},$$

where n represents the total number of credits taken by all students in a semester.

a. Suppose that in a particular institution students register for a total of 39,315 credits in one semester. What is the number of full-time-equivalent students in this institution?

b. Solve the formula for n.

c. Find the total number of credits students enroll in a semester if the number of full-time-equivalent students in this semester is 3254.

25. Suppose a cyclist can burn 530 calories in a 45-minute cycling session.

a. Write a formula that determines the number of calories C burned during two 45-minute sessions of cycling per day for d days.

b. According to this formula, how many calories would the cyclist burn in a week of cycling two 45-minute sessions per day?

26. *Refer to information given in problem 25.*

a. On average, a person loses 1 kilogram for every 7000 calories burned. Write a formula that calculates the number of kilograms K lost in d days of cycling two 45-minute sessions per day.

b. How many kilograms could the cyclist lose in 30 days? *Round the answer to the nearest half of a kilogram.*

27. Express the width L of a rectangle in terms of its perimeter P and length W. *Here "in terms of P and W" means using an expression that involves only variables P and W.*

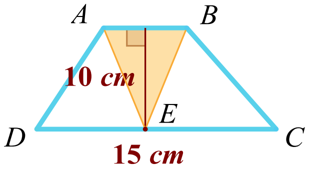
28. Express the area A of a circle in terms of its diameter d. *Here "in terms of d" means using an expression that involves only variable d.*

29. **a.** Solve the formula $I = Prt$ for t.

b. Using the formula from (**a**) determine how long it will take a deposit of \$125 to make the interest of \$15 when invested at 4% simple interest.

30. *Refer to information given in the accompanying figure.*

Find the area of the trapezoid $ABCD$ if its area is three times as large as the area of the shaded triangle ABE.



31. The number N of plastic bottles used each year is directly proportional to the number of people P using them.
  a. Assuming that 150 people use 18,000 bottles in one year, find the variation constant and state the direct variation equation.
  b. How many bottles are used each year in Vancouver, BC, which has a population of 631,490?

32. Under certain conditions, the volume V of a fixed amount of gas varies directly as its temperature T, in Kelvin degrees.
  a. Assuming the gas in a hot-air balloon occupies 120 m^3 at 200 K, find the direct variation equation.
  b. If the pressure of the gas remains constant, what would the volume of the gas be at 250 Kelvin degrees?

33. The recommended daily intake of fat varies directly as the number of calories consumed per day. Alice is on a 1200 diet and her healthy intake of fat is about 40 grams per day. Christopher needs about 2000 calories per day. To the nearest gram, what is his recommended daily intake of fat?

34. The distance covered by a falling object varies directly as the square of the elapsed time of the fall. A person flying in a hot-air balloon accidentally dropped a camera. Suppose the camera fell 16 meters during the first 2 seconds of the fall. If the camera hits the ground 5 seconds after it was dropped, how high was the balloon?

35. The air distance between Vancouver and Warsaw is 8,225 kilometers. The two cities are 45.2 centimeters apart on a desk globe. The air distance between Paris and Warsaw is 1367 kilometers. To the nearest millimeter, how far is Paris from Warsaw on this globe?

36. The stopping distance for a car varies directly as the square of its speed. If a car travelling 50 kilometers per hour requires 40 meters to stop, what would be the stopping distance for a car travelling 80 kilometers per hour?

37. The simple interest I varies jointly as the interest rate r and the principal P. Mother and daughter invested some money in simple interest accounts for the same period of time t. The daughter earns \$130 interest on the investment of \$2000 at 3.25%. What was the amount that the mother invested at 3.75% if she earns \$225 interest?

38. The lateral surface area (*surface area excluding the bases*) of a cylinder is jointly proportional to the height and radius of the cylinder. If a cylinder with radius 5 cm and height 8 cm has a lateral surface area of approximately 250 cm^2, what is the approximate lateral surface area of a can with a diameter of 6 cm and height of 12 cm?

39. The area of a triangle is jointly proportional to the height and the base of the triangle. If the base is increased by 50% and the height is decreased by 50%, how would the area of the triangle change?

40. The volume V of wood in a tree is directly proportional to the height h and the square of the **girth** (*circumference around the trunk*), g. Suppose the volume of a 20 meters tall tree with the girth of 1 meter is 64 cubic meters. To the nearest meter, find the height of a tree with a volume of 250 cubic meters and girth of 1.8 meters?

41. The number of barrels of oil used by a ship travelling at a constant speed is jointly proportional to the distance traveled and the square of the speed. If the ship uses 180 barrels of oil when travelling 200 miles at 40 miles per hour, find the number of barrels of oil needed for a ship that travels 300 miles at 25 miles per hour. *Round the answer to the nearest barrel.*

L3 Applications of Linear Equations

In this section, we study some strategies for solving problems with the use of linear equations, or well-known formulas. While there are many approaches to problem-solving, the following steps prove to be helpful.

Five Steps for Problem Solving
1. **Familiarize** yourself with the problem.
2. **Translate** the problem to a symbolic representation (usually an **equation** or an **inequality**).
3. **Solve** the equation(s) or the inequality(s).
4. **Check** if the answer makes sense in the original problem.
5. **State the answer** to the original problem clearly.

Here are some hints of how to **familiarize** yourself with the problem:

- **Read** the problem carefully a few times. In the first reading focus on the general setting of the problem. See if you can identify this problem as one of a motion, investment, geometry, age, mixture or solution, work, or a number problem, and draw from your experiences with these types of problems. During the second reading, focus on the specific information given in the problem, skipping unnecessary words, if possible.
- **List the information** given, including **units**, and check **what the problem asks for**.
- If applicable, **make a diagram** and label it with the given information.
- **Introduce a variable**(s) for the unknown quantity(ies). Make sure that the variable(s) is/are clearly defined (including units) by writing a "let" statement or labeling appropriate part(s) of the diagram. Choose descriptive letters for the variable(s). For example, let l be the length in centimeters, let t be the time in hours, etc.
- Express **other unknown values** in terms of the already introduced variable(s).
- Write applicable **formulas**.
- **Organize your data** in a meaningful way, for example by filling in a table associated with the applicable formula, inserting the data into an appropriate diagram, or listing them with respect to an observed pattern or rule.
- **Guess** a possible answer and check your guess. Observe the way in which the guess is checked. This may help you translate the problem into an equation.

Translation of English Phrases or Sentences to Expressions or Equations

One of the important phases of problem-solving is **translating** English words into a **symbolic representation**.

Here are the most commonly used **key words** suggesting a particular operation:

ADDITION (+)	SUBTRACTION (−)	MULTIPLICATION (·)	DIVISION (÷)
sum	**difference**	**product**	**quotient**
plus	minus	multiply	divide
add	**subtract from**	times	ratio
total	**less than**	**of**	**out of**
more than	less	half of	per
increase by	decrease by	half as much as	shared
together	diminished	twice, triple	cut into
perimeter	shorter	area	

Example 1 ► Translating English Words to an Algebraic Expression or Equation

Translate the word description into an algebraic expression or equation.

a. The sum of half of a number and two
b. The square of a difference of two numbers
c. Triple a number, increased by five, is one less than twice the number.
d. The quotient of a number and seven diminished by the number
e. The quotient of a number and seven, diminished by the number
f. The perimeter of a rectangle is four less than its area.
g. In a package of 12 eggs, the ratio of white to brown eggs is one out of three.
h. Five percent of the area of a triangle whose base is one unit shorter than the height

Solution ►

a. Let x represents "*a number*". Then

*The **sum of** **half of** a number **and** two* translates to $\frac{1}{2}x+2$

Notice that the word "***sum***" indicates addition sign at the position of the word "***and***". Since addition is a binary operation (needs two inputs), we reserve space for "*half of a number*" on one side and "*two*" on the other side of the addition sign.

b. Suppose x and y are the "*two numbers*". Then

***The square of** a **difference of** two numbers* translates to $(x-y)^2$

Notice that we are squaring everything that comes after "*the square of*".

c. Let x represents "*a number*". Then

***Triple** a number, increased by five, **is** one **less than** **twice** the number.*

translates to the equation: $3x+5=2x-1$

This time, we translated a sentence that results in an equation rather than expression. Notice that the "equal" sign is used in place of the word "is". Also, remember that phrases "less than" or "subtracted from" work "backwards". For example, A **less than** B or A **subtracted from** B translates to $B-A$. However, the word "less" is used in the usual direction, from left to right. For example, A **less** B translates to $A-B$.

d. Let x represent "*a number*". Then

***The quotient of** a number **and** seven **diminished** by the number* translates to $\frac{x}{7-x}$

Notice that "**the** number" refers to the same number x.

e. Let x represent "*a number*". Then

***The quotient of** a number **and** seven**,** **diminished** by the number* translates to $\frac{x}{7}-x$

Here, the **comma** indicates the end of the "**quotient section**". So, we diminish the quotient rather than diminishing the seven, as in *Example 1d*.

f. Let l and w represent the length and the width of a rectangle. Then

*The **perimeter** of a rectangle **is** four **less than** its **area**.*

translates to the equation: $\mathbf{2l + 2w = lw - 4}$

Here, we use a formula for the perimeter ($\mathbf{2l + 2w}$) and for the area ($\mathbf{lw}$) of a rectangle.

g. Let w represent the number of white eggs in a package of 12 eggs. Then $(12 - w)$ represents the number of brown eggs in this package. Therefore,

*In a package of 12 eggs, **the ratio of the number of white eggs to the number of brown eggs is the same as two to three**.*

translates to the equation: $\mathbf{\frac{w}{12-w} = \frac{2}{3}}$

Here, we expressed the unknown number of brown eggs $(12 - w)$ in terms of the number w of white eggs. Also, notice that the order of listing terms in a proportion is essential. Here, the first terms of the two ratios are written in the numerators (in blue) and the second terms (in brown) are written in the denominators.

h. Let h represent the height of a triangle. Since the base is *one unit shorter than the height,* we express it as $(h - 1)$. Using the formula $\frac{1}{2}bh$ for the area of a triangle, we translate

five percent of the area of a triangle** whose **base is one unit shorter than the height

to the expression: $\mathbf{0.05 \cdot \frac{1}{2}(h - 1)h}$

Here, we convert *five percent* to the number 0.05, as *per-cent* means *per hundred*, which tells us to divide 5 by a hundred.
Also, observe that the above word description is not a sentence, even though it contains the word "*is*". Therefore, the resulting symbolic form is an expression, not an equation. The word "*is*" relates the base and the height, which in turn allows us to substitute $(h - 1)$ in place of b, and obtain an expression in one variable.

So far, we provided some hints of how to familiarize ourself with a problem, we worked through some examples of how to translate word descriptions to a symbolic form, and we reviewed the process of solving linear equations (*see Section L1*). In the rest of this section, we will show various methods of solving commonly occurring types of problems, using representative examples.

Number Relation Problems

In number relation type of problems, we look for relations between quantities. Typically, we introduce a variable for one quantity and express the other quantities in terms of this variable following the relations given in the problem.

Example 2 ► **Solving a Number Relation Problem with Three Numbers**

The sum of three numbers is thirty-four. The second number is twice the first number, and the third number is one less than the second number. Find the three numbers.

Solution ► There are three unknown numbers such that their sum is thirty-four. This information allows us to write the equation

$$1^{\text{st}} \text{ number} + 2^{\text{nd}} \text{ number} + 3^{\text{rd}} \text{ number} = 34.$$

To solve such an equation, we wish to express all three unknown numbers in terms of one variable. Since the second number refers to the first, and the third number refers to the second, which in turn refers to the first, it is convenient to introduce a variable for the first number.
So, let n represent the first number.
The second number is twice the first, so $2n$ represents the second number.
The third number is one less than the second number, so $2n - 1$ represents the third number.

Therefore, our equation turns out to be

$$\begin{aligned} n + 2n + (2n - 1) &= 34 \\ 5n - 1 &= 34 \\ 5n &= 35 \\ n &= 7. \end{aligned}$$

Hence, the first number is **7**, the second number is $2n = 2 \cdot 7 = \mathbf{14}$, and the third number is $2n - 1 = 14 - 1 = \mathbf{13}$.

Consecutive Numbers Problems

Since **consecutive numbers** differ by one, we can represent them as $\boldsymbol{n, n+1, n+2}$, and so on.

Consecutive even or **consecutive odd** numbers differ by two, so both types of numbers can be represented by $\boldsymbol{n, n+2, n+4}$, and so on.

Notice that if the first number n is even, then $n + 2, n + 4$, ... are also even; however, if the first number n is odd then $n + 2, n + 4$, ... are also odd.

Example 3 ► **Solving a Consecutive Odd Integers Problem**

Find three consecutive odd integers such that three times the middle integer is five less than double the sum of the first and the third integer.

Solution ► Let the three consecutive odd numbers be called $n, n + 2$, and $n + 4$. We translate *three times the middle integer is five less than double the sum of the first and the third integer* into the equation

$$3(n + 2) = 2[n + (n + 4)] - 5$$

which gives

$$3n + 6 = 4n + 3$$

$$n = 3$$

Hence, the first number is **3**, the second number is $n + 2 = \mathbf{5}$, and the third number is $n + 4 = \mathbf{7}$.

Percent Problems

Rules to remember when solving percent problems:

$$1 = 100\% \quad \text{and} \quad \frac{\textbf{is} \textit{ a part}}{\textbf{of} \textit{ a whole}} = \frac{\%}{\mathbf{100}}$$

Also, remember that

$$\textit{\textbf{percent increase}}(\textit{decrease}) = \frac{\textit{\textbf{last}} - \textit{\textbf{first}}}{\textit{\textbf{first}}} \cdot 100\%$$

Example 4 ► **Finding the Amount of Tax**

Kristin bought a new fridge for \$1712.48, including 12% PST and GST tax. How much tax did she pay?

Solution ► Suppose the fridge costs p dollars. Then the tax paid for this fridge is 12% of p dollars, which can be represented by the expression $0.12p$. Since the total cost of the fridge including tax is \$1712.48, we set up the equation

$$p + 0.12p = 1712.48$$

which gives us

$$1.12p = 1712.48$$

$$p = 1529$$

The question calls for the amount of tax, so we calculate $0.12p = 0.12 \cdot 1529 = 183.48$.

Kristin paid \$183.48 in tax for the fridge.

Example 5 ► **Solving a Percent Increase Problem**

Susan got her hourly salary raised from \$11.50 per hour to \$12.75 per hour. To the nearest tenths of a percent, what was the percent increase in her hourly wage?

Solution ► We calculate the percent increase by following the rule $\frac{\textit{\textbf{last}}-\textit{\textbf{first}}}{\textit{\textbf{first}}} \cdot 100\%$.

So, Susan's hourly wage was increased by $\frac{12.75-11.50}{11.50} \cdot 100\% \approx \mathbf{10.9\%}$.

Investment Problems

When working with investment problems we often use the simple interest formula $\boldsymbol{I = Prt}$, where I represents the amount of interest, P represents the principal (amount of money invested), r represents the interest rate, and t stands for the time in years.

Also, it is helpful to organize data in a diagram like this:

$$\textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{I}\ +\ \textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{II}\ =\ \textbf{\textit{total interest}}$$

or

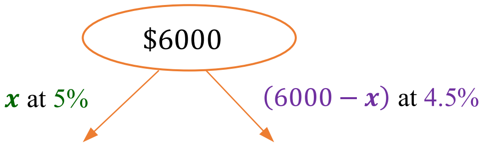
$$\textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{I}\ =\ \textbf{\textit{interest}}\ \textit{from account}\ \boldsymbol{II}$$

Example 6 ► **Solving an Investment Problem**

A student took out two student loans for a total of \$6000. One loan is at 5% annual interest and the other at 4.5% annual interest. If the total interest paid in a year is \$281, find the amount of each loan.

Solution ► To solve this problem in one equation, we would like to introduce only one variable. Suppose $\boldsymbol{x}$ is the amount of the first loan. Then the amount of the second loan is the remaining portion of the \$6000. So, it is $(6000 - \boldsymbol{x})$.

Using the simple interest formula $I = Prt$, for $t = 1$, we calculate the interest obtained from the 5% to be $0.05 \cdot \boldsymbol{x}$ and from the 4.5% account to be $0.045(6000 - \boldsymbol{x})$. Since the total interest equals to \$281, we set the equation as indicated in the diagram below.



$$0.05 \cdot \boldsymbol{x}\ +\ 0.045(6000 - \boldsymbol{x})\ =\ 281$$

For easier calculations, we may want to clear decimals by multiplying this equation by 1000.
This gives us

$$\begin{aligned} 50x + 45(6000 - x) &= 281000 \\ 50x + 270000 - 45x &= 281000 \\ 5x &= 11000 \end{aligned}$$

and finally

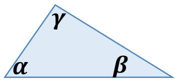
$$x = \$2200$$

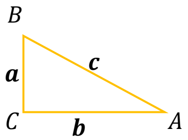
Thus, the first loan is **\$2200** and the second loan is $6000 - x = 6000 - 2200 = \$\mathbf{3800}$.

Geometry Problems

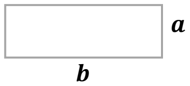
In geometry problems, we often use well-known formulas or facts that pertain to geometric figures. Here is a list of facts and formulas that are handy to know when solving various problems.

- The **sum of angles** in a triangle equals **180°**.

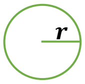


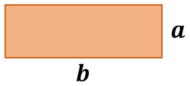


- The lengths of sides in a right-angle triangle ABC satisfy the **Pythagorean equation** $\boldsymbol{a^2 + b^2 = c^2}$, where c is the hypotenuse of the triangle.

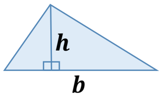


- The **perimeter of a rectangle** with sides a and b is given by the formula $\boldsymbol{2a + 2b}$.
- The **circumference** of a circle with radius r is given by the formula $\boldsymbol{2\pi r}$.

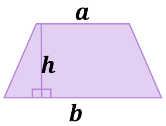




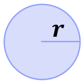
- The **area of** a **rectangle** or a **parallelogram** with base b and height h is given by the formula $\boldsymbol{bh}$.



- The **area of** a **triangle** with base b and height h is given by the formula $\boldsymbol{\frac{1}{2}bh}$.



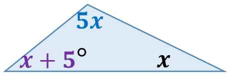
- The **area of** a **trapezoid** with bases a and b, and height h is given by the formula $\boldsymbol{\frac{1}{2}(a+b)h}$.
- The **area of** a **circle** with radius r is given by the formula πr^2.



Example 7 ► **Finding the Measure of Angles in a Triangle**

A cross section of a roof has a shape of a triangle. The largest angle of this triangle is five times as large as the smallest angle. The remaining angle is 5° greater than the smallest angle. Find the measure of each angle.

Solution ► Observe that the size of the largest and the remaining angle is compared to the size of the smallest angle. Therefore, it is convenient to introduce a variable, $\boldsymbol{x}$, for the measure of the smallest angle. Then, the expression for the measure of the largest angle, which is *five times as large as the smallest one*, is $\boldsymbol{5x}$ and the expression for the measure of the remaining angle, which is *5° greater than the smallest one*, is $\boldsymbol{x + 5°}$. To visualize the situation, it might be helpful to draw a triangle and label the three angles.



Since the sum of angles in any triangle is equal to 180°, we set up the equation

$$x + 5x + x + 5° = 180°$$

This gives us

$$7x + 5° = 180°$$

$$7x = 175°$$

$$x = 25°$$

So, the measure of the smallest angle is $\mathbf{25}°$,
the measure of the largest angle is $5x = 5 \cdot 25° = \mathbf{125}°$, and
the measure of the remaining angle is $x + 5° = 25° + 5° = \mathbf{30}°$.

Total Value Problems

When solving total value types of problems, it is helpful to organize the data in a table that compares the number of items and the value of these items. For example:

	item ***A***	item ***B***	total
number of items			
value of items			

Example 8 **Solving a Coin Problem**

The value of twenty-four coins consisting of dimes and quarters is \$3.75. How many quarters are in the collection of coins?

Solution Suppose the number of quarters is n. Since the whole collection contains 24 coins, then the number of dimes can be represented by $24 - n$. Also, in cents, the value of n quarters is $25n$, while the value of $24 - n$ dimes is $10(24 - n)$. We can organize this information as in the table below.

	dimes	quarters	Total
number of coins	$24 - n$	n	24
value of coins (in cents)	$10(24 - n)$	$25n$	375

The value is written in cents!

Using the last row of this table, we set up the equation

$$10(24 - n) + 25n = 375$$

and then solve it for n.

$$240 - 10n + 25n = 375$$
$$15n = 135$$
$$n = 9$$

So, there are **9** quarters in the collection of coins.

Mixture-Solution Problems

When solving total mixture or solution problems, it is helpful to organize the data in a table that follows one of the formulas

$unit\ price \cdot number\ of\ units = total\ value$ or $percent \cdot volume = content$

	unit price ·	# of units	= value
type I			
type II			
mix			

	% ·	volume	= content
type I			
type II			
solution			

Example 9 ► Solving a Mixture Problem

Dark chocolate kisses costing \$13.50 per kilogram are going to be mixed with white chocolate kisses costing \$7.00 per kilogram. How many kilograms of each type of chocolate kisses should be used to obtain 30 kilograms of a mixture that costs \$10.90 per kilogram?

Solution ►

In this problem, we mix two types of chocolate kisses: dark and white. Let x represent the number of kilograms of dark chocolate kisses. Since there are 30 kilograms of the mixture, we will express the number of kilograms of the white chocolate kisses as $30 - x$.

The information given in the problem can be organized as in the following table.

	unit price ·	# of units	= value (in \$)
dark kisses	13.50	x	$13.5x$
white kisses	7.00	$30 - x$	$7(30 - x)$
mix	10.90	30	327

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$13.5x + 7(30 - x) = 327$$

and then solve it for x.

$$\begin{aligned} 13.5x + 210 - 7x &= 327 \\ 6.5x &= 117 \\ x &= 18 \end{aligned}$$

So, the mixture should consist of **18** kilograms of dark chocolate kisses and $30 - x = 30 - 18 = \mathbf{12}$ kilograms of white chocolate kisses.

Example 10 ► Solving a Solution Problem

How many milliliters of pure alcohol should be added to 80 ml of a 20% alcohol solution to make a 50% alcohol solution?

Solution ►

Let x represent the volume of pure alcohol, in milliliters. The 50% solution is made by combining x ml of the pure alcohol with 80 ml of a 20% alcohol solution. So, the volume of the 50% solution can be expressed as $x + 80$.

Now, let us organize this information in the table below.

	% ·	volume	= acid
pure alcohol	1	x	x
20% solution	0.2	80	16
50% solution	0.5	$x + 80$	$0.5(x + 80)$

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$x + 16 = 0.5(x + 80)$$

and then solve it for x.

$$x + 16 = 0.5x + 40$$
$$0.5x = 24$$
$$x = 12$$

So, there should be added **12** milliliters of pure alcohol.

Motion Problems

When solving motion problems, refer to the formula

$$\boldsymbol{Rate \cdot Time = Distance}$$

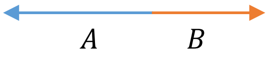
and organize date in a table like this:

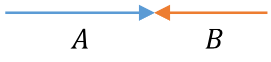
	R ·	T	= D
motion I			
motion II			
total			

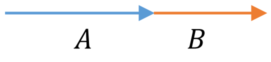
Some boxes in the "total" row are often left empty. For example, in motion problems, we usually do not add rates. Sometimes, the "total" row may not be used at all.

If two moving object (or two components of a motion) are analyzed, we usually encounter the following situations:

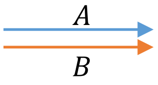
1. The two objects A and B move apart, approach each other, or move successively in the same direction (see the diagram below). In these cases, it is likely we are interested in the **total distance** covered. So, the last row in the above table will be useful to record the total values.

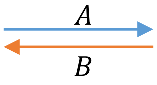






2. Both objects follow the same pathway. Then the **distances** corresponding to the two motions are the same and we may want to **equal** them. In such cases, there may not be any total values to consider, so the last row in the above table may not be used at all.

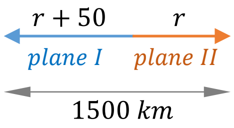




Example 11 ► **Solving a Motion Problem where Distances Add**

Two private planes take off from the same town and fly in opposite directions. The first plane is flying 50 km/h faster than the second one. In 3 hours, the planes are 1500 kilometers apart. Find the rate of each plane.

Solution ►



The rates of both planes are unknown. However, since the rate of the first plane is 50 km/h faster than the rate of the second plane, we can introduce only one variable. For example, suppose r represents the rate of the second plane. Then the rate of the first plane is represented by the expression $r + 50$.

In addition, notice that 1500 kilometers is the **total distance** covered by both planes, and 3 hours is the flight time of each plane.

Now, we can complete a table following the formula $R \cdot T = D$.

	R ·	T	= D
plane I	$r + 50$	3	$3(r + 50)$
plane II	r	3	$3r$
total			1500

Notice that neither the total rate nor the total time was included here. This is because these values are not relevant to this particular problem. The equation that relates distances comes from the last column:

$$3(r + 50) + 3r = 1500$$

After solving it for r,

$$3r + 150 + 3r = 1500$$

$$6r = 1350$$

we obtain

$$r = 225$$

Therefore, the speed of the first plane is $r + 50 = 225 + 50 = \mathbf{275\ km/h}$ and the speed of the second plane is $\mathbf{225\ km/h}$.

Example 12 ► **Solving a Motion Problem where Distances are the Same**

A police officer spotted a speeding car moving at 120 km/h. Ten seconds later, the police officer starts chasing the car, travelling on a motorcycle at 140 km/h. How long does it take the police officer to catch the car?

Solution ► Let t represent the time, in minutes, needed for the police officer to catch the car. The time that the speeding car drives is 10 seconds longer than the time that the police officer drives. To match the denominations, we convert 10 seconds to $\frac{10}{60} = \frac{1}{6}$ of a minute. So, the time used by the car is $t + \frac{1}{6}$.

In addition, the rates are given in kilometers per hour, but we need to have them in kilometers per minute. So, we convert $\frac{120\ km}{1\ h} = \frac{120\ km}{60\ min} = 2\frac{km}{min}$, and similarly $\frac{140\ km}{1\ h} = \frac{140\ km}{60\ min} = \frac{7}{3}\frac{km}{min}$.

Now, we can complete a table that follows the formula $R \cdot T = D$.

	R ·	T	= D
car	2	$t + \frac{1}{6}$	$2(t + \frac{1}{6})$
police	$\frac{7}{3}$	t	$\frac{7}{3}t$

Notice that this time there is no need for the "total" row.

Since distances covered by the car and the police officer are the same, we set up the equation

$$2\left(t + \frac{1}{6}\right) = \frac{7}{3}t$$

To solve it for t, we may want to clear some fractions first. After multiplying by 3, we obtain

$$6\left(t+\frac{1}{6}\right)=7t$$

which becomes

$$6t+1=7t$$

and finally

$$1=t$$

So, the police officer needs one minute to catch this car.

Even though the above examples show a lot of ideas and methods used in solving specific types of problems, we should keep in mind that the best way to learn problem-solving is to **solve a lot of problems**. This is because every problem might present slightly different challenges than the ones that we have seen before. The more problems we solve, the more experience we gain, and with time, problem-solving becomes easier.

L.3 Exercises

Translate each word description into an algebraic expression or equation.

1. A number less seven

2. A number less than seven

3. Half of the sum of two numbers

4. Two out of all apples in the bag

5. The difference of squares of two numbers

6. The product of two consecutive numbers

7. The sum of three consecutive integers is 30.

8. Five more than a number is double the number.

9. The quotient of three times a number and 10

10. Three percent of a number decreased by a hundred

11. Three percent of a number, decreased by a hundred

12. The product of 8 more than a number and 5 less than the number

13. A number subtracted from the square of the number

14. The product of six and a number increased by twelve

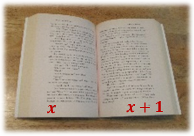
Solve each problem.

15. When the quotient of a number and 4 is added to twice the number, the result is 10 more than the number. Find the number.

16. When 25% of a number is added to 9, the result is 3 more than the number. Find the number.

17. The numbers on two adjacent safety deposit boxes add to 477. What are the numbers?

18. The sum of page numbers on two consecutive pages of a book is 543. What are the page numbers?



19. The number of international students at UFV, Abbotsford, BC, increased from 914 in the school year 2013/14 to 1708 in 2017/18. To the nearest tenths of a percent, what was the percent increase in enrollment during this time?

20. The number of domestic students at UFV, Abbotsford, BC, declined from 13762 in the school year 2013/14 to 12864 in 2017/18. To the nearest tenths of a percent, what was the percent decrease in enrollment during this time?

21. Find three consecutive odd integers such that the sum of the first, three times the second, and two times the third is 80.

22. Find three consecutive even integers such that the sum of the first, two times the second, and five times the third is 120.

23. Twice the sum of three consecutive odd integers is 210. Find the three integers.

24. Stephano paid $30,495 for a new Honda Civic. If this amount includes 7% of the sales tax, what is the cost of this car before tax?

25. After a 4% raise, the new monthly salary of a factory worker is $1924. What was the old monthly salary of this worker?

26. Jason bought a discounted fridge. The regular price of the fridge was $790.00, but he only paid $671.50. What was the percent discount?

27. The U.S. government issued about 156,000 patents in 2015. This was a decrease of about 1.7% from the number of patents issued in 2014. To the nearest hundred, how many patents were issued in 2014?

28. An investor has some funds at a 4% simple interest account and some at a 5% simple interest account. If the overall investment of $25000 gains $1134.00 of interest in one year, find the amount invested at each rate.

29. Jessie has $51,000 to invest. She plans to invest part of the money in an account paying 3% simple interest and the rest of the money into bonds paying 6.5% simple interest. How much should she invest at each rate to gain $3000 interest in a year?

30. Jan invested some money at 2.5% simple interest and twice this amount at 3.25%. Her total annual interest was $405. How much was invested at each rate?

31. Peter invested some money at 4.5% simple interest, and $2000 more than this amount at 5.25%. His total annual interest was $690. How much was invested at each rate?

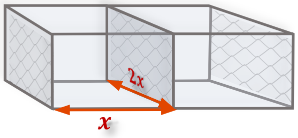
32. Daria invested $15,000 in bonds paying 6.5%. If she had some additional funds, she could invest in a saving account paying 2.5% simple interest. How much money would have to be invested at 2.5% for the average return on the two investments to be 5%?

33. Jack received a bonus payment of $12,000 and invested it in bonds paying 4.5% simple interest. If he had some additional funds, he could invest in a saving account paying 2.75% simple interest. How much additional money should he deposit in the 2.75% account so that his return on the two investments will be 4%?

34. A 126 cm long wire is cut into two pieces. Each piece is bent to form an equilateral triangle. If one triangle is twice as large as the other, how long are the sides of the triangles?

35. The measure of the smallest angle in a triangle is half the measure of the largest angle. The third angle is 15° less than the largest angle. Find the measure of each angle.

36. 35 ft of molding was used to trim a garage door. If the longer side of the door was 3 ft longer than twice the length of each of the shorter sides, then what are the dimensions of the door?



37. Billy plans to construct two adjacent rectangular outdoor cages for his rabbits. The cages would have open tops and bottoms, and share their longer side, as on the accompanying diagram. Each cage is planned to be twice as long as it is wide. If Billy has 80 ft of fencing, how large can the cages be?

38. The perimeter of a tennis court is 76 meters. The width of the court is 14 meters less than the length. Find the dimensions of the court.

39. Teresa inserted 16 coins into a vending machine to purchase a chocolate bar for $1.25. If she used only dimes and nickels, how many of each type of coins did she use?

40. Robert used 12 coins consisting of dimes, nickels, and quarters to buy the *Vancouver Sun* for $1.50. If he had twice as many dimes as nickels and the same many nickels as quarters, how many of each type of coins did he use?

41. A 30-kilogram mixture at $25.28 per kilogram consists of pecans at $27.50 per kilogram and cashews at $23.80 per kilogram. How many kilograms of each were used to make the mixture?

42. A store owner bought 15 kilograms of peanuts for $72. He wants to mix these peanuts with raisins costing $7.50 per kilogram to get a mixture costing $6 per kilogram. How many kilograms of raisins should he use?

43. Tickets to a movie theatre cost $8.50 for an adult and $3.50 for a child. If $1253 were collected for selling a total of 178 tickets, how many of each type of tickets were sold?

44. Find the unit cost of a sunscreen made from 160 milliliters of lotion that cost $1.49 per milliliter and 90 milliliters of lotion that cost $2.49 per milliliter.

45. A tea mixture was prepared by mixing 20 kg of tea costing $10.80 per kilogram with 30 kg of tea costing $6.50 per pound. Find the unit cost of the tea mixture.

46. A pharmacist has 150 milliliters of a solution that contains 80% of a particular medication. How much pure water should he add to change the concentration of the medication to 25%?

47. How many grams of a 50% gold alloy must be mixed with 100 grams of an 80% gold alloy to make a 75% gold alloy?

48. A jeweller mixed 40 g of an 80% silver alloy with 60 g of a 25% silver alloy. What percent of silver contains the resulting alloy?

49. How many milliliters of water must be added to 250 ml of a 7% hydrogen peroxide solution to make a 3% hydrogen peroxide solution?

50. A car radiator contains 9 liters of a 50% antifreeze solution. How many liters need to be replaced with pure antifreeze to bring the antifreeze concentration to 80%?

51. Jessica is working with sulphuric acid solutions in a lab. She needs to dilute 50 milliliters of a 70% sulphuric acid solution to a 50% solution by mixing it with a 25% sulphuric acid solution. How many milliliters of a 25% solution should she use?

52. Two planes fly towards each other starting from two cities that are 4200 km apart. If one plane is travelling 150 km/h faster than the other and they pass each other after 2.5 hours, what is the speed of each plane?

53. A plane flies at 630 km/h in still air. To the nearest minute, how long will it take the plane to travel 1000 kilometers

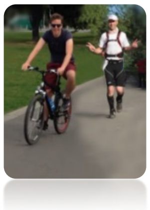
a. into a 90-km/h headwind?

b. with a 90-km/h tailwind?

54. At 7:00 am, Jacob left his house jogging at 10 km/h to a nearby park for his routine morning exercises. Six minutes later, his brother Andrew followed him using the same route. Running at 15 km/h, in how many minutes will Andrew catch up with his brother?

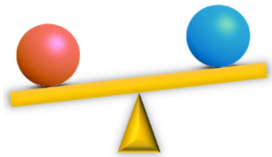
55. Tina walked at a rate of 8 km/h from home to a bike shop. She bought a bike there and rode it back home at a rate of 24 km/h. If the total time spent travelling was one hour, how far from Tina's home was the bike shop?

56. A jogger and a cyclist went to a park for their morning exercise. They start moving on a trail loop from the same point and in the same direction. On average, the cyclist travels three times as fast as the jogger. If after 20 minutes the cyclist has finished his first loop and the jogger has still 6 km to complete the loop, how long is the trail loop?



57. A 2 km long freight train is moving at 35 km/h on a straight segment of a train-track. Suppose a car is moving at 65 km/h on a parallel road, in the opposite direction. How long would it take the car to pass from the front till the end of the train?

L4 Linear Inequalities and Interval Notation



Mathematical inequalities are often used in everyday life situations. We observe speed limits on highways, minimum payments on credit card bills, maximum smartphone data usage per month, the amount of time we need to get from home to school, etc. When we think about these situations, we often refer to limits, such as "a speed limit of 100 kilometers per hour" or "a limit of 1 GB of data per month." However, we don't have to travel at exactly 100 kilometers per hour on the highway or use exactly 1 GB of data per month. The limit only establishes a boundary for what is allowable. For example, a driver travelling x kilometers per hour is obeying the speed limit of 100 kilometers per hour if $x \leq 100$ and breaking the speed limit if $x > 100$. A speed of $x = 100$ represents the boundary between obeying the speed limit and breaking it. Solving linear inequalities is closely related to solving linear equations because equality is the boundary between *greater than* and *less than*. In this section, we discuss techniques needed to solve linear inequalities and ways of presenting these solutions.

Linear Inequalities

Definition 4.1 ► A **linear inequality** is an inequality with only **constant** or **linear terms**.
A linear inequality in one variable can be written in one of the following forms:

$$\boldsymbol{Ax+B>0},\quad \boldsymbol{Ax+B\geq 0},\quad \boldsymbol{Ax+B<0},\quad \boldsymbol{Ax+B\leq 0},\quad \boldsymbol{Ax+B\neq 0},$$

for some real numbers A and B, and a variable x.

A variable value that makes an inequality true is called a **solution** to this inequality. We say that such a variable value **satisfies** the inequality.

Example 1 ► **Determining if a Given Number is a Solution of an Inequality**

Determine whether each of the given values is a solution of the inequality.

a. $3x - 7 > -2$; 2, 1 **b.** $\frac{y}{2} - 6 \geq -3$; 8, 6

Solution ►

a. To check if 2 is a solution of $3x - 7 > -2$, replace x by 2 and determine whether the resulting inequality $3 \cdot 2 - 7 > -2$ is a true statement. Since $6 - 7 = -1$ is indeed larger than -2, then 2 satisfies the inequality. So 2 is a solution of $3x - 7 > -2$.

After replacing x by 1, we obtain $3 \cdot 1 - 7 > -2$, which simplifies to the false statement $-4 > -2$. This shows that 1 is not a solution of the given inequality.

b. To check if 8 is a solution of $\frac{y}{2} - 6 \geq -3$, substitute $y = 8$. The inequality becomes $\frac{8}{2} - 6 \geq -3$, which simplifies to $-2 \geq -3$. Since this is a true statement, 8 is a solution of the given inequality.

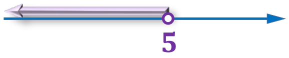
Similarly, after substituting $y = 6$, we obtain a true statement $\frac{6}{2} - 6 \geq -3$, as the left side of this inequality equals to -3. This shows that -3 is also a solution to the original inequality.

Usually, an inequality has an infinite number of solutions. For example, one can check that the inequality

$$2x - 10 < 0$$

is satisfied by -5, 0, 1, 3, 4, 4.99, and generally by any number that is smaller than 5. So in the above example, the set of all solutions, called the **solution set**, is infinite. Generally, the solution set to a linear inequality in one variable can be stated either using **set-builder notation**, or **interval notation**. Particularly, the solution set of the above inequality could be stated as $\{x | x < 5\}$, or as $(-\infty, 5)$.

In addition, it is often beneficial to visualize solution sets of inequalities in one variable as graphs on a number line. The solution set to the above example would look like this:



For more information about presenting solution sets of inequalities in the form of a graph or interval notation, refer to *Example 3* and the subsection on "*Interval Notation*" in *Section R2* of the *Review* chapter.

To solve an inequality means to find all the variable values that satisfy the inequality, which in turn means to find its solution set. Similarly as in the case of equations, we find these solutions by producing a sequence of simpler and simpler inequalities preserving the solution set, which eventually result in an inequality of one of the following forms:

$$x > constant, \quad x \geq constant, \quad x < constant, \quad x \leq constant, \quad x \neq constant.$$

Definition 4.2 ► **Equivalent inequalities** are inequalities with the same solution set.

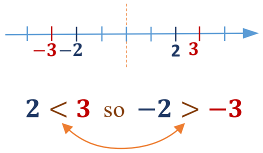
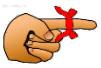


Figure 1

Generally, we create equivalent inequalities in the same way as we create equivalent equations, **except for *multiplying* or *dividing* an inequality by a *negative* number**. Then, we ***reverse the inequality symbol***, as illustrated in *Figure 1*.

So, if we multiply (or divide) the inequality

$$-x \geq 3$$



by -1, then we obtain an equivalent inequality

$$x \leq -3.$$

***multiplying* or *dividing* by a *negative* reverses the inequality sign**

We encourage the reader to confirm that the solution set to both of the above inequalities is $(-\infty, -3]$.

Multiplying or dividing an inequality by a positive number leaves the inequality sign unchanged.

The table below summarizes the basic inequality operations that can be performed to produce equivalent inequalities, starting with $A < B$, where A and B are any algebraic expressions. Suppose C is a real number or another algebraic expression. Then, we have:

Inequality operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) < 1+3$ can be written as $2x-6<4$
Addition	if $A < B$ then $A + C < B + C$	if $2x - 6 < 4$ then $2x - 6 + 6 < 4 + 6$
Subtraction	if $A < B$ then $A - C < B - C$	if $2x < x + 4$ $2x - x < x + 4 - x$
Multiplication **when multiplying by a negative value, reverse the inequality sign**	if $C > 0$ and $A < B$ then $CA < CB$ if $C < 0$ and $A < B$ then $CA > CB$	if $2x < 10$ then $\frac{1}{2} \cdot 2x < \frac{1}{2} \cdot 10$ if $-x < -5$ then $x > 5$
Division **when dividing by a negative value, reverse the inequality sign**	if $C > 0$ and $A < B$ then $\frac{A}{C} < \frac{B}{C}$ if $C < 0$ and $A < B$ then $\frac{A}{C} > \frac{B}{C}$	if $2x < 10$ then $\frac{2x}{2} < \frac{10}{2}$ if $-2x < 10$ then $x > -5$

Example 2 ► **Using Inequality Operations to Solve Linear Inequalities in One Variable**

Solve the inequalities. Graph the solution set on a number line and state the answer in interval notation.

a. $\frac{3}{4}x + 3 > 15$

b. $-2(x+3) > 10$

c. $\frac{1}{2}x - 3 \le \frac{1}{4}x + 2$

d. $-\frac{2}{3}(2x-3) - \frac{1}{2} \ge \frac{1}{2}(5-x)$

Solution ►

a. To isolate x, we apply inverse operations in reverse order. So, first we subtract the 3, and then we multiply the inequality by the reciprocal of the leading coefficient. Thus,

$$\frac{3}{4}x + 3 > 15$$

$$\frac{3}{4}x > 12$$

$$x > \frac{\overset{4}{\cancel{12}} \cdot 4}{\cancel{3}} = 16$$

To visualize the solution set of the inequality $x > 16$ on a number line, we graph the interval of all real numbers that are greater than 16.

Finally, we give the answer in interval notation by stating $\boldsymbol{x} \in (\mathbf{16}, \infty)$. This tells us that any x-value greater than 16 satisfies the original inequality.

Note: The answer can be stated as $x \in (16, \infty)$, or simply as $(16, \infty)$. Both forms are correct.

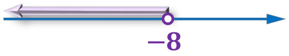
b. Here, we will first simplify the left-hand side expression by expanding the bracket and then follow the steps as in *Example 2a.* Thus,

REVERSE the inequality when **dividing by a negative!**

$$-2(x+3) > 10$$
$$-2x - 6 > 10$$
$$-2x > 16$$
$$x < -8$$

divide both sides by -2

The corresponding graph looks like this:

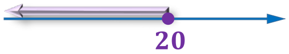


The solution set in interval notation is $(-\infty, -\mathbf{8})$.

c. To solve this inequality, we will collect and combine linear terms on the left-hand side and free terms on the right-hand side of the inequality.

$$\frac{1}{2}x - 3 \le \frac{1}{4}x + 2$$
$$\frac{1}{2}x - \frac{1}{4}x \le 5$$
$$\frac{1}{4}x \le 5$$
$$x \le 20$$

This can be graphed as



and stated in interval notation as $(-\infty, \mathbf{20}]$.

d. To solve this inequality, it would be beneficial to clear the fractions first. So, we will multiply the inequality by the LCD of 3 and 2, which is 6.

remember to multiply each term by 6, but only once!

$$-\frac{2}{3}(2x-3) - \frac{1}{2} \le \frac{1}{2}(5-x)$$
$$-\frac{2\cdot \overset{2}{\cancel{6}}}{\cancel{3}}(2x-3) - \frac{1\cdot \overset{3}{\cancel{6}}}{\cancel{2}} \le \frac{1\cdot \overset{3}{\cancel{6}}}{\cancel{2}}(5-x)$$
$$-4(2x-3) - 3 \le 3(5-x)$$

$$-8x + 12 - 3 \leq 15 - 3x$$
$$-8x + 9 \leq 15 - 3x$$

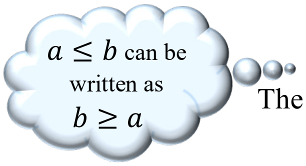
At this point, we could collect linear terms on the left or on the right-hand side of the inequality. Since it is easier to work with a positive coefficient by the x-term, let us move the linear terms to the right-hand side this time.

So, we obtain

$$-6 \leq 5x$$
$$\frac{-6}{5} \leq x,$$

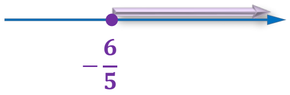
which after writing the inequality from right to left, gives us the final result

$$x \geq -\frac{6}{5}$$



$a \leq b$ can be written as $b \geq a$

The solution set can be graphed as



which means that all real numbers $x \in \left[-\frac{6}{5}, \infty\right)$ satisfy the original inequality.

Example 3 ► **Solving Special Cases of Linear Inequalities**

Solve each inequality.

a. $-2(x - 3) > 5 - 2x$ **b.** $-12 + 2(3 + 4x) < 3(x - 6) + 5x$

Solution ► **a.** Solving the inequality

$$-2(x - 3) > 5 - 2x$$
$$-2x + 6 > 5 - 2x$$
$$6 > 5,$$

leads us to a true statement that does not depend on the variable x. This means that any real number x satisfies the inequality. Therefore, the solution set of the original inequality is equal to all real numbers $\mathbb{R}$. This could also be stated in interval notation as $(-\infty, \infty)$.

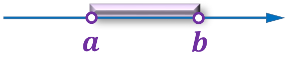
b. Solving the inequality

$$-12 + 2(3 + 4x) < 3(x - 6) + 5x$$
$$-12 + 6 + 8x < 3x - 18 + 5x$$
$$-6 + 8x < 8x - 18$$
$$-6 < -18$$

leads us to a false statement that does not depend on the variable x. This means that no real number x would satisfy the inequality. Therefore, the solution set of the original inequality is an empty set $\emptyset$. We say that the inequality has **no solution**.

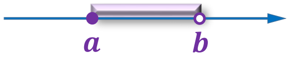
Three-Part Inequalities

The fact that an unknown quantity x lies between two given quantities a and b, where $a < b$, can be recorded with the use of the three-part inequality $a < x < b$. We say that x is enclosed by the values (or oscillates between the values) a and b. For example, the systolic high blood pressure p oscillates between 120 and 140 mm Hg. It is convenient to record this fact using the three-part inequality $120 < p < 140$, rather than saying that $p < 140$ and at the same time $p > 120$. The solution set of the three-part inequality $a < x < b$ or $b > x > a$ is a **bounded** interval (a, b) that can be graphed as



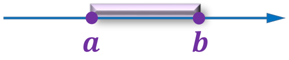
The hollow (open) dots indicate that the endpoints do not belong to the solution set. Such interval is called **open**.

If the inequality symbol includes equation ($\leq$ or $\geq$), the corresponding endpoint of the interval is included in the solution set. On a graph, this is reflected as a solid (closed) dot. For example, the solution set of the three-part inequality $a \leq x < b$ is the interval $[a, b)$, which is graphed as



Such interval is called **half-open** or **half-closed**.

An interval with both endpoints included is referred to as a **closed** interval. For example, $[a, b]$ is a closed interval and its graph looks like this



Any three-part inequality of the form

$$\textit{constant } \boldsymbol{a} < (\leq)\ \textit{one variable linear } \boldsymbol{\textit{expression}} < (\leq)\ \textit{constant } \boldsymbol{b},$$

where $a \leq b$, can be solved similarly as a single inequality, by applying inequality operations to all of the three parts. When solving such an inequality, the goal is to isolate the variable in the middle part by moving all constants to the outside parts.

Example 4 ► **Solving Three-Part Inequalities**

Solve each three-part inequality. Graph the solution set on a number line and state the answer in interval notation.

a. $-2 \leq 1 - 3x \leq 3$ **b.** $-3 < \frac{2x-3}{4} \leq 6$

Solution ► **a.** To isolate x from the expression $1 - 3x$, subtract 1 first, and then divide by -3. These operations must be applied to all three parts of the inequality. So, we have

Remember to **reverse** both inequality symbols when **dividing by a negative** number!

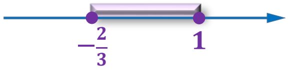
$$-2 \le 1 - 3x \le 3$$

$$-3 \le -3x \le 2$$

divide all sides by -3

$$1 \ge x \ge -\frac{2}{3}$$

The result can be graphed as



The inequality is satisfied by all $x \in \left[-\frac{2}{3}, 1\right]$.

b. To isolate x from the expression $\frac{2x-3}{4}$, we first multiply by 4, then add 3, and finally divide by 2.

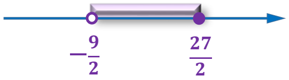
$$-3 < \frac{2x-3}{4} \le 6$$

$$-12 < 2x - 3 \le 24$$

$$-9 < 2x \le 27$$

$$-\frac{9}{2} < x \le \frac{27}{2}$$

The result can be graphed as



The inequality is satisfied by all $x \in \left(-\frac{9}{2}, \frac{27}{2}\right]$.

Inequalities in Application Problems

Linear inequalities are often used to solve problems in areas such as business, construction, design, science, or linear programming. The solution to a problem involving an inequality is generally an interval of real numbers. We often ask for the range of values that solve the problem.

Below is a list of common words and phrases indicating the use of particular inequality symbols.

Word expression	Interpretation
a **is** less (smaller) than ***b***	$a < b$
a **is** less than or equal to ***b***	$a \le b$
a **is** greater (more, bigger) than ***b***	$a > b$
a **is** greater than or equal to ***b***	$a \ge b$
a **is** at least ***b***	$a \ge b$
a **is** at most ***b***	$a \le b$
a **is** no less than ***b***	$a \ge b$
a **is** no more than ***b***	$a \le b$
a exceeds ***b***	$a > b$
a **is** different than ***b***	$a \ne b$
x **is** between ***a*** and ***b***	$a < x < b$
x **is** between ***a*** and ***b*** inclusive	$a \le x \le b$

Example 5 ► Translating English Words to an Inequality

Translate the word description into an inequality and then solve it.

a. Twice a number, increased by 3 is at most 9.
b. Two diminished by five times a number is between -4 and 7

Solution ► **a.** Twice a number, increased by 3 translates to $2x + 3$. Since "at most" corresponds to the symbol "≤", the inequality to solve is

$$2x + 3 \leq 9$$
$$2x \leq 6$$
$$x \leq 3$$

So, all $\boldsymbol{x} \in (-\infty, \mathbf{3}]$ satisfy the condition of the problem.

b. Two diminished by five times a number translates to $2 - 5x$. The phrase "between -4 and 7" tells us that the expression $2 - 5x$ is enclosed by the numbers -4 and 7, but not equal to these numbers. So, the inequality to solve is

$$-4 < 2 - 5x < 7$$
$$-6 < -5x < 5$$
$$\frac{6}{5} > x > -1$$

Therefore, the solution set to this problem is the interval of numbers $\left(\mathbf{-1}, \frac{\mathbf{6}}{\mathbf{5}}\right)$.

Remember: To record an interval, list its endpoints in ***increasing order*** (from the smaller to the larger number.)

Example 6 ► Using a Linear Inequality to Compare Cellphone Charges

A cellphone company advertises two pricing plans for day-time minutes. The first plan costs \$14.99 per month with 30 free day-time minutes and \$0.36 per minute after that. The second plan costs \$24.99 per month with 20 free day-time minutes and \$0.25 per minute after that. A customer figured that he will pay less by choosing the first plan. What could be the maximum number of day-time minutes that he predicts to use per month?

Solution ► Let n represent the number of cellphone minutes used per month. In the first plan, since the first 30 minutes are free, the number of paid-minutes can be represented by $n - 30$. Hence, the total charge according to the first plan is $14.99 + 0.36(n - 30)$. Similarly, in the second plan, the number of paid-minutes can be represented by $n - 20$. Therefore the total charge according to the second plan is $24.99 + 0.25(n - 20)$.

Since the first plan is to be cheaper, the inequality to solve is

$$14.99 + 0.36(n - 30) < 24.99 + 0.25(n - 20).$$

To work with 'nicer' numbers, such as integers, we may want to eliminate the decimals by multiplying the above inequality by 100 first. Then, after removing the brackets via distribution, we obtain

$$1499 + 36n - 1080 < 2499 + 25n - 500$$
$$419 + 11n < 1999$$
$$11n < 1580$$
$$n < \frac{1580}{11} \approx 143.6$$

So, the maximum number of day-time minutes for the first plan to be cheaper is **143**.

Example 7 ► Finding the Test Score Range of the Missing Test

Arek obtained 71% on a midterm test. If he wishes to bring his course mark to a B, the average of his midterm and final exam marks must be between 73% and 76%, inclusive. What range of scores on the final exam would guarantee Arek a mark of B in this course?

Solution ►

Let n represent Arek's score on his final exam. Then, the average of his midterm and final exam is represented by the expression

$$\frac{71+n}{2}.$$

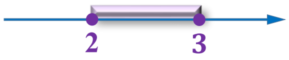
Since this average must be between 73% and 76% inclusive, we need to solve the three-part inequality

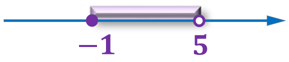
$$73 \le \frac{71+n}{2} \le 76$$
$$146 \le 71 + n \le 152$$
$$75 \le n \le 81$$

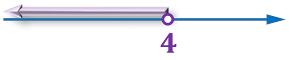
To attain a final grade of a B, Arek's score on his final exam should fall **between 75% and 81%**, inclusive.

L.4 Exercises

Using interval notation, record the set of numbers represented by each graph. (Refer to the part "Interval Notation" in Section R2 of the Review chapter, if needed.)

1. 

2. 

3. 

4. 

Graph *each solution set. For each interval write the corresponding* ***inequality*** *(or inequalities), and for each inequality, write the solution set in* ***interval notation****. (Refer to the part "Interval Notation" in Section R2 of the Review chapter, if needed.)*

5. $(3, \infty)$ **6.** $(-\infty, 2]$ **7.** $[-7,5]$ **8.** $[-1, 4)$

9. $x \geq -5$ **10.** $x > 6$ **11.** $x < -2$ **12.** $x \leq 0$

13. $-4 < x < 1$ **14.** $3 \leq x < 7$ **15.** $-5 < x \leq -2$ **16.** $0 \leq x \leq 1$

Determine whether or not the given value is a solution to the inequality.

17. $4n + 15 > 6n + 20;\ \ -5$ **18.** $16 - 5a > 2a + 9;\ \ 1$

19. $\frac{x}{4} + 7 \geq 5;\ \ -8$ **20.** $6y - 7 \leq 2 - y;\ \ \frac{2}{3}$

Solve *each inequality.* ***Graph*** *the solution set and write the solution using* ***interval notation****.*

21. $2 - 3x \geq -4$ **22.** $4x - 6 > 12 - 10x$

23. $\frac{3}{5}x > 9$ **24.** $-\frac{2}{3}x \leq 12$

25. $5(x + 3) - 2(x - 4) \geq 2(x + 7)$ **26.** $5(y + 3) + 9 < 3(y - 2) + 6$

27. $2(3x - 4) - 4x \leq 2x + 3$ **28.** $7(4 - x) + 5x > 2(16 - x)$

29. $\frac{4}{5}(7x + 6) > 40$ **30.** $\frac{2}{3}(4x - 3) \leq 30$

31. $\frac{5}{2}(2a - 3) < \frac{1}{3}(6 - 2a)$ **32.** $\frac{2}{3}(3x - 1) \geq \frac{3}{2}(2x - 3)$

33. $\frac{5-2x}{2} \geq \frac{2x+1}{4}$ **34.** $\frac{3x-2}{-2} \geq \frac{x-4}{-5}$

35. $0.05 + 0.08x < 0.01x - 0.04(3 - 3x)$ **36.** $-0.2(5x + 2) > 0.4 + 1.5x$

37. $-\frac{1}{4}(p + 6) + \frac{3}{2}(2p - 5) \leq 10$ **38.** $\frac{3}{5}(t - 2) - \frac{1}{4}(2t - 7) \leq 3$

39. $-6 \leq 5x - 7 \leq 4$ **40.** $-10 < 3b - 5 < -1$

41. $2 \leq -3m - 7 \leq 4$ **42.** $4 < -9x + 5 < 8$

43. $-\frac{1}{2} < \frac{1}{4}x - 3 < \frac{1}{2}$ **44.** $-\frac{2}{3} \leq 4 - \frac{1}{4}x \leq \frac{2}{3}$

45. $-3 \leq \frac{7-3x}{2} < 5$ **46.** $-7 < \frac{3-2x}{3} \leq -2$

Using interval notation, state the set of numbers satisfying each description.

47. The sum of a number and 5 exceeds 12.

48. 5 times a number, decreased by 6, is smaller than -16.

49. 2 more than three times a number is at least 8.

50. Triple a number, subtracted from 5, is at most 7.

51. Half of a number increased by 3 is no more than 12.

52. Twice a number increased by 1 is different than 14.

53. Double a number is between -6 and 8.

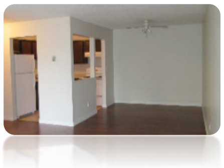
54. Half a number, decreased by 3, is between 1 and 12.

Solve each problem.

55. There are three major tests in the algebra course that Nicole takes. She already wrote the first two tests and received 79% and 89% respectively. What score must she aim for when writing her third test to keep an average test mark of 85% or higher?

56. To receive a B in a university course, the average mark needs to be between 73 and 76, inclusive. Suppose a final grade in a particular course is calculated by taking average of the four major tests, including the final exam. On the first three tests, a student obtained the following scores: 59, 71, and 86. What range of scores on the final exam will guarantee the student a B in this course?

57. A marketing company has a budget of $1400 to run an advertisement on a particular website. The website charges $10.50 per day to display the add and $220 set up fee. Maximally, how many days the ad can be posted on this site?



58. Ken plans to paint a room with 340 square feet of wall area. He needs to buy some masking tape, drop sheets, and paint brushes for a total of $32. A gallon of paint covers 100 square feet of area, and the paint is sold only in gallons. If Ken plans to stay within $150 for the whole job what is the maximum cost per gallon of paint that he can afford?

59. Last week, the temperature in Banff, BC, ranged between 23°F and 68°F. Using the formula $F = \frac{9}{5}C + 32$, find the temperature range in degrees Celsius.

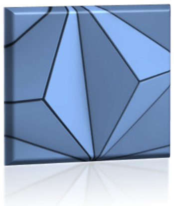
60. One day, the temperature range in Whistler, BC, was between -2°C and 18°C. Using the formula $C = \frac{5}{9}(F - 32)$, find the temperature range in degrees Fahrenheit.

61. Adon makes $1600 a month with an additional commission of 7% of his sales. This month, Adon's earnings were higher than $3700. How high must have been his sales?

62. Suppose a particular bank offers two chequing accounts. The first account charges $5 per month and $0.75 per cheque after the first 10 cheques. The second account charges $12.50 per month with unlimited cheque writing. What is the maximum number of cheques processed for a customer who chooses the first account as the better option?

63. The Toronto Dominion Bank offers a chequing account that charges $10.95 per month plus $1.25 per cheque after the first 25 cheques. A competitor bank is offering an account for $7.95 per month plus $1.30 per cheque after the first 25 cheques. If a business chooses the first account as the cheaper option, what is the minimum number of cheques that the business predicts to write monthly?

L5 Operations on Sets and Compound Inequalities

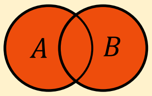


In the previous section, it was shown how to solve a three-part inequality. In this section, we will study how to solve systems of inequalities, often called **compound** inequalities, that consist of two linear inequalities joined by the words "*and*" or "*or*". Studying compound inequalities is an extension of studying three-part inequalities. For example, the three-part inequality, $2 < x \le 5$, is in fact a system of two inequalities, $2 < x$ *and* $x \le 5$. The solution set to this system of inequalities consists of all numbers that are larger than 2 *and* at the same time are smaller or equal to 5. However, notice that the system of the same two inequalities connected by the word "*or*", $2 < x$ *or* $x \le 5$, is satisfied by any real number. This is because any real number is either larger than 2 *or* smaller than 5. Thus, to find solutions to compound inequalities, aside for solving each inequality individually, we need to pay attention to the joining words, "*and*" or "*or*". These words suggest particular operations on the sets of solutions.

Operations on Sets

Sets can be added, subtracted, or multiplied.

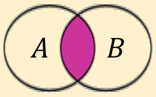
Definition 5.1 ►



The result of the addition of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **union** (or sum), symbolized by $\boldsymbol{A \cup B}$, and defined as:

$$\boldsymbol{A \cup B = \{x | x \in A \text{ or } x \in B\}}$$

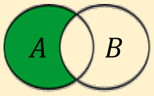
This is the set of all elements that belong to either $\boldsymbol{A}$ ***or*** $\boldsymbol{B}$.



The result of the multiplication of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called an **intersection** (or product, or **common part**), symbolized by $\boldsymbol{A \cap B}$, and defined as:

$$\boldsymbol{A \cap B = \{x | x \in A \text{ and } x \in B\}}$$

This is the set of all elements that belong to both $\boldsymbol{A}$ ***and*** $\boldsymbol{B}$.



The result of the subtraction of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **difference**, symbolized by $\boldsymbol{A \setminus B}$, and defined as:

$$\boldsymbol{A \setminus B = \{x | x \in A \text{ and } x \notin B\}}$$

This is the set of all elements that belong to $\boldsymbol{A}$ ***and*** do not belong to $\boldsymbol{B}$.

Example 1 ► **Performing Operations on Sets**

Suppose $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, and $C = \{6\}$. Find the following sets:

a. $A \cap B$

b. $A \cup B$

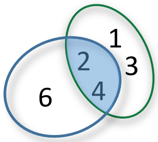
c. $A \setminus B$

d. $B \cup C$

e. $B \cap C$

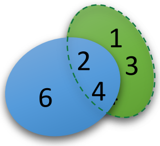
f. $A \cap C$

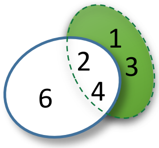
Solution



a. The intersection of sets $A = \{1, \textcircled{2}, 3, \textcircled{4}\}$ and $B = \{\textcircled{2}, \textcircled{4}, 6\}$ consists of numbers that belong to both sets, so $\boldsymbol{A \cap B = \{2, 4\}}$.

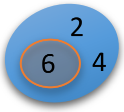
b. The union of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{A \cup B = \{1, 2, 3, 4, 6\}}$.



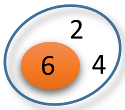


c. The difference $A \setminus B$ of sets $A = \{1, \cancel{2}, 3, \cancel{4}\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to the set A but do not belong to the set B, so $\boldsymbol{A \setminus B = \{1, 3\}}$.

d. The union of sets $B = \{2, 4, 6\}$, and $C = \{6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{B \cup C = \{2, 4, 6\}}$.



Notice that $B \cup C = B$. This is because C is a **subset** of B.

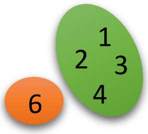


e. The intersection of sets $B = \{2, 4, 6\}$ and $C = \{6\}$ consists of numbers that belong to both sets, so $\boldsymbol{B \cap C = \{6\}}$.

Notice that $B \cap C = C$. This is because C is a **subset** of B.

f. Since the sets $A = \{1, 2, 3, 4\}$ and $C = \{6\}$ do not have any common elements, then $\boldsymbol{A \cap C = \emptyset}$.

Recall: Two sets with no common part are called **disjoint**.
Thus, the sets A and C are disjoint.



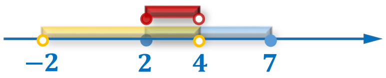
Example 2 ► Finding Intersections and Unions of Intervals

Write the result of each set operation as a single interval, if possible.

a.	$(-2,4) \cap [2,7]$	**b.**	$(-\infty, 1] \cap (-\infty, 3)$
c.	$(-1,3) \cup (1, 6]$	**d.**	$(3, \infty) \cup [5, \infty)$
e.	$(-\infty,3) \cup (4, \infty)$	**f.**	$(-\infty,3) \cap (4, \infty)$
g.	$(-\infty,5) \cup (4, \infty)$	**h.**	$(-\infty,3] \cap [3, \infty)$

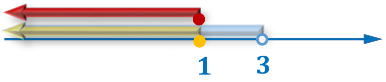
Solution

a. The interval of points that belong to both, the interval $(-2,4)$, in yellow, and the interval $[2,7]$, in blue, is marked in red in the graph below.



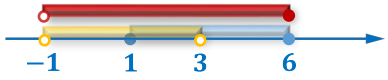
So, we write $(-2,4) \cap [2,7] = [2, 4)$.

b. As in problem **a.**, the common part $(-\infty, 1] \cap (-\infty, 3)$ is illustrated in red on the graph below.



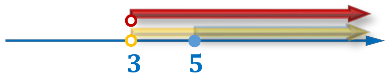
So, we write $(-\infty, 1] \cap (-\infty, 3) = (-\infty, 1]$.

c. This time, we take the union of the interval $(-1,3)$, in yellow, and the interval $(1,6]$, in blue. The result is illustrated in red on the graph below.



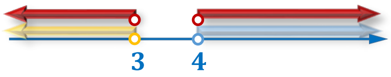
So, we write $(-1,3) \cup (1,6] = (-1,6]$.

d. The union $(3,\infty) \cup [5,\infty)$ is illustrated in red on the graph below.



So, we write $(3,\infty) \cup [5,\infty) = (3,\infty)$.

e. As illustrated in the graph below, this time, the union $(-\infty,3) \cup (4,\infty)$ can't be written in the form of a single interval.

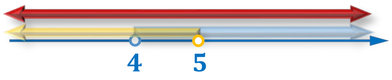


So, the expression $(-\infty,3) \cup (4,\infty)$ cannot be simplified.

f. As shown in the graph below, the interval $(-\infty,3)$ has no common part with the interval $(4,\infty)$.

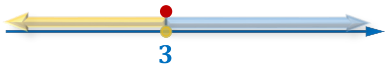
Therefore, $(-\infty,3) \cap (4,\infty) = \emptyset$

g. This time, the union $(-\infty,5) \cup (4,\infty)$ covers the entire number line.



Therefore, $(-\infty,5) \cup (4,\infty) = (-\infty,\infty)$

h. As shown in the graph below, there is only one point common to both intervals, $(-\infty,3]$ and $[3,\infty)$. This is the number 3.



Since a single number is not considered to be an interval, we should use a set rather than interval notation when recording the answer. So, $(-\infty,3] \cap [3,\infty) = \{3\}$.

Compound Inequalities

The solution set to a system of two inequalities joined by the word *and* is the intersection of solutions of each inequality in the system. For example, the solution set to the system

$$\begin{cases} x > 1 \\ x \le 4 \end{cases}$$

is the intersection of the solution set for $x > 1$ and the solution set for $x \le 4$. This means that the solution to the above system equals $(1, \infty) \cap (-\infty, 4] = (1, 4]$, as illustrated in the graph below.

The solution set to a system of two inequalities joined by the word *or* is the union of solutions of each inequality in the system. For example, the solution set to the system

$$x \le 1 \text{ or } x > 4$$

is the union of the solution of $x \le 1$ and, the solution of $x > 4$. This means that the solution to the above system equals $(-\infty, 1) \cup [4, \infty)$, as illustrated in the graph below.

Example 3 ► **Solving Compound Linear Inequalities**

Solve each compound inequality. Pay attention to the joining word *and* or *or* to find the overall solution set. Give the solution set in both interval and graph form.

a. $3x + 7 \ge 4 \quad and \quad 2x - 5 < -1$

b. $-2x - 5 \ge 1 \quad or \quad x - 5 \ge -3$

c. $3x - 11 < 4 \quad or \quad 4x + 9 \ge 1$

d. $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$

e. $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$

f. $4x - 2 < -8 \quad or \quad 5x - 3 < 12$

Solution ► **a.** To solve this system of inequalities, first, we solve each individual inequality, keeping in mind the joining word *and*. So, we have

$$\begin{array}{rcc} 3x + 7 \ge 4 & and & 2x - 5 < -1 \\ 3x \ge -3 & and & 2x < 4 \\ x \ge -1 & and & x < 2 \end{array}$$

The joining word *and* indicates that we look for the intersection of the obtained solutions. These solutions (in yellow and blue) and their intersection (in red) are shown in the graph below.

Therefore, the system of inequalities is satisfied by all $x \in [\mathbf{-1}, \mathbf{2})$.

b. As in the previous example, first, we solve each individual inequality, except this time we keep in mind the joining word *or*. So, we have

$$-2x - 5 \ge 1 \qquad or \qquad x - 5 \ge -3$$

reverse the signs

$$-2x \geq 6 \qquad or \qquad x \geq 2$$

$$x \leq -3$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.

Therefore, the system of inequalities is satisfied by all $x \in (-\infty, \mathbf{-3}] \cup [\mathbf{2}, \infty)$.

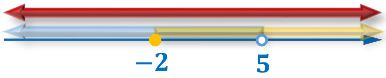
c. As before, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$3x - 11 < 4 \qquad or \qquad 4x + 9 \geq 1$$

$$3x < 15 \qquad or \qquad 4x \geq -8$$

$$x < 5 \qquad or \qquad x \geq -2$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the system of inequalities is satisfied by all real numbers. The solution set equals to $\mathbb{R}$.

d. Any three-part inequality is a system of inequalities with the joining word *and*. The system $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$ could be written as

$$-2 < 3 - \frac{1}{4}x \quad and \quad 3 - \frac{1}{4}x < \frac{1}{2}$$

and solved as in *Example 3a*. Alternatively, it could be solved in the three-part form, similarly as in *Section L4, Example 4*. Here is the three-part form solution.

$$-2 < 3 - \frac{1}{4}x < \frac{1}{2}$$

$$-5 < -\frac{1}{4}x < \frac{1}{2} - \frac{3 \cdot 2}{2}$$

$$-5 < -\frac{1}{4}x < -\frac{5}{2}$$

reverse the signs !

$$20 > x > \frac{5 \cdot 4}{2}$$

$$20 > x > 10$$

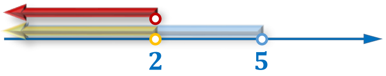
So the solution set is the interval $(10, 20)$, visualized in the graph below.

Remark: Solving a system of inequalities in three-part form has its benefits. First, the same operations are applied to all three parts, which eliminates the necessity of repeating the solving process for the second inequality. Second, the solving process of a three-part inequality produces the final interval of solutions rather than two intervals that need to be intersected to obtain the final solution set.

e. The system $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$ consists of two inequalities joined by the word *and*. So, we solve it similarly as in *Example 3a*.

$$\begin{array}{rcr} 4x - 7 < 1 & \textit{and} & 7 - 3x > -8 \\ 4x < 8 & \textit{and} & -3x > -15 \\ x < 2 & \textit{and} & x < 5 \end{array}$$

These solutions of each individual inequality (in yellow and blue) and the intersection of these solutions (in red) are indicated in the graph below.

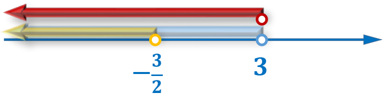


Therefore, the interval $(-\infty, 2)$ is the solution to the whole system.

f. As in *Example 3b* and *3c*, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$\begin{array}{rcr} 4x - 2 < -8 & \textit{or} & 5x - 3 < 12 \\ 4x \geq -6 & \textit{or} & 5x \geq 15 \\ x < -\frac{3}{2} & \textit{or} & x < 3 \end{array}$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the interval $(-\infty, 3)$ is the solution to the whole system.

Compound Inequalities in Application Problems

Compound inequalities are often used to solve problems that ask for a range of values satisfying certain conditions.

Example 4 ► **Finding the Range of Values Satisfying Conditions of a Problem**

The equation $P = 1 + \frac{d}{11}$ gives the pressure P, in atmospheres (atm), at a depth of d meters in the ocean. Atlantic cod occupy waters with pressures between 1.6 and 7 atmospheres. To the nearest meter, what is the depth range at which Atlantic cod should be searched for?

Solution ► The pressure P suitable for Atlantic cod is between 1.6 to 7 atmospheres. We record this fact in the form of the three-part inequality $1.6 \leq P \leq 7$. To find the corresponding depth d, in meters, we substitute $P = 1 + \frac{d}{11}$ and solve the three-part inequality for d. So, we have

$$1.6 \leq 1 + \frac{d}{11} \leq 7$$

$$0.6 \leq \frac{d}{11} \leq 6$$

$$6.6 \leq d \leq 66$$

Thus, Atlantic cod should be searched for between 7 and 66 meters below the surface.

Example 5 ► **Using Set Operations to Solve Applied Problems Involving Compound Inequalities**

Given the information in the table,

Film	**Admissions** (in millions)	**Adjusted Gross Income** (in millions of dollars)
Gone With the Wind	202	1825
Star Wars	178	1608
The Sound of Music	142	1286
Titanic	136	1224
Avatar	97	878

list the films that belong to each set.

a. The set of films with admissions greater than 150,000,000 *and* adjusted gross income greater than \$1,000,000,000.

b. The set of films with admissions greater than 150,000,000 *or* adjusted gross income greater than \$1,000,000,000.

c. The set of films with admissions smaller than 150,000,000 *and* adjusted gross income greater than \$1,000,000,000.

Solution ►

a. The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind* and *Star Wars*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind, Star Wars, The Sound of Music,* and *Titanic*. Therefore, the set of films satisfying both of these properties contains of ***Gone With the Wind*** and ***Star Wars***.

b. The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind* and *Star Wars*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind, Star Wars, The Sound of Music,* and *Titanic*. Therefore, the set of films satisfying at least one of these properties consists of ***Gone With the Wind, Star Wars, The Sound of Music***, and ***Titanic***.

c. The set of films with admissions smaller than 150,000,000 consists of *The Sound of Music, Titanic,* and *Avatar.* The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind, Star Wars, The Sound of Music,* and *Titanic*. Therefore, the set of films satisfying both of these properties consists of ***The Sound of Music***, and ***Titanic***.

L.5 Exercises

Let $A = \{1, 2, 3, 4\}$, $B = \{1,\ 3,\ 5\}$, $C = \{5\}$. *Find each set.*

1. $A \cap B$

2. $A \cup B$

3. $B \cup C$

4. $A \setminus B$

5. $A \cap C$

6. $A \cup B \cup C$

7. $B \cap C$

8. $A \cup C$

Write the result of each set operation as a single interval, if possible.

9. $(-7, 3] \cap [1, 6]$

10. $(-8, 5] \cap (-1, 13)$

11. $(0, 3) \cup (1, 7]$

12. $[-7, 2] \cup (1, 10)$

13. $(-\infty, 13) \cup (1, \infty)$

14. $(-\infty, 1) \cap (2, \infty)$

15. $(-\infty, 1] \cap [1, \infty)$

16. $(-\infty, -1] \cup [1, \infty)$

17. $(-2, \infty) \cup [3, \infty)$

18. $(-2, \infty) \cap [3, \infty)$

Solve each compound inequality. Give the solution set in both interval and graph form.

19. $x + 1 > 6 \quad or \quad 1 - x > 3$

20. $-3x \geq -6 \quad and \quad -2x \leq 12$

21. $4x + 1 < 5 \quad and \quad 4x + 7 > -1$

22. $3y - 11 > 4 \quad or \quad 4y + 9 \leq 1$

23. $3x - 7 < -10 \quad and \quad 5x + 2 \leq 22$

24. $\frac{1}{4}y - 2 < -3 \quad or \quad 1 - \frac{3}{2}y \geq 4$

25. $\begin{cases} 1 - 7x \leq -41 \\ 3x + 1 \geq -8 \end{cases}$

26. $\begin{cases} 2(x + 1) < 8 \\ -2(x - 4) > -2 \end{cases}$

27. $-\frac{2}{3} \leq 3 - \frac{1}{2}a < \frac{2}{3}$

28. $-4 \leq \frac{7 - 3a}{5} \leq 4$

29. $5x + 12 > 2 \quad or \quad 7x - 1 < 13$

30. $4x - 2 > 10 \quad and \quad 8x + 2 \leq -14$

31. $7t - 1 > -1 \quad and \quad 2t - 5 \geq -10$

32. $7z - 6 > 0 \quad or \quad -\frac{1}{2}z \leq 6$

33. $\frac{5x+4}{2} \geq 7 \quad or \quad \frac{7-2x}{3} \geq 2$

34. $\frac{2x-5}{-2} \geq 2 \quad and \quad \frac{2x+1}{3} \geq 0$

35. $13 - 3x > -8 \quad and \quad 12x + 7 \geq -(1 - 10x)$

36. $1 \leq -\frac{1}{3}(4b - 27) \leq 3$

Solve each problem.

37. Two friends plan to drive between 680 and 920 kilometers per day. If they estimate that their average driving speed will be 80 km/h, how many hours per day will they be driving?

38. A substance is in a liquid state if its temperature is between its melting point and its boiling point. The melting point of phosphorus is $44°C$ and its boiling point is $280.5°C$. Using the conversion formula $C = \frac{5}{9}(F - 32)$, determine the rage of temperatures in $°F$ for which phosphorus assumes a liquid state.

39. Kevin's birthday party will cost \$400 to rent a banquet hall and an additional \$25 for every guest. If Kevin wants to keep the cost of the party between \$750 and \$1000, how many guests could he invite?

40. Michael works for \$15 per hour plus \$18 per every overtime hour after the first 40 hours per week. How many hours of overtime must he work to earn between \$800 and \$1000 per week?

41. The table below shows average tuition fees for full-time international undergraduate and graduate students during the 20018/19 academic year, by field of study.

Field of Study	Undergraduate	Graduate
Education	\$19,461	\$15,236
Humanities	\$26,175	\$13,520
Business	\$26,395	\$22,442
Mathematics	\$30,187	\$15,553
Dentistry	\$55,802	\$21,635
Nursing	\$20,354	\$13,713
Veterinary Medicine	\$60,458	\$9,088

Which fields of study belong to the following sets:

a. the set of the fields of study that cost less than \$25000 for undergraduates *or* less than \$15,000 for graduates

b. the set of the fields of study that cost less than \$25000 for undergraduates *and* less than \$15,000 for graduates

c. the set of the fields of study that cost more than \$25000 for undergraduates *or* more than \$15,000 for graduates

d. the set of the fields of study that cost more than \$25000 for undergraduates *and* more than \$15,000 for graduates

L6 Absolute Value Equations

The concept of **absolute value** (also called **numerical value**) was introduced in *Section R2*. Recall that when using geometrical visualisation of real numbers on a number line, the absolute value of a number x, denoted $|x|$, can be interpreted as the distance of the point x from zero. Since distance cannot be negative, the result of absolute value is always nonnegative. In addition, the distance between points x and a can be recorded as $|x-a|$ (see *Definition 2.2* in *Section R2*), which represents the nonnegative difference between the two quantities. In this section, we will take a closer look at absolute value properties, and then apply them to solve absolute value equations.

Properties of Absolute Value

The formal definition of absolute value

$$|x| \overset{def}{=} \begin{cases} x, & \text{if} \quad x \geq 0 \\ -x, & \text{if} \quad x < 0 \end{cases}$$

tells us that, when x is nonnegative, the absolute value of x is the same as x, and when x is negative, the absolute value of it is the **opposite** of x.

So, $|2| = 2$ and $|-2| = -(-2) = 2$. Observe that this complies with the notion of a distance from zero on a number line. Both numbers, 2 and -2 are at a distance of 2 units from zero. They are both solutions to the equation $|x| = 2$.

Since $|x|$ represents the distance of the number x from 0, which is never negative, we can claim the first absolute value property:

$|x| \geq 0$, for any real x

Here are several other absolute value properties that allow us to simplify algebraic expressions.

Let x and y be any real numbers. Then

$|x| = 0$ if and only if $x = 0$

Only zero is at the distance zero from zero.

$|-x| = |x|$

The distance of opposite numbers from zero is the same.

$|xy| = |x||y|$

Absolute value of a product is the product of absolute values.

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for $y \neq 0$

Absolute value of a quotient is the quotient of absolute values.

Attention: *Absolute value doesn't 'split' over addition or subtraction! That means*

$$|\boldsymbol{x} \pm \boldsymbol{y}| \neq |\boldsymbol{x}| \pm |\boldsymbol{y}|$$

For example, $|2+(-3)| = 1 \neq 5 = |2| + |-3|$.

Example 1 ► **Simplifying Absolute Value Expressions**

Simplify, leaving as little as possible inside each absolute value sign.

a. $|-2x|$

b. $|3x^2y|$

c. $\left|\frac{-a^2}{2b}\right|$

d. $\left|\frac{-1+x}{4}\right|$

Solution ► **a.** Since absolute value can 'split' over multiplication, we have

$$|-2x| = |-2||x| = \boldsymbol{2|x|}$$

b. Using the multiplication property of absolute value and the fact that x^2 is never negative, we have

$$|3x^2y| = |3||x^2||y| = \boldsymbol{3x^2|y|}$$

c. Using properties of absolute value, we have

$$\left|\frac{-a^2}{2b}\right| = \frac{|-1||a^2|}{|2||b|} = \boldsymbol{\frac{a^2}{2|b|}}$$

d. Since absolute value does not 'split' over addition, the only simplification we can perform here is to take 4 outside of the absolute value sign. So, we have

$$\left|\frac{-1+x}{4}\right| = \boldsymbol{\frac{|x-1|}{4}} \text{ or equivalently } \boldsymbol{\frac{1}{4}|x-1|}$$

Remark: *Convince yourself that* $|x-1|$ *is not equivalent to* $x+1$ *by evaluating both expressions at, for example,* $x = 1$.

Absolute Value Equations

The formal definition of absolute value (see *Definition 2.1* in *Section R2*) applies not only to a single number or a variable x but also to any algebraic expression. Generally, we have

$$|\boldsymbol{expr.}| \overset{\boldsymbol{def}}{=} \begin{cases} \boldsymbol{expr.}, & \boldsymbol{if} \quad \boldsymbol{expr.} \geq \boldsymbol{0} \\ -(\boldsymbol{expr.}), & \boldsymbol{if} \quad \boldsymbol{expr.} < \boldsymbol{0} \end{cases}$$

This tells us that, when an *expression* is nonnegative, the absolute value of the *expression* is the **same** as the ***expression***, and when the *expression* is negative, the absolute value of the *expression* is the **opposite** of the ***expression***.

For example, to evaluate $|x-1|$, we consider when the expression $x-1$ is nonnegative and when it is negative. Since $x-1 \geq 0$ for $x \geq 1$, we have

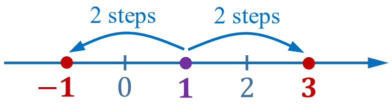
$$|x-1| = \begin{cases} x-1, & for\ \ x \geq 1 \\ -(x-1), & for\ \ x < 1 \end{cases}$$

Notice that both expressions, $x-1\ for\ x \geq 1$ and $-(x-1)\ for\ x < 1$ produce nonnegative values that represent the distance of a number x from 0.

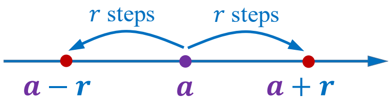
In particular,

if $x = 3$, then $|\boldsymbol{x} - \mathbf{1}| = x - 1 = 3 - 1 = \mathbf{2}$,
and
if $x = -1$, then $|\boldsymbol{x} - \mathbf{1}| = -(x-1) = -(-1-1) = -(-2) = \mathbf{2}$.

As illustrated on the number line below, both numbers, **3** and **−1** are at the distance of **2** units from **1**.



Generally, the equation $|\boldsymbol{x} - \boldsymbol{a}| = \boldsymbol{r}$ tells us that the distance between $\boldsymbol{x}$ and $\boldsymbol{a}$ is equal to $\boldsymbol{r}$. This means that $\boldsymbol{x}$ is $\boldsymbol{r}$ units away from number $\boldsymbol{a}$, in either direction.



Therefore, $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{r}$ and $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{r}$ are the solutions of the equation $|\boldsymbol{x} - \boldsymbol{a}| = \boldsymbol{r}$.

Example 2 ► **Solving Absolute Value Equations Geometrically**

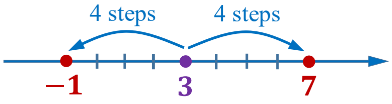
For each equation, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x-3| = 4$

b. $|x+5| = 3$

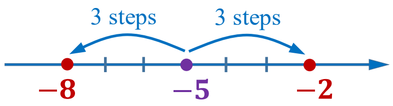
Solution ►

a. Geometrically, $|x-3|$ represents the distance between x and 3. Thus, in $|x-3| = 4$, x is a number whose distance from 3 is 4. So, $x = 3 \pm 4$, which equals either -1 or 7.



Therefore, the solution set is $\{\mathbf{-1}, \mathbf{7}\}$.

b. By rewriting $|x+5|$ as $|x-(-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x+5| = 3$, x is a number whose distance from -5 is 3. Thus, $x = -5 \pm 3$, which results in -8 or -2.



Therefore, the solution set is $\{\boldsymbol{-8}, \boldsymbol{-2}\}$.

Although the geometric interpretation of absolute value proves to be very useful in solving some of the equations, it can be handy to have an algebraic method that will allow us to solve any type of absolute value equation.

Suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.}| = \boldsymbol{r}, \quad \text{where } r > 0$$

We have two possibilities. Either the $\boldsymbol{expression}$ inside the absolute value bars is nonnegative, or it is negative. By definition of absolute value, if the $\boldsymbol{expression}$ is nonnegative, our equation becomes

$$\boldsymbol{expr.} = \boldsymbol{r}$$

If the $\boldsymbol{expression}$ is negative, then to remove the absolute value bar, we must change the sign of the $\boldsymbol{expression}$. So, our equation becomes

$$-\boldsymbol{expr.} = \boldsymbol{r},$$

which is equivalent to

$$\boldsymbol{expr.} = -\boldsymbol{r}$$

In summary, for $\boldsymbol{r} > 0$, the equation

$$|\boldsymbol{expr.}| = \boldsymbol{r}$$

$$\boldsymbol{expr.} = \boldsymbol{r} \quad \text{or} \quad \boldsymbol{expr.} = -\boldsymbol{r},$$

is equivalent to the system of equations with the connecting word *or*.

If $\boldsymbol{r} = 0$, then $|\boldsymbol{expr.}| = \boldsymbol{0}$ is equivalent to the equation $\boldsymbol{expr.} = \boldsymbol{0}$ with no absolute value.

If $\boldsymbol{r} < 0$, then $|\boldsymbol{expr.}| = \boldsymbol{r}$ has **NO SOLUTION**, as an absolute value is never negative.

Now, suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.A}| = |\boldsymbol{expr.B}|$$

Since both expressions, $\boldsymbol{A}$ and $\boldsymbol{B}$, can be either nonnegative or negative, when removing absolute value bars, we have four possibilities:

$$\begin{array}{ccc} \boldsymbol{expr.A = expr.B} & \text{or} & \boldsymbol{expr.A = -expr.B} \\ \boldsymbol{-expr.A = -expr.B} & \text{or} & \boldsymbol{-expr.A = expr.B} \end{array}$$

However, observe that the equations in blue are equivalent. Also, the equations in green are equivalent. So, in fact, it is enough to consider just the first two possibilities.

Therefore, the equation

$$\begin{tikzcd} & {|expr.A| = |expr.B|} \arrow[dl] \arrow[dr] & \\ {expr.A = expr.B} & \text{or} & {expr.A = -(expr.B),} \end{tikzcd}$$

is equivalent to the system of equations with the connecting word *or*.

Example 3 ► Solving Absolute Value Equations Algebraically

Solve the following equations.

a. $|2 - 3x| = 7$

b. $5|x| - 3 = 12$

c. $\left|\frac{1-x}{4}\right| = 0$

d. $|6x + 5| = -4$

e. $|2x - 3| = |x + 5|$

f. $|x - 3| = |3 - x|$

Solution ►

a. To solve $|2 - 3x| = 7$, we remove the absolute value bars by changing the equation into the corresponding system of equations with no absolute value anymore. Then, we solve the resulting linear equations. So, we have

$$\begin{tikzcd} & {|2 - 3x| = 7} \arrow[dl] \arrow[dr] & \\ {2 - 3x = 7} & \text{or} & {2 - 3x = -7} \\ {2 - 7 = 3x} & \text{or} & {2 + 7 = 3x} \\ {x = \frac{-5}{3}} & \text{or} & {x = \frac{9}{3} = 3} \end{tikzcd}$$

Therefore, the solution set of this equation is $\left\{-\frac{\mathbf{5}}{\mathbf{3}}, \mathbf{3}\right\}$.

b. To solve $5|x| - 3 = 12$, first, we **isolate the absolute value**, and then replace the equation by the corresponding system of two linear equations.

$$5|x| - 3 = 12$$
$$5|x| = 15$$
$$\begin{tikzcd} & {|x| = 3} \arrow[dl] \arrow[dr] & \\ {x = 3} & \text{or} & {x = -3} \end{tikzcd}$$

So, the solution set of the given equation is $\{\mathbf{-3}, \mathbf{3}\}$.

c. By properties of absolute value, $\left|\frac{1-x}{4}\right| = 0$ if and only if $\frac{1-x}{4} = 0$, which happens when the numerator $1 - x = 0$. So, the only solution to the given equation is $x = \mathbf{1}$.

d. Since an absolute value is never negative, the equation $|6x + 5| = -4$ does not have any solution.

e. To solve $|2x-3|=|x+5|$, we remove the absolute value symbols by changing the equation into the corresponding system of linear equations with no absolute value. Then, we solve the resulting equations. So, we have

$$\begin{tikzcd} & |2x-3|=|x+5| \arrow[dl] \arrow[dr] & \\ 2x-3=x+5 & \mathit{or} & 2x-3=-(x+5) \end{tikzcd}$$

$$\begin{array}{ccc} 2x-x=5+3 & \mathit{or} & 2x-3=-x-5 \\ x=8 & \mathit{or} & 3x=-2 \\ & & x=\frac{-2}{3} \end{array}$$

Therefore, the solution set of this equation is $\left\{-\frac{2}{3},\mathbf{8}\right\}$.

f. We solve $|x-3|=|3-x|$ as in *Example 3e*.

$$\begin{tikzcd} & |x-3|=|3-x| \arrow[dl] \arrow[dr] & \\ x-3=3-x & \mathit{or} & x-3=-(3-x) \end{tikzcd}$$

$$\begin{array}{ccc} 2x=6 & \mathit{or} & x-3=-3+x \\ x=3 & \mathit{or} & 0=0 \end{array}$$

Since the equation $0=0$ is always true, any real x-value satisfies the original equation $|x-3|=|3-x|$. So, the solution set to the original equation is $\mathbb{R}$.

Remark: *Without solving the equation in Example 3f, one could observe that the expressions $x-3$ and $3-x$ are opposite to each other and as such, they have the same absolute value. Therefore, the equation is always true.*

Summary of Solving Absolute Value Equations

Step 1 **Isolate the absolute value** expression on one side of the equation.

Step 2 **Check for special cases**, such as

$$\begin{array}{l} |\boldsymbol{A}|=\mathbf{0} \leftrightarrow \boldsymbol{A}=\mathbf{0} \\ |\boldsymbol{A}|=-\boldsymbol{r} \rightarrow \textbf{No solution} \end{array}$$

Step 3 **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations with the joining word *or*,

$$\begin{tikzcd} & |\boldsymbol{A}|=\boldsymbol{r} \quad (\boldsymbol{r}>\mathbf{0}) \arrow[dl] \arrow[dr] & \\ \boldsymbol{A}=\boldsymbol{r} & \mathit{or} & \boldsymbol{A}=-\boldsymbol{r} \end{tikzcd} \qquad \begin{tikzcd} & |\boldsymbol{A}|=|\boldsymbol{B}| \arrow[dl] \arrow[dr] & \\ \boldsymbol{A}=\boldsymbol{B} & \mathit{or} & \boldsymbol{A}=-\boldsymbol{B} \end{tikzcd}$$

Step 4 **Solve** the resulting equations.

Step 5 **State the solution set** as a union of the solutions of each equation in the system.

L.6 Exercises

Simplify, if possible, leaving as little as possible inside the absolute value symbol.

1. $|-2x^2|$

2. $|3x|$

3. $\left|\frac{-5}{y}\right|$

4. $\left|\frac{3}{-y}\right|$

5. $|7x^4y^3|$

6. $|-3x^5y^4|$

7. $\left|\frac{x^2}{y}\right|$

8. $\left|\frac{-4x}{y^2}\right|$

9. $\left|\frac{-3x^3}{6x}\right|$

10. $\left|\frac{5x^2}{-25x}\right|$

11. $|(x-1)^2|$

12. $|x^2-1|$

13. *In each situation, find the* ***number of solutions*** *for the equation* $|\boldsymbol{ax}+\boldsymbol{b}|=\boldsymbol{k}$.

a. $k<0$

b. $k=0$

c. $k>0$

Solve each equation.

14. $|x|=7$

15. $|-x|=4$

16. $|5x|=20$

17. $|y-3|=8$

18. $|2y+5|=9$

19. $7|3x-5|=35$

20. $-3|2x-7|=-12$

21. $\left|\frac{1}{2}x+3\right|=11$

22. $\left|\frac{2}{3}x-1\right|=5$

23. $|2x-5|=-1$

24. $|7x+11|=0$

25. $2+3|a|=8$

26. $10-|2a-1|=4$

27. $\left|\frac{2x-1}{3}\right|=5$

28. $\left|\frac{3-5x}{6}\right|=3$

29. $|2p+4|=|3p-1|$

30. $|5-q|=|q+7|$

31. $\left|\frac{1}{2}x+3\right|=\left|\frac{1}{5}x-1\right|$

32. $\left|\frac{2}{3}x-8\right|=\left|\frac{1}{6}x+3\right|$

33. $\left|\frac{3x-6}{2}\right|=\left|\frac{5+x}{5}\right|$

34. $\left|\frac{6-5x}{4}\right|=\left|\frac{7+3x}{3}\right|$

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Graphs and Linear Functions

A 2-dimensional graph is a visual representation of a relationship between two variables given by an equation or an inequality. Graphs help us solve algebraic problems by analysing the geometric aspects of a problem. While equations are more suitable for precise calculations, graphs are more suitable for showing patterns and trends in a relationship. To fully utilize what graphs can offer, we must first understand the concepts and skills involved in graphing that are discussed in this chapter.

G1 System of Coordinates, Graphs of Linear Equations and the Midpoint Formula

In this section, we will review the rectangular coordinate system, graph various linear equations and inequalities, and introduce a formula for finding coordinates of the midpoint of a given segment.

The Cartesian Coordinate System

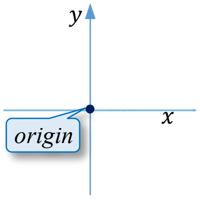


Figure 1a

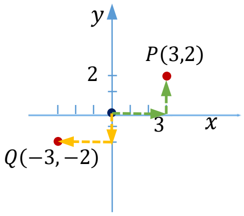


Figure 1b

A rectangular coordinate system, also called a **Cartesian coordinate system** (in honor of French mathematician, ***René Descartes)***, consists of two perpendicular number lines that cross each other at point zero, called the **origin**. Traditionally, one of these number lines, usually called the **x-axis**, is positioned horizontally and directed to the right (see *Figure 1a*). The other number line, usually called **y-axis**, is positioned vertically and directed up. Using this setting, we identify each point P of the plane with an **ordered pair** of numbers (x, y), which indicates the location of this point with respect to the origin. The first number in the ordered pair, the **x-coordinate**, represents the horizontal distance of the point P from the origin. The second number, the **y-coordinate**, represents the vertical distance of the point P from the origin. For example, to locate point $P(3,2)$, we start from the origin, go 3 steps to the right, and then two steps up. To locate point $Q(-3,-2)$, we start from the origin, go 3 steps to the left, and then two steps down (see *Figure 1b*).

Observe that the coordinates of the origin are $(0,0)$. Also, the second coordinate of any point on the x-axis as well as the first coordinate of any point on the y-axis is equal to zero. So, points on the x-axis have the form $(x, 0)$, while points on the y-axis have the form of $(0, y)$.

To **plot** (or **graph**) an ordered pair (x, y) means to place a dot at the location given by the ordered pair.

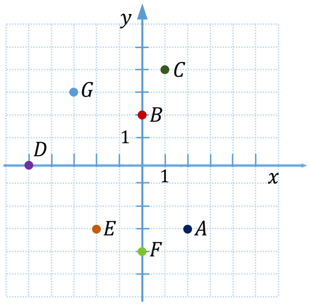
Example 1 ► **Plotting Points in a Cartesian Coordinate System**

Plot the following points:

$A(2,-3)$, $B(0,2)$, $C(1,4)$, $D(-5,0)$,
$E(-2,-3)$, $F(0,-4)$, $G(-3,3)$

Solution ►

Remember! *The order of numbers in an ordered pair is important! The **first** number represents the **horizontal** displacement and the **second** number represents the **vertical** displacement from the origin.*



Graphs of Linear Equations

A **graph of an equation** in two variables, x and y, is the set of points corresponding to **<u>all</u> ordered pairs** $(\boldsymbol{x}, \boldsymbol{y})$ that **satisfy** the equation (make the equation true). This means that a graph of an equation is the visual representation of the **solution set** of this equation.

To determine if a point (a, b) belongs to the graph of a given equation, we check if the equation is satisfied by $x = a$ and $y = b$.

Example 2 ► **Determining if a Point is a Solution of a Given Equation**

Determine if the points $(5,3)$ and $(-3,-2)$ are solutions of $2x - 3y = 0$.

Solution ► After substituting $x = 5$ and $y = 3$ into the equation $2x - 3y = 0$, we obtain

$$\begin{aligned} 2 \cdot 5 - 3 \cdot 3 &= 0 \\ 10 - 9 &= 0 \\ 1 &= 0, \end{aligned}$$

which is not true. Since the coordinates of the point $(5,3)$ do not satisfy the given equation, the point $(\mathbf{5}, \mathbf{3})$ **is not a solution** of this equation.

Note: The fact that the point $(5,3)$ **does not satisfy** the given equation indicates that it **does not belong to the graph** of this equation.

However, after substituting $x = -3$ and $y = -2$ into the equation $2x - 3y = 0$, we obtain

$$\begin{aligned} 2 \cdot (-3) - 3 \cdot (-2) &= 0 \\ -6 + 6 &= 0 \\ 0 &= 0, \end{aligned}$$

which is true. Since the coordinates of the point $(-3,-2)$ satisfy the given equation, the point $(-\mathbf{3}, -\mathbf{2})$ **is a solution** of this equation.

Note: The fact that the point $(-3,-2)$ **satisfies** the given equation indicates that it **belongs to the graph** of this equation.

To find a solution to a given equation in two variables, we choose a particular value for one of the variables, substitute it into the equation, and then solve the resulting equation for the other variable.

For example, to find a solution to $3x + 2y = 6$, we can choose for example $x = 0$, which leads us to

$$\begin{aligned} 3 \cdot 0 + 2y &= 6 \\ 2y &= 6 \\ y &= 3. \end{aligned}$$

This means that the point $(0, 3)$ satisfies the equation and therefore belongs to the graph of this equation. If we choose a different x-value, for example $x = 1$, the corresponding y-value becomes

$$\begin{aligned} 3 \cdot 1 + 2y &= 6 \\ 2y &= 3 \\ y &= \tfrac{3}{2}. \end{aligned}$$

So, the point $\left(1, \frac{3}{2}\right)$ also belongs to the graph.

Since any real number could be selected for the x-value, there are infinitely many solutions to this equation. Obviously, we will not be finding all of these infinitely many ordered pairs of numbers in order to graph the solution set to an equation. Rather, based on the location of several solutions that are easy to find, we will look for a pattern and predict the location of the rest of the solutions to complete the graph.

To find more points that belong to the graph of the equation in our example, we might want to solve $3x + 2y = 6$ for y. The equation is equivalent to

$$2y = -3x + 6$$

$$y = -\tfrac{3}{2}x + 3$$

Observe that if we choose x-values to be multiples of 2, the calculations of y-values will be easier in this case. Here is a table of a few more (x, y) points that belong to the graph:

x	$y = -\frac{3}{2}x + 3$	(x, y)
-2	$-\frac{3}{2}(-2) + 3 = 6$	$(-2, 6)$
2	$-\frac{3}{2}(2) + 3 = 0$	$(2, 0)$
4	$-\frac{3}{2}(4) + 3 = -3$	$(4, -3)$

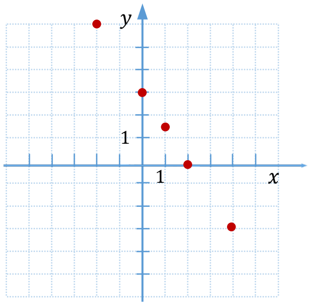


Figure 2a

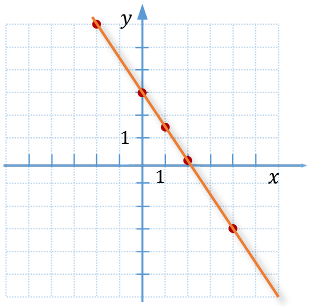


Figure 2b

After plotting the obtained solutions, $(-2, 6)$, $(0, 3)$, $\left(1, \frac{3}{2}\right)$, $(2, 0)$, $(4, -3)$, we observe that the points appear to lie on the same line (see *Figure 2a*). If all the ordered pairs that satisfy the equation $3x + 2y = 6$ were graphed, they would form the line shown in *Figure 2b*. Therefore, if we knew that the graph would turn out to be a line, it would be enough to find just two points (solutions) and draw a line passing through them.

How do we know whether or not the graph of a given equation is a line? It turns out that:

For any equation in two variables, the graph of the equation is a **line** if and only if (iff) the equation is **linear**.

So, the question is how to recognize a linear equation?

Definition 1.1 ► Any equation that can be written in the form

$$\boldsymbol{Ax + By = C}, \text{ where } A, B, C \in \mathbb{R}, \text{ and } A \text{ and } B \text{ are not both } 0,$$

is called a **linear equation** in two variables.

The form $\boldsymbol{Ax + By = C}$ is called **standard form** of a linear equation.

Example 3 ► **Graphing Linear Equations Using a Table of Values**

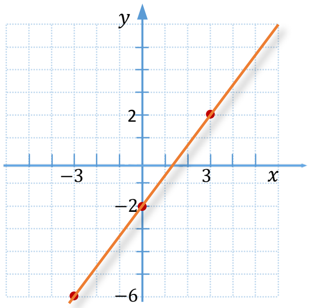
Graph $4x - 3y = 6$ using a table of values.

Solution ► Since this is a linear equation, we expect the graph to be a line. While finding two points satisfying the equation is sufficient to graph a line, it is a good idea to use a third point to guard against errors. To find several solutions, first, let us solve $4x - 3y = 6$ for y:

$$\begin{aligned} -3y &= -4x + 6 \\ y &= \tfrac{4}{3}x - 2 \end{aligned}$$

We like to choose x-values that will make the calculations of the corresponding y-values relatively easy. For example, if x is a multiple of 3, such as -3, 0 or 3, the denominator of $\frac{4}{3}$ will be reduced. Here is the table of points satisfying the given equation and the graph of the line.

$\boldsymbol{x}$	$\boldsymbol{y = \frac{4}{3}x - 2}$	$\boldsymbol{(x, y)}$
$\boldsymbol{-3}$	$\frac{4}{3}(-3) - 2 = -6$	$(-3, -6)$
$\boldsymbol{0}$	$\frac{4}{3}(0) - 2 = -2$	$(0, -2)$
$\boldsymbol{3}$	$\frac{4}{3}(3) - 2 = 2$	$(3, 2)$



To graph a linear equation in standard form, we can develop a table of values as in *Example 2*, or we can use the x- and y-intercepts.

Definition 1.2 ► The **$\boldsymbol{x}$-intercept** is the point (if any) where the graph intersects the x-axis. So, the x-intercept has the form $(\boldsymbol{x}, \boldsymbol{0})$.

The **$\boldsymbol{y}$-intercept** is the point (if any) where the graph intersects the y-axis. So, the y-intercept has the form $(\boldsymbol{0}, \boldsymbol{y})$.

Example 4 ► **Graphing Linear Equations Using $\boldsymbol{x}$- and $\boldsymbol{y}$-intercepts**

Graph $5x - 3y = 15$ by finding and plotting the x- and y-intercepts.

Solution ► To find the x-intercept, we substitute $y = 0$ into $5x - 3y = 15$, and then solve the resulting equation for y. So, we have

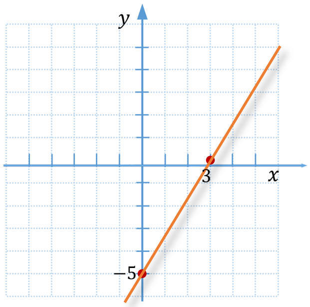
$$5x = 15$$
$$x = 3.$$

To find y-intercept, we substitute $x = 0$ into $5x - 3y = 15$, and then solve the resulting equation for x. So,

$$-3y = 15$$
$$y = -5.$$

Hence, we have

	x	y
x-intercept	**3**	**0**
y-intercept	**0**	**−5**



To find several points that belong to the graph of a linear equation in two variables, it was easier to solve the standard form $Ax + By = C$ for y, as follows

$$By = -Ax + C$$

$$y = -\frac{A}{B}x + \frac{C}{B}.$$

This form of a linear equation is also very friendly for graphing, as the graph can be obtained without any calculations. See *Example 4*.

Any equation $Ax + By = C$, where $B \neq 0$ can be written in the form

$$\boldsymbol{y = mx + b},$$

which is referred to as the **slope-intercept form** of a linear equation.
The value $\boldsymbol{m} = -\frac{\boldsymbol{A}}{\boldsymbol{B}}$ represents the **slope** of the line. Recall that ***slope*** $= \frac{\boldsymbol{rise}}{\boldsymbol{run}}$.
The value $\boldsymbol{b}$ represents the y-intercept, so the point $(\mathbf{0}, \boldsymbol{b})$ belongs to the graph of this line.

Example 5 ► Graphing Linear Equations Using Slope and y-intercept

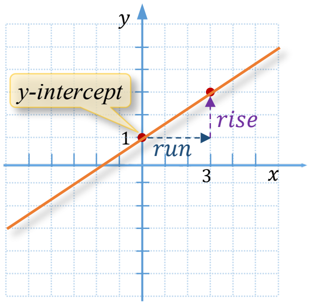
Determine the slope and y-intercept of each line and then graph it.

a. $y = \frac{2}{3}x + 1$ **b.** $5x + 2y = 8$

Solution ► **a.** The slope is the coefficient by x, so it is $\frac{2}{3}$.

The y-intercept is $(0,1)$.

So we plot point $(0,1)$ and then, since $\frac{2}{3} = \frac{rise}{run}$, we rise 2 units and run 3 units to find the next point that belongs to the graph.

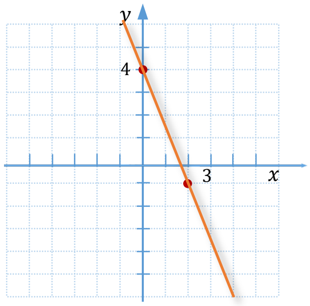


b. To see the slope and y-intercept, we solve $5x + 2y = 8$ for y.

$$2y = -5x + 8$$

$$y = \frac{-5}{2}x + 4$$

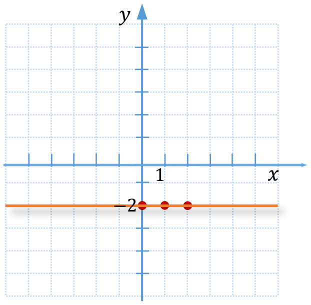
So, the slope is $\frac{-5}{2}$ and the y-intercept is $(0,4)$. We start from $(0,4)$ and then run 2 units and fall 5 units (because of -5 in the numerator).



Note: Although we can *run* to the right or to the left, depending on the sign in the denominator, we usually **keep the denominator positive and always *run* forward** (to the right). If the slope is negative, we **keep the negative sign in the numerator** and either ***rise*** or ***fall***, depending on this sign. However, when finding additional points of the line, sometimes we can repeat the *run/rise* movement in either way, to the right, or to the left from one of the already known points. For example, in *Example 4a*, we could find the additional point at $(-3,-1)$ by *running* 3 units to the left and 2 units down from $(0,1)$, as the slope $\frac{2}{3}$ can also be seen as $\frac{-2}{-3}$, if needed.

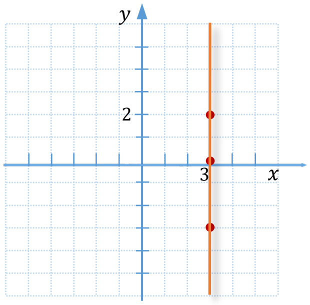
Some linear equations contain just one variable. For example, $x = 3$ or $y = -2$. How would we graph such equations in the xy-plane?

Observe that $\boldsymbol{y = -2}$ can be seen as $y = 0x - 2$, so we can graph it as before, using the **slope** of **zero** and the y-intercept of $(0,-2)$. The graph consists of all points that have y-coordinates equal to -2. Those are the points of type $(x,-2)$, where x is any real number. The graph is a **horizontal line** passing through the point $(0,-2)$.



Note: The horizontal line $\boldsymbol{y = 0}$ is the x-axis.

The equation $\boldsymbol{x = 3}$ doesn't have a slope-intercept representation, but it is satisfied by any point with x-coordinate equal to 3. So, by plotting several points of the type $(3,y)$, where y is any real number, we obtain a **vertical line** passing through the point $(3,0)$. This particular line doesn't have a y-intercept, and its ***slope*** $= \frac{\textit{rise}}{\textit{run}}$ is considered to be **undefined**. This is because the "*run*" part calculated between any two points on the line is equal to zero and we can't perform division by zero.



Note: The vertical line $\boldsymbol{x = 0}$ is the y-axis.

In general, the graph of any equation of the type

$$\boldsymbol{y = b}, \text{ where } \boldsymbol{b} \in \mathbb{R}$$

is a **horizontal line** with y-intercept $(0, \boldsymbol{b})$. The **slope** of such a line is **zero**.

The graph of any equation of the type

$$\boldsymbol{x = a}, \text{ where } \boldsymbol{a} \in \mathbb{R}$$

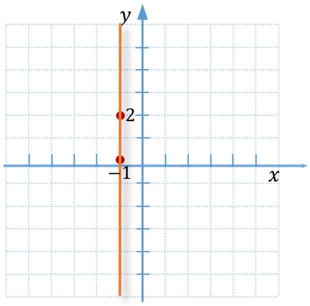
is a **vertical line** with x-intercept $(\boldsymbol{a}, \boldsymbol{0})$. The **slope** of such a line is **undefined**.

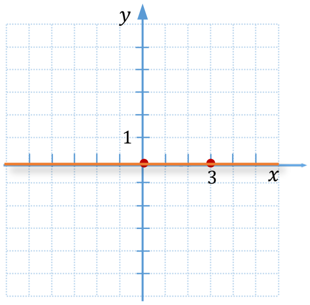
Example 6 ► **Graphing Special Types of Linear Equations**

Graph each equation and state its slope.

a. $x = -1$ **b.** $y = 0$ **c.** $y = x$

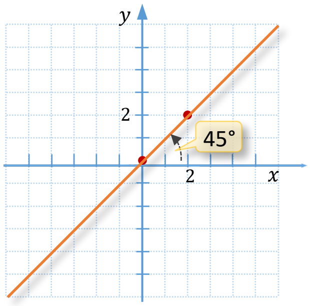
Solution ► **a.** The solutions to the equation $x = -1$ are all pairs of the type $(-1, y)$, so after plotting points like $(-1,0)$, $(-1,2)$, etc., we observe that the graph is a **vertical line** intercepting the x-axis at $x = -1$. So the **slope** of this line is **undefined**.





b. The solutions to the equation $y = 0$ are all pairs of the type $(x, 0)$, so after plotting points like $(0,0)$, $(0,3)$, etc., we observe that the graph is a **horizontal line** following the x-axis. The **slope** of this line is **zero**.

c. The solutions to the equation $y = x$ are all pairs of the type (x, x), so after plotting points like $(0,0)$, $(2,2)$, etc., we observe that the graph is a **diagonal line**, passing through the origin and making a 45° angle with the x-axis. The **slope** of this line is **1**.

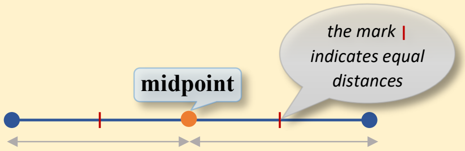


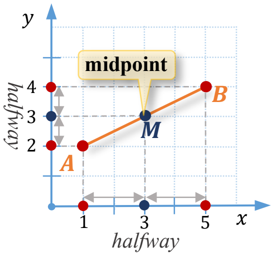
Observation: *A graph of any equation of the type* $\boldsymbol{y = mx}$ *is a line passing through the origin, as the point* $(0,0)$ *is one of the solutions.*

Midpoint Formula

To find a representative value of a list of numbers, we often calculate the average of these numbers. Particularly, to find an average of, for example, two test scores, 72 and 84, we take half of the sum of these scores. So, the average of 72 and 84 is equal to $\frac{72+84}{2} = \frac{156}{2} = 78$. Observe that 78 lies on a number line exactly halfway between 72 and 84. The idea of taking an average is employed in calculating coordinates of the midpoint of any line segment.

Definition 1.3 ► The **midpoint** of a line segment is the point of the segment that is equidistant from both ends of this segment.





Suppose $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $\boldsymbol{M}$ is the **midpoint** of the line segment $\overline{AB}$. Then the x-coordinate of M lies halfway between the two end x-values, x_1 and x_2, and the y-coordinate of M lies halfway between the two end y-values, y_1 and y_2. So, the coordinates of the midpoint are **averages** of corresponding x-, and y-coordinates:

$$\boldsymbol{M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)} \tag{1}$$

Example 7 ► Finding Coordinates of the Midpoint

Find the midpoint M of the line segment connecting $P = (-3,7)$ and $Q = (5, -12)$.

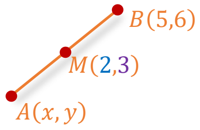
Solution ► The coordinates of the midpoint M are averages of the x- and y-coordinates of the endpoints. So,

$$M = \left(\frac{-3+5}{2}, \frac{7+(-12)}{2}\right) = \boldsymbol{\left(1, -\frac{5}{2}\right)}.$$

Example 8 ► Finding Coordinates of an Endpoint Given the Midpoint and the Other Endpoint

Suppose segment AB has its midpoint M at $(2,3)$. Find the coordinates of point A, knowing that $B = (5,6)$.

Solution ► Let $A = (x, y)$ and $B = (5,6)$. Since $M = (2,3)$ is the midpoint of $\overline{AB}$, by formula (1), the following equations must hold:



$$\frac{x+5}{2} = 2 \quad \text{and} \quad \frac{y+6}{2} = 3$$

Multiplying these equations by 2, we obtain

$$x + 5 = 4 \quad \text{and} \quad y + 6 = 6,$$

which results in

$$x = -1 \quad \text{and} \quad y = 0.$$

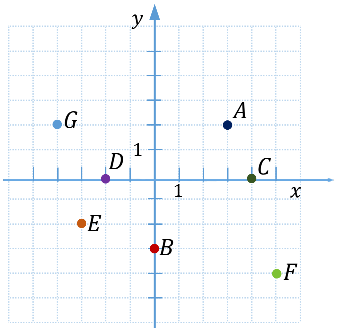
Hence, the coordinates of point A are $(\boldsymbol{-1, 0})$.

G.1 Exercises

1. Plot each point in a rectangular coordinate system.

a. $(1,2)$ **b.** $(-2,0)$ **c.** $(0,-3)$ **d.** $(4,-1)$ **e.** $(-1,-3)$

2. State the coordinates of each plotted point.



Determine if the given ordered pair is a solution of the given equation.

3. $(-2,2)$; $y = \frac{1}{2}x + 3$

4. $(4,-5)$; $3x + 2y = 2$

5. $(5,4)$; $4x - 5y = 1$

Graph each equation using the suggested table of values.

6. $y = 2x - 3$

x	y
0	
1	
2	
3	

7. $y = -\frac{1}{3}x + 2$

x	y
−3	
0	
3	
6	

8. $x + y = 3$

x	y
0	
	0
−1	
	1

9. $4x - 5y = 20$

x	y
0	
	0
2	
	−3

Graph each equation using a table of values.

10. $y = \frac{1}{3}x$

11. $y = \frac{1}{2}x + 2$

12. $6x - 3y = -9$

13. $6x + 2y = 8$

14. $y = \frac{2}{3}x - 1$

15. $y = -\frac{3}{2}x$

16. $3x + y = -1$

17. $2x = -5y$

18. $-3x = -3$

19. $6y - 18 = 0$

20. $y = -x$

21. $2y - 3x = 12$

Determine the ***x-*** *and* ***y-intercepts*** *of each line and then graph it. Find additional points, if needed.*

22. $5x + 2y = 10$

23. $x - 3y = 6$

24. $8y + 2x = -4$

25. $3y - 5x = 15$

26. $y = -\frac{2}{5}x - 2$

27. $y = \frac{1}{2}x - \frac{3}{2}$

28. $2x - 3y = -9$

29. $2x = -y$

*Determine the **slope** and **y-intercept** of each line and then graph it.*

30. $y = 2x - 3$ **31.** $y = -3x + 2$ **32.** $y = -\frac{4}{3}x + 1$ **33.** $y = \frac{2}{5}x + 3$

34. $2x + y = 6$ **35.** $3x + 2y = 4$ **36.** $-\frac{2}{3}x - y = 2$ **37.** $2x - 3y = 12$

38. $2x = 3y$ **39.** $y = \frac{3}{2}$ **40.** $y = x$ **41.** $x = 3$

Find the midpoint of each segment with the given endpoints.

42. $(-8,\ 4)$ and $(-2,-6)$ **43.** $(4,-3)$ and $(-1,3)$ **44.** $(-5,-3)$ and $(7,5)$

45. $(-7,\ 5)$ and $(-2,11)$ **46.** $\left(\frac{1}{2},\ \frac{1}{3}\right)$ and $\left(\frac{3}{2},-\frac{5}{3}\right)$ **47.** $\left(\frac{3}{5},-\frac{1}{3}\right)$ and $\left(\frac{1}{2},-\frac{5}{2}\right)$

Segment AB has the given coordinates for the endpoint A and for its midpoint M. Find the coordinates of the endpoint B.

48. $A(-3,2)$, $M(3,-2)$ **49.** $A(7,\ 10)$, $M(5,3)$

50. $A(5,-4)$, $M(0,6)$ **51.** $A(-5,-2)$, $M(-1,4)$

G2 Slope of a Line and Its Interpretation

Slope (steepness) is a very important concept that appears in many branches of mathematics as well as statistics, physics, business, and other areas. In algebra, slope is used when graphing lines or analysing linear equations or functions. In calculus, the concept of slope is used to describe the behaviour of many functions. In statistics, slope of a regression line explains the general trend in the analysed set of data. In business, slope plays an important role in linear programming. In addition, slope is often used in many practical ways, such as the slope of a road (*grade*), slope of a roof (*pitch*), slope of a ramp, etc.

In this section, we will define, calculate, and provide some interpretations of slope.

Slope

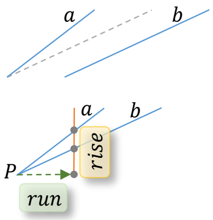


Figure 1a

Given two lines, a and b, how can we tell which one is steeper? One way to compare the steepness of these lines is to move them closer to each other, so that a point of intersection, P, can be seen, as in *Figure 1a*. Then, after running horizontally a few steps from the point P, draw a vertical line to observe how high the two lines have risen. The line that crosses this vertical line at a higher point is steeper. So, for example in *Figure 1a*, line a is steeper than line b. Observe that because we run the same horizontal distance for both lines, we could compare the steepness of the two lines just by looking at the vertical rise. However, since the *run* distance can be chosen arbitrarily, to represent the steepness of any line, we must look at the *rise* (vertical change) in respect to the *run* (horizontal change). This is where the concept of slope as a ratio of *rise* to *run* arises.

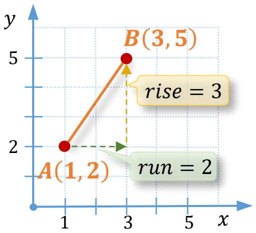


Figure 1b

To measure the slope of a line or a line segment, we choose any two distinct points of such a figure and calculate the ratio of the **vertical change** (***rise***) to the **horizontal change** (***run***) between the two points. For example, the slope between points $A(1,2)$ and $B(3,5)$ equals

$$\frac{rise}{run} = \frac{3}{2},$$

as in *Figure 1a*. If we rewrite this ratio so that the denominator is kept as one,

$$\frac{3}{2} = \frac{1.5}{1} = 1.5,$$

we can think of slope as of the **rate of change in y-values with respect to x-values**. So, a slope of 1.5 tells us that the y-value increases by 1.5 units per every increase of one unit in x-value.

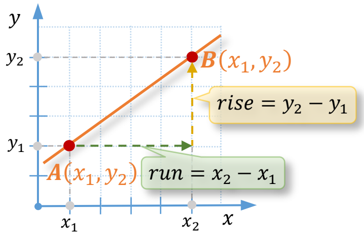


Figure 1c

Generally, the slope of a line passing through two distinct points, $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$, is the **ratio** of the change in y-values, $y_2 - y_1$, to the change in x-values, $x_2 - x_1$, as presented in *Figure 1c*. Therefore, the formula for calculating slope can be presented as

$$\frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x},$$

where the Greek letter Δ (delta) is used to denote the change in a variable.

Definition 2.1 ► Suppose a line passes through two distinct points (x_1, y_1) and (x_2, y_2).

If $x_1 \neq x_2$, then the **slope** of this line, often denoted by $\boldsymbol{m}$, is equal to

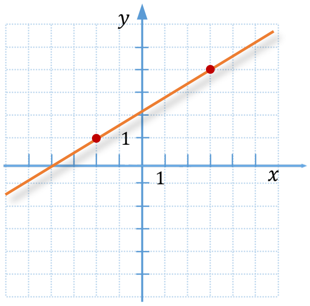
$$\boxed{m = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.}$$

If $x_1 = x_2$, then the **slope** of the line is said to be **undefined**.

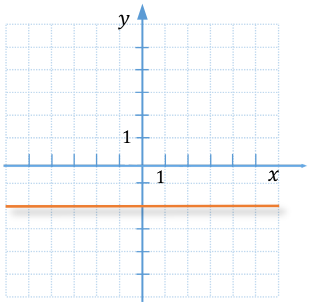
Example 1 ► Determining Slope of a Line, Given Its Graph

Determine the slope of each line.

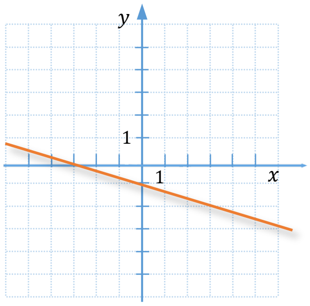
a.



b.

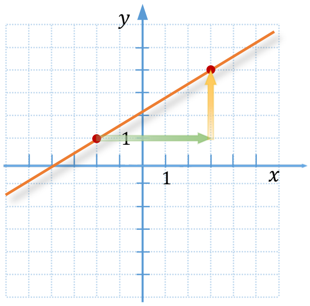


c.



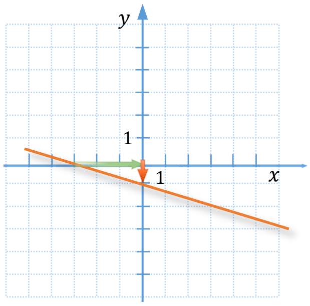
Solution ►

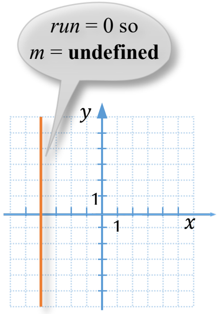
a. To read the slope we choose two distinct points with integral coefficients (often called ***lattice points***), such as the points suggested in the graph. Then, starting from the first point $(-2,1)$ we *run* 5 units and *rise* 3 units to reach the second point $(3,4)$. So, the slope of this line is $m = \frac{3}{5}$.



b. This is a horizontal line, so the *rise* between any two points of this line is zero. Therefore the slope of such a line is also **zero**.

c. If we refer to the lattice points $(-3,0)$ and $(0,-1)$, then the *run* is 3 and the *rise* (or rather ***fall***) is -1. Therefore the slope of this line is $m = -\frac{1}{3}$.





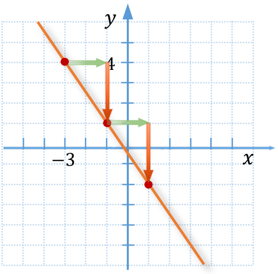
Observation:

A line that **increases** from left to right has a **positive slope**.
A line that **decreases** from left to right has a **negative slope**.
The slope of a **horizontal** line is **zero**.
The slope of a **vertical** line is **undefined**.

Example 2 ► Graphing Lines, Given Slope and a Point

Graph the line with slope $-\frac{3}{2}$ that passes through the point$(-3,4)$.

Solution ► First, plot the point $(-3,4)$. To find another point that belongs to this line, start at the plotted point and run 2 units, then fall 3 units. This leads us to point $(-1,1)$. For better precision, repeat the movement (two across and 3 down) to plot one more point, $(1,-2)$. Finally, draw a line connecting the plotted points.



Example 3 ► Calculating Slope of a Line, Given Two Points

Determine the slope of a line passing through the points $(-3,5)$ and $(7,-11)$.

Solution ► The slope of the line passing through $(-3,5)$ and $(7,-11)$ is the quotient

$$\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{5-(-11)}{-3-7}=\frac{5+11}{-10}=-\frac{16}{10}=\mathbf{-1.6}$$

Example 4 ► Determining Slope of a Line, Given Its Equation

Determine the slope of a line given by the equation $2x-5y=7$.

Solution ► To see the slope of a line in its equation, we change the equation to its slope-intercept form, $y=mx+b$. The slope is the coefficient m. When solving $2x-5y=7$ for y, we obtain

$$-5y=-2x+7$$

$$y=\frac{\mathbf{2}}{\mathbf{5}}x-\frac{7}{5}.$$

So, the slope of this line is equal to $\frac{\mathbf{2}}{\mathbf{5}}$.

Example 5 ► Interpreting Slope as an Average Rate of Change

The value of a particular stock has increased from \$156.60 on February 10, 2018, to \$187.48 on November 10, 2018. What is the average rate of change of the value of this stock per month for the given period of time?

Solution ► The average value of the stock has increased by $187.48 - 156.60 = 30.88$ dollars over the 9 months (from February 10 to November 10). So, the slope of the line segment connecting the the values of the stock on these two days (as marked on the above chart) equals

$$\frac{30.88}{9} \cong \mathbf{3.43}\ \$/month$$

This means that the value of the stock was increasing on average by 3.43 dollars per month between February 10, 2018, and November 10, 2018.

Observe that the change in value was actually different in each month. Sometimes the change was larger than the calculated slope, but sometimes the change was smaller or even negative. However, the **slope** of the above segment gave us the information about the **average rate of change** in the stock's value during the stated period.

Parallel and Perpendicular Lines

Since slope measures the steepness of lines, and **parallel lines** have the same steepness, then the **slopes** of **parallel lines** are **equal**.

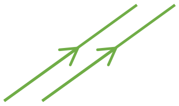


Figure 2

To indicate on a diagram that lines are parallel, we draw on each line arrows pointing in the same direction (see *Figure 2*). To state in mathematical notation that two lines are parallel, we use the $\parallel$ sign.

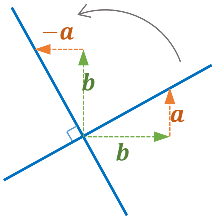


Figure 3

To see how the slopes of perpendicular lines are related, rotate a line with a given slope $\frac{a}{b}$ (where $b \neq 0$) by 90°, as in *Figure 3*. Observe that under this rotation the vertical change $\boldsymbol{a}$ becomes the horizontal change but in opposite direction ($-\boldsymbol{a}$), and the horizontal change $\boldsymbol{b}$ becomes the vertical change. So, the **slope** of the **perpendicular line** is $-\frac{b}{a}$. In other words, **slopes** of **perpendicular lines** are **opposite reciprocals**. Notice that the **product of perpendicular slopes**, $\frac{a}{b} \cdot \left(-\frac{b}{a}\right)$, is equal to $\mathbf{-1}$.

In the case of $b = 0$, the slope is undefined, so the line is vertical. After rotation by 90°, we obtain a horizontal line, with a slope of zero. So a line with a zero slope and a line with an "undefined" slope can also be considered perpendicular.

To indicate on a diagram that two lines are perpendicular, we draw a square at the intersection of the two lines, as in *Figure 3*. To state in mathematical notation that two lines are perpendicular, we use the $\perp$ sign.

In summary, if $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ are **slopes** of two lines, then the lines are:

- **parallel** iff $\boldsymbol{m_1 = m_2}$, and
- **perpendicular** iff $\boldsymbol{m_1 = -\frac{1}{m_2}}$ (or equivalently, if $\boldsymbol{m_1 \cdot m_2 = -1}$)
  In addition, a **horizontal** line (with a slope of **zero**) is **perpendicular** to a **vertical** line (with **undefined** slope).

Example 6 ► **Determining Whether the Given Lines are Parallel, Perpendicular, or Neither**

For each pair of linear equations, determine whether the lines are parallel, perpendicular, or neither.

a. $3x + 5y = 7$
$5x - 3y = 4$

b. $y = x$
$2x - 2y = 5$

c. $y = 5$
$y = 5x$

Solution ►

a. As seen in *Section G1*, the slope of a line given by an equation in standard form, $Ax + By = C$, is equal to $-\frac{A}{B}$. One could confirm this by solving the equation for y and taking the coefficient by x for the slope.
Using this fact, the slope of the line $3x + 5y = 7$ is $-\frac{\mathbf{3}}{\mathbf{5}}$, and the slope of $5x - 3y = 4$ is $\frac{\mathbf{5}}{\mathbf{3}}$. Since these two slopes are opposite reciprocals of each other, the two lines are **perpendicular**.

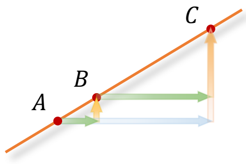
b. The slope of the line $y = x$ is $\mathbf{1}$ and the slope of $2x - 2y = 5$ is also $\frac{2}{2} = \mathbf{1}$. So, the two lines are parallel.

c. The line $y = 5$ can be seen as $y = 0x + 5$, so its slope is $\mathbf{0}$. The slope of the second line, $y = 5x$, is $\mathbf{5}$. So, the two lines are neither parallel nor perpendicular.

Collinear Points

Definition 2.2 ► Points that lie on the same line are called **collinear**.

Two points are always collinear because there is only one line passing through these points. The question is how could we check if a third point is collinear with the given two points? If we have an equation of the line passing through the first two points, we could plug in the coordinates of the third point and see if the equation is satisfied. If it is, the third point is collinear with the other two. But, can we check if points are collinear without referring to an equation of a line?



Notice that if several points lie on the same line, the slope between any pair of these points will be equal to the slope of this line. So, these slopes will be the same. One can also show that if the slopes between any two points in the group are the same, then such points lie on the same line. So, they are collinear.

Points are **collinear** iff the **slope** between each pair of points **is the same**.

Example 7 ► **Determine Whether the Given Points are Collinear**

Determine whether the points $A(-3,7)$, $B(-1,2)$, and $C = (3,-8)$ are collinear.

Solution ► Let m_{AB} represent the slope of $\overline{AB}$ and m_{BC} represent the slope of $\overline{BC}$. Since

$$m_{AB} = \frac{2-7}{-1-(-3)} = -\frac{5}{2} \text{ and } m_{BC} = \frac{-8-2}{3-(-1)} = -\frac{10}{4} = -\frac{5}{2},$$

Then all points A, B, and C lie on the same line. Thus, they are collinear.

Example 8 ► Finding the Missing Coordinate of a Collinear Point

For what value of y are the points $P(2,2)$, $Q(-1,y)$, and $R(1,6)$ collinear?

Solution ► For the points P, Q, and R to be collinear, we need the slopes between any two pairs of these points to be equal. For example, the slope m_{PQ} should be equal to the slope m_{PR}. So, we solve the equation

$$m_{PQ} = m_{PR}$$

for y:

$$\frac{y-2}{-1-2} = \frac{6-2}{1-2}$$

$$\frac{y-2}{-3} = -4$$

$$y - 2 = 12$$

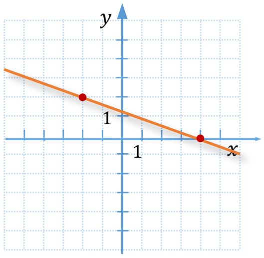
$$y = 14$$

Thus, point Q is collinear with points P and R, if $y = \mathbf{14}$.

G.2 Exercises

Given the graph, find the slope of each line.

1.



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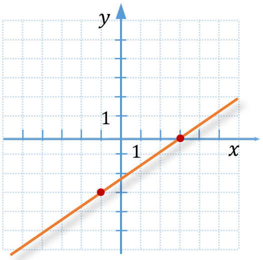

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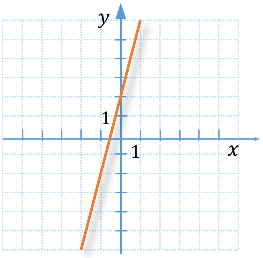
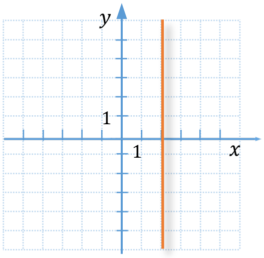


4.



Given the equation, find the slope of each line.

5. $y = \frac{1}{2}x - 7$

6. $y = -\frac{1}{3}x + 5$

7. $4x - 5y = 2$

8. $3x + 4y = 2$

9. $x = 7$

10. $y = -\frac{3}{4}$

11. $y + x = 1$

12. $-8x - 7y = 24$

13. $-9y - 36 + 4x = 0$

Graph each line satisfying the given information.

14. passing through $(-2, -4)$ with slope $m = 4$

15. passing through $(-1, -2)$ with slope $m = -3$

16. passing through $(-3, 2)$ with slope $m = \frac{1}{2}$

17. passing through $(-3,4)$ with slope $m = -\frac{2}{5}$

18. passing through $(2, -1)$ with undefined slope

19. passing through $(2, -1)$ with slope $m = 0$

20. Which of the following formulas represent slope?

a. $m = \frac{y_1 - y_2}{x_2 - x_1}$

b. $m = \frac{y_1 - y_2}{x_1 - x_2}$

c. $m = \frac{x_2 - x_1}{y_2 - y_1}$

d. $m = \frac{y_2 - y_1}{x_2 - x_1}$

Find the slope of the line through each pair of points.

21. $(-2,2), \ (4,5)$

22. $(8,7), \ (2,-1)$

23. $(9,-4), \ (3,-8)$

24. $(-5,2), \ (-9,5)$

25. $(-2,3), \ (7,-12)$

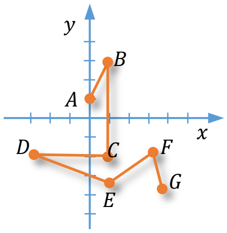
26. $(3,-1), \ \left(-\frac{1}{2}, \frac{1}{5}\right)$

27. $(-5,2), \ (8,2)$

28. $(-3,4), \ (-3,10)$

29. $\left(\frac{1}{2}, 6\right), \ \left(-\frac{2}{3}, \frac{5}{2}\right)$

30. List the line segments in the accompanying figure with respect to their slopes, from the smallest to the largest slope. List the segment with an undefined slope as last.



31. *Refer to the graphs of a stock price observed during a 3-month period. Match each situation in **a–d** with the most appropriate graph in **A–D**.*

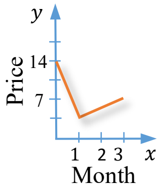
a. The price rose sharply during the first month, remained stable during the second month before falling to the original price by the end of the third month.

b. The price fell sharply during the first month and then rose slowly for the next two months.

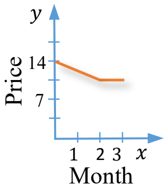
c. The price rose sharply during the first month, fell slowly during the second month before rising slowly to its highest peak during the third month.

d. The price fell slowly during the first two months before stabilizing in the third month.

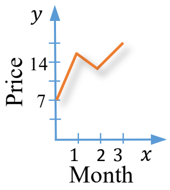
A.



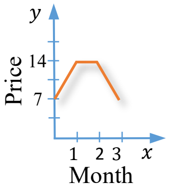
B.



C.

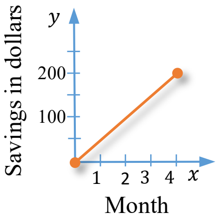


D.

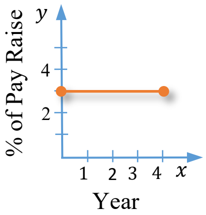


Find and interpret the average rate of change illustrated in each graph.

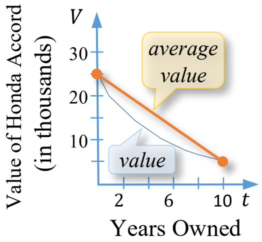
32.



33.



34.



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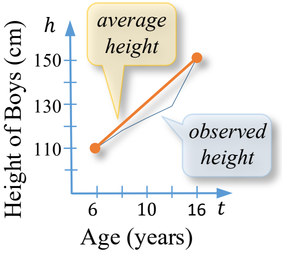


In problems #43-46, sketch a graph depicting each situation. Assume that the roads in each problem are straight.

36. The distance that a driver is from home if he starts driving home from a town that is 30 kilometers away and he drives at a constant speed for half an hour.

37. The distance that a cyclist is from home if he bikes away from home at 30 kilometers per hour for 30 minutes and then bikes back home at 15 kilometers per hour.

38. The distance that Alice is from home if she walks 4 kilometers from home to a shoping centre, stays there for 1.5 hours, and then walks back home. Assume that Alice walks at a constant speed for 30 minutes each way.

39. The amount of water in a 500 liters outdoor pool for kids that is filled at the rate of 1500 liters per hour, left full for 4 hours, and then drained at the rate of 3000 liters per hour.

Solve each problem.

40. At 6:00 a.m. a 60,000-liter swimming pool was $\frac{1}{3}$ full and at 9:00 a.m. the pool was filled up to $\frac{3}{4}$ of its capacity. Find the rate of filling the pool with the assumption that the rate was constant.

41. Jan and Bill plan to drive to Kelowna that is 324 kilometers away. Jan noticed that during one hour they change their location from being $\frac{1}{3}$ of the way to being $\frac{2}{3}$ of the way. Assuming that they drive at a constant rate, what is their average speed of driving?

42. Suppose we see a road sign informing that a road grade is 7% for the next 1.5 kilometers. In meters, what would the expected change in elevation be 1.5 kilometers down the road?

Decide whether each pair of lines is parallel, perpendicular, or neither.

43. $y = x$
$y = -x$

44. $y = 3x - 6$
$y = -\frac{1}{3}x + 5$

45. $2x + y = 7$
$-6x - 3y = 1$

46. $x = 3$
$x = -2$

47. $3x + 4y = 3$
$3x - 4y = 5$

48. $5x - 2y = 3$
$2x - 5y = 1$

49. $y - 4x = 1$
$x + 4y = 3$

50. $y = \frac{2}{3}x - 2$
$-2x + 3y = 6$

Solve each problem.

51. Check whether or not the points $(-2,7)$, $(1,5)$, and$(3,4)$ are collinear.

52. The following points,$(2,2)$, $(-1,k)$, and$(1,6)$ are collinear. Find the value of k.

G3 Forms of Linear Equations in Two Variables

Linear equations in two variables can take different forms. Some forms are easier to use for graphing, while others are more suitable for finding an equation of a line given two pieces of information. In this section, we will take a closer look at various forms of linear equations and their utilities.

Forms of Linear Equations

The form of a linear equation that is most useful for graphing lines is the slope-intercept form, as introduced in *Section G1*.

Definition 3.1 ► The **slope-intercept form** of the equation of a line with **slope m** and **y-intercept** $(0, b)$ is

$$y = mx + b.$$

Example 1 ► **Writing and Graphing Equation of a Line in Slope-Intercept Form**

Write the equation in slope-intercept form of the line satisfying the given conditions, and then graph this line.

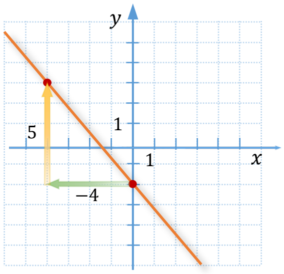
a. slope $-\frac{5}{4}$ and y-intercept $(0, -2)$

b. slope $\frac{1}{2}$ and passing through $(-2, -5)$

Solution ►

a. To write this equation, we substitute $m = -\frac{5}{4}$ and $b = -2$ into the slope-intercept form. So, we obtain

$$y = -\frac{5}{4}x - 2.$$



To graph this line, we start with plotting the y-intercept $(0, -2)$. To find the second point, we follow the slope, as in *Example 2, Section G2*. According to the slope $-\frac{5}{4} = \frac{-5}{4}$, starting from $(0, -2)$, we could run 4 units to the right and 5 units down, but then we would go out of the grid. So, this time, let the negative sign in the slope be kept in the denominator, $\frac{5}{-4}$. Thus, we run 4 units to the left and 5 units up to reach the point $(-4, 3)$. Then we draw the line by connecting the two points.

b. Since $m = \frac{1}{2}$, our equation has a form $y = \frac{1}{2}x + b$. To find b, we substitute point $(-2, -5)$ into this equation and solve for b. So

$$-5 = \frac{1}{2}(-2) + b$$

gives us

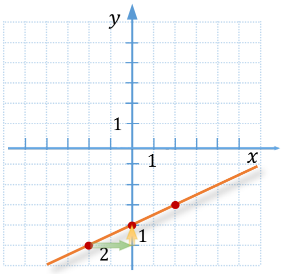
$$-5 = -1 + b$$

and finally

$$b = -4.$$

Therefore, our equation of the line is $\boldsymbol{y = \frac{1}{2}x - 4}$.

We graph it, starting by plotting the given point $(-2, -5)$ and finding the second point by following the slope of $\frac{1}{2}$, as described in *Example 2, Section G2*.



The form of a linear equation that is most useful when writing equations of lines with unknown y-intercept is the slope-point form.

Definition 3.2 ► The **slope-point form** of the equation of a line with slope $\boldsymbol{m}$ and passing through the point $(\boldsymbol{x_1}, \boldsymbol{y_1})$ is

$$\boldsymbol{y - y_1 = m(x - x_1)}.$$

This form is based on the defining property of a line. A line can be defined as a set of points with a constant slope m between any two of these points. So, if (x_1, y_1) is a given (fixed) point of the line and (x, y) is any (variable) point of the line, then, since the slope is equal to m for all such points, we can write the equation

$$m = \frac{y - y_1}{x - x_1}.$$

After multiplying by the denominator, we obtain the slope-point formula, as in *Definition 3.2*.

Example 2 ► **Writing Equation of a Line Using Slope-Point Form**

Use the slope-point form to write an equation of the line satisfying the given conditions. Leave the answer in the slope-intercept form and then graph the line.

a. slope $-\frac{2}{3}$ and passing through $(1, -3)$

b. passing through points $(2,5)$ and $(-1, -2)$

Solution ►

a. To write this equation, we plug the slope $m = -\frac{2}{3}$ and the coordinates of the point $(1, -3)$ into the slope-point form of a line. So, we obtain

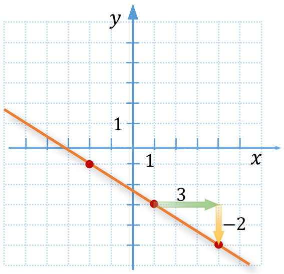
$$y - (-3) = -\frac{2}{3}(x - 1)$$

$$y + 3 = -\frac{2}{3}x + \frac{2}{3}$$

$$y = -\frac{2}{3}x + \frac{2}{3} - \frac{9}{3}$$

$$\boldsymbol{y = -\frac{2}{3}x - \frac{7}{3}}$$

To graph this line, we start with plotting the point $(1,-3)$ and then apply the slope of $-\frac{2}{3}$ to find additional points that belong to the line.



b. This time the slope is not given, so we will calculate it using the given points, $(2,5)$ and $(-1,-2)$. Thus,

$$m=\frac{\Delta y}{\Delta x}=\frac{-2-5}{-1-2}=\frac{-7}{-3}=\frac{7}{3}$$

Then, using the calculated slope and one of the given points, for example $(2,5)$, we write the slope-point equation of the line

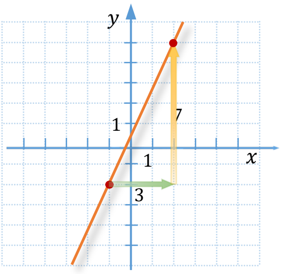
$$y-5=\frac{7}{3}(x-2)$$

and solve it for y:

$$y-5=\frac{7}{3}x-\frac{14}{3}$$

$$y=\frac{7}{3}x-\frac{14}{3}+\frac{15}{3}$$

$$\boldsymbol{y=\frac{7}{3}x+\frac{1}{3}}$$



To graph this line, it is enough to connect the two given points.

One of the most popular forms of a linear equation is the standard form. This form is helpful when graphing lines based on x- and y-intercepts, as illustrated in *Example 3, Section G1*.

Definition 3.3 ► The **standard form** of a linear equation is

$$\boldsymbol{Ax+By=C},$$

Where $A,B,C\in\mathbb{R}$, A and B are not both 0, and $A\geq 0$.

When writing linear equations in standard form, the expectation is to use a **nonnegative coefficient A** and **clear any fractions**, if possible. For example, to write $-x+\frac{1}{2}y=3$ in standard form, we multiply the equation by (-2), to obtain $2x-y=-6$. In addition, we prefer to write equations in simplest form, where the greatest common factor of A,B, and C is 1. For example, we prefer to write $2x-y=-6$ rather than any multiple of this equation, such as $4x-2y=-12$, or $6x-3y=-18$.

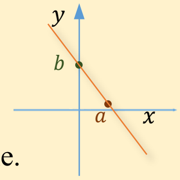
Observe that if $B\neq 0$ then the **slope** of the line given by the equation $\boldsymbol{Ax+By=C}$ is $-\frac{\boldsymbol{A}}{\boldsymbol{B}}$.
This is because after solving this equation for y, we obtain $\boldsymbol{y=-\frac{A}{B}x+\frac{C}{B}}$.
If $B=0$, then the slope is **undefined**, as we are unable to divide by zero.

The form of a linear equation that is most useful when writing equations of lines based on their x- and y-intercepts is the intercept form.

Definition 3.4 ► The **intercept form** of a linear equation is

$$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1,}$$

where $(a, 0)$ is the ***x*-intercept** and $(0, b)$ is the ***y*-intercept** of the line.



We should be able to convert a linear equation from one form to another.

Example 3 ► **Converting a Linear Equation to a Different Form**

a. Write the equation $3x + 7y = 2$ in slope-intercept form.
b. Write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form.
c. Write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form.

Solution ► **a.** To write the equation $3x + 7y = 2$ in slope-intercept form, we solve it for y.

$$3x + 7y = 2$$
$$7y = -3x + 2$$
$$\boldsymbol{y = -\frac{3}{7}x + \frac{2}{7}}$$

b. To write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form, we bring the x-term to the left side of the equation and multiply the equation by the LCD, with the appropriate sign.

$$y = \frac{3}{5}x + \frac{7}{2}$$
$$-\frac{3}{5}x + y = \frac{7}{2}$$
$$\boldsymbol{6x - 10y = -35}$$

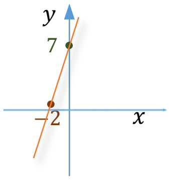
c. To write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form, we multiply it by the LCD, with the appropriate sign.

$$\frac{x}{4} - \frac{y}{3} = 1$$
$$\boldsymbol{3x - 4y = 12}$$

Example 4 ► **Writing Equation of a Line Using Intercept Form**

Write an equation of the line passing through points $(0,7)$ and $(-2,0)$. Leave the answer in standard form.

Solution ►

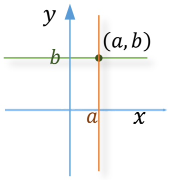


Since point $(0,7)$ is the y-intercept and point $(-2,0)$ is the x-intercept of our line, to write the equation of the line we can use the intercept form with $a = -2$ and $b = 7$. So, we have

$$\frac{x}{-2} + \frac{y}{7} = 1.$$

To change this equation to standard form, we multiply it by the LCD $= -14$. Thus,

$$\mathbf{7x - 2y = -14.}$$



Equations representing horizontal or vertical lines are special cases of linear equations in standard form, and as such, they deserve special consideration.

The **horizontal line** passing through the point (a, b) has equation $\boldsymbol{y = b}$, while the **vertical line** passing through the same point has equation $\boldsymbol{x = a}$.

The equation of a **horizontal line**, $\boldsymbol{y = b}$, can be shown in standard form as $0x + y = b$. Observe, that the slope of such a line is $-\frac{0}{1} = 0$.

The equation of a **vertical line**, $\boldsymbol{x = a}$, can be shown in standard form as $x + 0y = a$. Observe, that the slope of such a line is $-\frac{1}{0} =$ undefined.

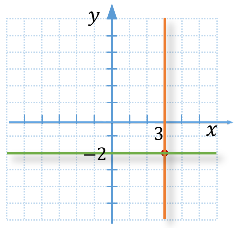
Example 5 ► Writing Equations of Horizontal and Vertical Lines

Find equations of the vertical and horizontal lines that pass through the point $(3, -2)$. Then, graph these two lines.

Solution ►

Since x-coordinates of all points of the vertical line, including $(3, -2)$, are the same, then these x-coordinates must be equal to 3. So, the equation of the vertical line is $x = 3$.

Since y-coordinates of all points of a horizontal line, including $(3, -2)$, are the same, then these y-coordinates must be equal to -2. So, the equation of the horizontal line is $y = -2$.



Here is a summary of the various forms of linear equations.

Forms of Linear Equations

Equation	Description	When to Use
$\boldsymbol{y = mx + b}$	**Slope-Intercept Form** slope is $\boldsymbol{m}$ y-intercept is $(0, \boldsymbol{b})$	This form is ideal for graphing by using the y-intercept and the slope.
$\boldsymbol{y - y_1 = m(x - x_1)}$	**Slope-Point Form** slope is $\boldsymbol{m}$ the line passes through $(\boldsymbol{x_1}, \boldsymbol{y_1})$	This form is ideal for finding the equation of a line if the slope and a point on the line, or two points on the line, are known.

$\boldsymbol{Ax + By = C}$	**Standard Form** slope is $-\frac{A}{B}$, if $B \neq 0$ x-intercept is $\left(\frac{C}{A}, 0\right)$, if $A \neq 0$. y-intercept is $\left(0, \frac{C}{B}\right)$, if $B \neq 0$.	This form is useful for graphing, as the x- and y-intercepts and the slope can be easily found by dividing appropriate coefficients.
$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1}$	**Intercept Form** slope is $-\frac{b}{a}$ x-intercept is $(a, 0)$ y-intercept is $(0, b)$	This form is ideal for graphing, using the x- and y-intercepts.
$\boldsymbol{y = b}$	**Horizontal Line** slope is 0 y-intercept is $(0, b)$	This form is used to write equations of, for example, horizontal asymptotes.
$\boldsymbol{x = a}$	**Vertical Line** slope is undefined x-intercept is $(a, 0)$	This form is used to write equations of, for example, vertical asymptotes.

Note: Except for the equations for a horizontal or vertical line, all of the above forms of linear equations can be converted into each other via algebraic transformations.

Writing Equations of Parallel and Perpendicular Lines

Recall that the slopes of parallel lines are the same, and slopes of perpendicular lines are opposite reciprocals. See *Section G2*.

Example 6 ► **Writing Equations of Parallel Lines Passing Through a Given Point**

Find the slope-intercept form of a line parallel to $y = -2x + 5$ that passes through the point $(-4,5)$. Then, graph both lines on the same grid.

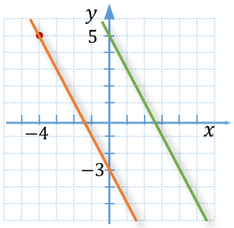
Solution ► Since the line is parallel to $y = -2x + 5$, its slope is -2. So, we plug the slope of -2 and the coordinates of the point $(-4,5)$ into the slope-point form of a linear equation.

$$y - 5 = -2(x + 4)$$

This can be simplified to the slope-intercept form, as follows:

$$y - 5 = -2x - 8$$

$$\boldsymbol{y = -2x - 3}$$



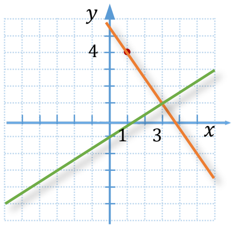
As shown in the accompanying graph, the line $y = -2x - 3$ (in orange) is parallel to the line $y = -2x + 5$ (in green) and passes through the given point $(-4,5)$.

Example 7 ► **Writing Equations of Perpendicular Lines Passing Through a Given Point**

Find the slope-intercept form of a line perpendicular to $2x - 3y = 6$ that passes through the point $(1,4)$. Then, graph both lines on the same grid.

Solution ► The slope of the given line, $2x - 3y = 3$, is $\frac{2}{3}$. To find the slope of a perpendicular line, we take the opposite reciprocal of $\frac{2}{3}$, which is $-\frac{3}{2}$. Since we already know the slope and the point, we can plug these pieces of information into the slope-point formula. So, we have

$$\begin{aligned} y - 4 &= -\tfrac{3}{2}(x - 1) \\ y - 4 &= -\tfrac{3}{2}x + \tfrac{3}{2} \\ y &= -\tfrac{3}{2}x + \tfrac{3}{2} + \tfrac{8}{2} \\ \boldsymbol{y} &= \boldsymbol{-\tfrac{3}{2}x + \tfrac{11}{2}} \end{aligned}$$



As shown in the accompanying graph, the line $2x - 3y = 6$ (in orange) is indeed perpendicular to the line $y = -\frac{3}{2}x + \frac{11}{2}$ (in green) and passes through the given point $(1,4)$.

Linear Equations in Applied Problems

Linear equations can be used to model a variety of applications in sciences, business, and other areas. Here are some examples.

Example 8 ► **Given the Rate of Change and the Initial Value, Determine the Linear Model Relating the Variables**

Lucy and Jack bought a sofa for \$1800. They have paid \$300 down and the rest is going to be paid by interest-free monthly payments of \$75 per month, until the bill is paid in full.

a. Write an equation to express the amount that is already paid off, A, in terms of the number of months, n, since their purchase.

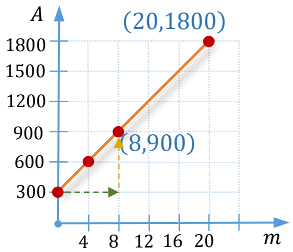
b. Graph the equation found in part **a.**

c. According to the graph, when will the bill be paid in full?

Solution ► **a.** Since each month the couple pays \$75, after n months, the amount paid off by the monthly installments is $75n$. If we add the initial payment of \$300, the equation representing the amount paid off can be written as

$$A = 75n + 300$$

b. To graph this equation, we use the slope-intercept method. Starting with the A-intercept of 300, we run 1 and rise 75, repeating this process as many times as needed to hit a lattice point on the chosen scale. As indicated in the accompanying graph, some of the points that the line passes through are $(0,300)$, $(4,600)$, $(8,900)$, and $(20,1800)$.

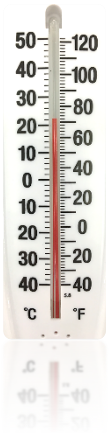


c. As shown in the graph, \$1800 will be paid off in 20 months.

Example 9 ► Finding a Linear Equation that Fits the Data Given by Two Ordered Pairs

In Fahrenheit scale, water freezes at 32°F and boils at 212°F. In Celsius scale, water freezes at 0°C and boils at 100°C. Write a linear equation that can be used to calculate the Celsius temperature, C, when the Fahrenheit temperature, F, is known.

Solution ►



To predict the Celsius temperature, C, knowing the Fahrenheit temperature, F, we treat the variable C as dependent on the variable F. So, we consider C as the second coordinate when setting up the ordered pairs, (F, C), of given data. The corresponding freezing temperatures give us the pair $(32,0)$ and the boiling temperatures give us the pair $(212,100)$. To find the equation of a line passing through these two points, first, we calculate the slope, and then, we use the slope-point formula. So, the slope is

$$m = \frac{C_2 - C_1}{F_2 - F_1} = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \boldsymbol{\frac{5}{9}},$$

and using the point $(32,0)$, the equation of the line is

$$\boldsymbol{C = \frac{5}{9}(F - 32)}$$

Example 10 ► Determining if the Given Set of Data Follows a Linear Pattern

Observe the data given in each table below. Do they follow a linear pattern? If "yes", find the slope-intercept form of an equation of the line passing through all the given points. If "no", explain why not.

a.

x	1	3	5	7	9
y	12	15	18	21	24

b.

x	5	10	15	20	25
y	15	21	26	30	35

Solution ►

a. The set of points follows a linear pattern if the slopes between consecutive pairs of these points are the same. These slopes are the ratios of increments in y-values to increments in x-values. Notice that the increases between successive x-values of the given points are constantly equal to 2. So, to check if the points follow a linear pattern, it is enough to check if the increases between successive y-values are also constant. Observe that the numbers in the list 12, 15, 18, 21, 24 steadily increase by 3. Thus, the given set of data follow a linear pattern.

To find an equation of the line passing through these points, we use the slope, which is $\frac{3}{2}$, and one of the given points, for example $(1,12)$. By plugging these pieces of information into the slope-point formula, we obtain

$$y - 12 = \frac{3}{2}(x - 1),$$

which after simplifying becomes

$$y - 12 = \frac{3}{2}x - \frac{3}{2}$$
$$\boldsymbol{y = \frac{3}{2}x + \frac{21}{2}}$$

b. Observe that the increments between consecutive x-values of the given points are constantly equal to 5, while the increments between consecutive y-values in the list 15, 21, 26, 30, 35 are 6, 5, 4, 5. So, they are not constant. Therefore, the given set of data does not follow a linear pattern.

Example 11 ► Finding a Linear Model Relating the Number of Items Bought at a Fixed Amount

At a local market, a farmer sells organically grown apples at \$0.50 each and pears at \$0.75 each.

a. Write a linear equation in standard form relating the number of apples, a, and pears, p, that can be bought for \$60.

b. Graph the equation from part (a).

c. Using the graph, find at least 2 points (a, p) satisfying the equation, and interpret their meanings in the context of the problem.

Solution ►

a. It costs $0.5a$ dollars to buy a apples. Similarly, it costs $0.75p$ dollars to buy p pears. Since the total charge is \$60, we have

$$0.5a + 0.75p = 60$$

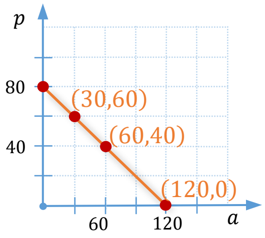
The coefficients can be converted into integers by multiplying the equation by a hundred. This would give us

$$50a + 75p = 6000,$$

which, after dividing by 25, turns into

$$\boldsymbol{2a + 3p = 240}.$$

b. To graph this equation, we will represent the number of apples, a, on the horizontal axis and the number of pears, p, on the vertical axis, respecting the alphabetical order of labelling the axes. Using the intercept method, we connect points $(120,0)$ and $(0,80)$.



c. Aside of the intercepts, $(120,0)$ and $(0,80)$, the graph shows us a few more points that satisfy the equation. In particular, $\boldsymbol{(30,60)}$ and $\boldsymbol{(60,40)}$ are points of the graph. If a point (a, p) of the graph has integral coefficients, it tells us that \$60 can buy a apples and p pears. For example, the point $\boldsymbol{(30,60)}$ tells us that **30 apples** and **60 pears** can be bought for **\$60**.

G.3 Exercises

*Write each equation in **standard form**.*

1. $y = -\frac{1}{2}x - 7$

2. $y = \frac{1}{3}x + 5$

3. $\frac{x}{5} + \frac{y}{-4} = 1$

4. $y - 7 = \frac{3}{2}(x - 3)$

5. $y - \frac{5}{2} = -\frac{2}{3}(x + 6)$

6. $2y = -0.21x + 0.35$

*Write each equation in **slope-intercept form**.*

7. $3y = \frac{1}{2}x - 5$

8. $\frac{x}{3} + \frac{y}{5} = 1$

9. $4x - 5y = 10$

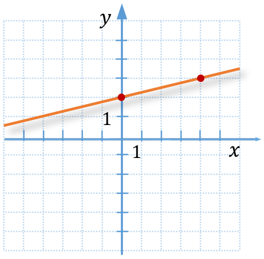
10. $3x + 4y = 7$

11. $y + \frac{3}{2} = \frac{2}{5}(x + 2)$

12. $y - \frac{1}{2} = -\frac{2}{3}\left(x - \frac{1}{2}\right)$

*Write an equation in **slope-intercept** form of the line shown in each graph.*

13.



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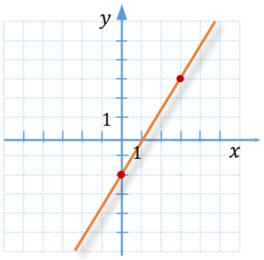

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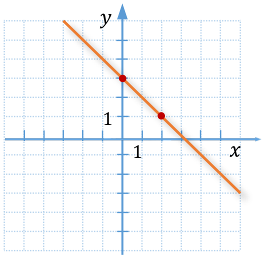
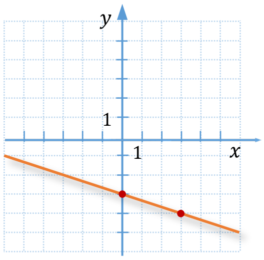


16.



*Find an equation of the line that satisfies the given conditions. Write the equation in **slope-intercept** and **standard** form.*

17. through $(-3,2)$, with slope $m = \frac{1}{2}$

18. through $(-2,3)$, with slope $m = -4$

19. with slope $m = \frac{3}{2}$ and y-intercept at -1

20. with slope $m = -\frac{1}{5}$ and y-intercept at 2

21. through $(-1,-2)$, with y-intercept at -3

22. through $(-4,5)$, with y-intercept at $\frac{3}{2}$

23. through $(2,-1)$ and $(-4,6)$

24. through $(3,7)$ and $(-5,1)$

25. through $\left(-\frac{4}{3},-2\right)$ and $\left(\frac{4}{5},\frac{2}{3}\right)$

26. through $\left(\frac{4}{3},\frac{3}{2}\right)$ and $\left(-\frac{1}{2},\frac{4}{3}\right)$

Find an equation of the line that satisfies the given conditions.

27. through $(-5,7)$, with slope 0

28. through $(-2,-4)$, with slope 0

29. through $(-1,-2)$, with undefined slope

30. through $(-3,4)$, with undefined slope

31. through $(-3,6)$ and horizontal

32. through $\left(-\frac{5}{3},-\frac{7}{2}\right)$ and horizontal

33. through $\left(-\frac{3}{4},-\frac{3}{2}\right)$ and vertical

34. through $(5,-11)$ and vertical

*Write an equation in **standard form** for each of the lines described. In each case make a sketch of the given line and the line satisfying the conditions.*

35. through $(7,2)$ and parallel to $3x - y = 4$

36. through $(4,1)$ and parallel to $2x + 5y = 10$

37. through $(-2,3)$ and parallel to $-x + 2y = 6$

38. through $(-1,-3)$ and parallel to $-x + 3y = 12$

39. through $(-1,2)$ and parallel to $y = 3$

40. through $(-1,2)$ and parallel to $x = -3$

41. through $(6,2)$ and perpendicular to $2x - y = 5$

42. through $(0,2)$ and perpendicular to $5x + y = 15$

43. through $(-2,4)$ and perpendicular to $3x + y = 6$

44. through $(-4,-1)$ and perpendicular to $x - 3y = 9$

45. through $(3,-4)$ and perpendicular to $x = 2$

46. through $(3,-4)$ and perpendicular to $y = -3$

*For each situation, write an equation in the form **y = mx + b**, and then answer the question of the problem.*

47. Membership in the Apollo Athletic Club costs \$80, plus \$49.95 per month. Express the cost C of the membership in terms of the number of months n that the membership is good for. What is the cost of the one-year membership?

48. A cellphone plan includes 1000 anytime minutes for \$55 per month, plus a one-time activation fee of \$75. Assuming that a cellphone is included in this plan at no additional charge, express the cost C of service in terms of the number of months n of this service. How much would a one-year contract plan cost for this cellphone?

49. An air compressor can be rented for \$23 per day and a \$60 deposit. Let d represent the number of days that the compressor is rented and C represent the total charge for renting, in dollars.
a. Write an equation that relates C with d.
b. Suppose Colin rented the air compressor and paid \$198. For how long did Colin rented the conpressor?

50. A car can be rented for \$75 plus \$0.15 per kilometer. Let d represent the number of kilometers driven and C represent the cost of renting, in dollars.
a. Write an equation that relates C with d.
b. How many kilometers was the car driven if the total cost of renting is \$101.40?

Solve each problem.

51. Originally there were 8 members of a local high school Math Circle. Three years later, the Math Circle counted 25 members. Assuming that the membership continues to grow at the same rate, find an equation that represents the number N of the Math Circle members t years after.

52. Driving on a highway, Steven noticed a 152-km marker on the side of the road. Ten minutes later, he noticed a 169-km marker. Find a formula that can be used to determine the distance driven d, in kilometers, in terms of the elapsed time t, in hours.

53. The table below shows the annual tuition and fees at Oxford University for out-of-state students.

Year y	**2007**	**2016**
Cost C	\$24400	\$31600

a. Find the slope-intercept form of a line that fits the given data.
b. Interpret the slope in the context of the problem.
c. Using the line from (a), find the predicted annual tuition and fees at Oxford University in 2022.

54. The life expectancy for a person born in 1900 was 48 years, and in 2000 it was 77 years. To the nearest year, estimate the life expectancy for someone born in 1970.

55. 3 years after Stan opened his mutual funds account, the amount in the account was \$2540. Two years later, the amount in the account was \$2900. Assuming a constant average increase in \$/year, find a linear equation that represents the amount A in Stan's account t years after it was opened.

56. Connor is a car salesperson in the local auto shop. His pay consists of a base salary and a 1.5% commission on sales. One month, his sales were \$165,000, and his total pay was \$3600.
a. Write an equation in slope-intercept form that shows Conner's total monthly income I in terms of his monthly sales s.
b. Graph the equation developed in (**a**).
c. What does the I-intercept represent in the context of the problem?
d. What does the slope represent in the context of the problem?

57. A taxi driver charges \$2.50 as his base fare and a constant amount for each kilometer driven. Helen paid \$7.75 for a 3-kilometer trip.
a. Find an equation in slope-intercept form that defines the total fare $f(k)$ as a function of the number k of kilometers driven.
b. Graph the equation found in (**a**).
c. What does the slope of this graph represent in the above situation?
d. How many kilometers were driven if a passenger pays \$23.50?

G4 Linear Inequalities in Two Variables

In many real-life situations, we are interested in a range of values satisfying certain conditions rather than in one specific value. For example, when exercising, we like to keep the heart rate between 120 and 140 beats per minute. The systolic blood pressure of a healthy person is usually between 100 and 120 mmHg (millimeters of mercury). Such conditions can be described using inequalities. Solving systems of inequalities has its applications in many practical business problems, such as how to allocate resources to achieve a maximum profit or a minimum cost. In this section, we study graphical solutions of linear inequalities.

Linear Inequalities in Two Variables

Definition 4.1 ► Any inequality that can be written as

$$\boldsymbol{Ax+By<C},\ \boldsymbol{Ax+By\le C},\ \boldsymbol{Ax+By>C},\ \boldsymbol{Ax+By\ge C}, \text{ or } \boldsymbol{Ax+By\ne C},$$

where $A, B, C \in \mathbb{R}$ and A and B are not both 0, is a **linear inequality in two variables**.

To **solve** an inequality in two variables, x and y, means to **find <u>all</u> ordered pairs** $(\boldsymbol{x}, \boldsymbol{y})$ satisfying the inequality.

Inequalities in two variables arise from many situations. For example, suppose that the number of full-time students, f, and part-time students, p, enrolled in upgrading courses at the University of the Fraser Valley is at most 1200. This situation can be represented by the inequality

$$f+p\le 1200.$$

Some of the solutions (f, p) of this inequality are: $(1000, 200)$, $(1000, 199)$, $(1000, 198)$, $(600, 600)$. $(550, 600)$, $(1100, 0)$, and many others.

The solution sets of inequalities in two variables contain infinitely many ordered pairs of numbers which, when graphed in a system of coordinates, fulfill specific regions of the coordinate plane. That is why it is more beneficial to present such solutions in the form of a graph rather than using set notation. To graph the region of points satisfying the inequality $f+p\le 1200$, we may want to solve it first for p,

$$p\le -f+1200,$$

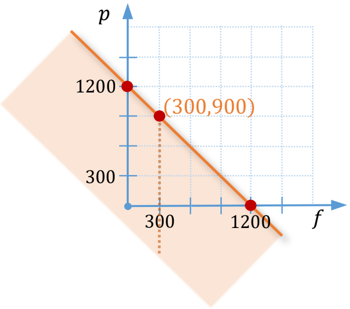


Figure 1a

and then graph the related equation, $p=-f+1200$, called the **boundary line**. Notice, that setting f to, for instance, 300 results in the inequality

$$p\le -300+1200=900.$$

So, any point with the first coordinate of 300 and the second coordinate of 900 or less satisfies the inequality (see the dotted half-line in *Figure 1a*). Generally, observe that any point with the first coordinate f and the second coordinate $-f+1200$ or less satisfies the inequality. Since the union of all half-lines that start from the boundary line and go down is the whole half-plane below the boundary line, we shade it as the solution set to the discussed inequality (see *Figure 1a*). The solution set also includes the points of the boundary line, as the inequality includes equation.

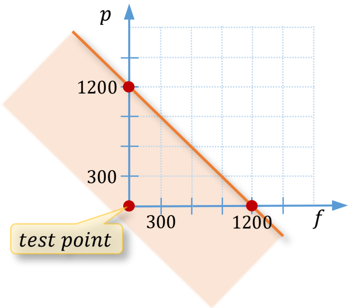


Figure 1b

The above strategy can be applied to any linear inequality in two variables. Hence, one can conclude that the solution set to a given linear inequality in two variables consists of **all points of one of the half-planes** obtained by cutting the coordinate plane by the corresponding boundary line. This fact allows us to find the solution region even faster. After graphing the boundary line, to know which half-plane to shade as the solution set, it is enough to check just one point, called a **test point**, chosen outside of the boundary line. In our example, it was enough to test for example point $(0,0)$. Since $0 \leq -0 + 1200$ is a true statement, then the point $(0,0)$ belongs to the solution set. This means that the half-plane containing this test point must be the solution set to the given inequality, so we shade it.

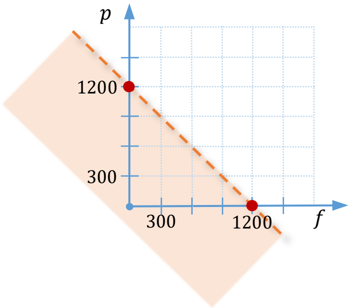


Figure 1c

The solution set of the strong inequality $p < -f + 1200$ consists of the same region as in *Figure 1b*, except for the points on the boundary line. This is because the points of the boundary line satisfy the equation $p = -f + 1200$, but not the inequality $p < -f + 1200$. To indicate this on the graph, we draw the boundary line using a dashed line (see *Figure 1c*).

In summary, to graph the solution set of a linear inequality in two variables, follow the steps:

1. Draw the graph of the corresponding **boundary line**.
   Make the line **solid** if the inequality involves ≤ or ≥.
   Make the line **dashed** if the inequality involves < or >.
2. Choose a **test point** outside of the line and substitute the coordinates of that point into the inequality.
3. If the test point satisfies the original inequality, **shade the half-plane containing the point.**
   If the test point does not satisfy the original inequality, **shade the other half-plane** (the one that does not contain the point).

Example 1 ► Determining if a Given Ordered Pair of Numbers is a Solution to a Given Inequality

Determine if the points $(3,1)$ and $(2,1)$ are solutions to the inequality $5x - 2y > 8$.

Solution ► An ordered pair is a solution to the inequality $5x - 2y > 8$ if its coordinates satisfy this inequality. So, to determine whether the pair $(3,1)$ is a solution, we substitute 3 for x and 1 for y. The inequality becomes

$$5 \cdot 3 - 2 \cdot 1 > 8,$$

which simplifies to the true inequality $13 > 8$.
Thus, $(3,1)$ is a solution to $5x - 2y > 8$.

However, replacing x by 2 and y by 1 results in $5 \cdot 2 - 2 \cdot 1 > 8$, or equivalently $8 > 8$. Since 8 is not larger than 8, the point $(2,1)$ does not satisfy the inequality. Thus, $(2,1)$ is not a solution to $5x - 2y > 8$.

Example 2 ► **Graphing Linear Inequalities in Two Variables**

Graph the solution set of each inequality in two variables.

a. $2x - 3y < 6$

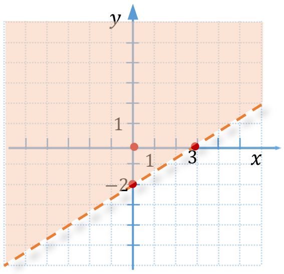
b. $y \le 3x - 1$

c. $x \ge -3$

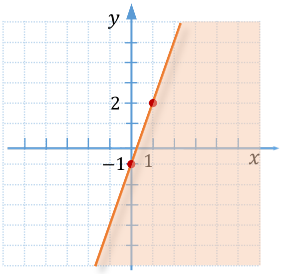
d. $y \ne x$

Solution ►

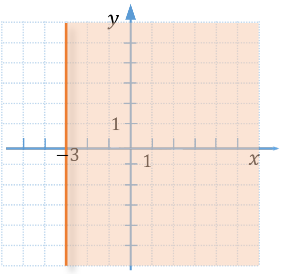
a. First, we graph the boundary line $2x - 3y = 6$, using the x- and y-intercepts: $(3,0)$ and $(0,-2)$. Since the inequality $<$ does not involve an equation, the line is marked as **dashed**, which indicates that the points on the line are not the solutions of the inequality. Then, we choose the point $(0,0)$ for the test point. Since $2 \cdot 0 - 3 \cdot 0 < 6$ is a true statement, then we shade the half-plane containing $(0,0)$ as the solution set.



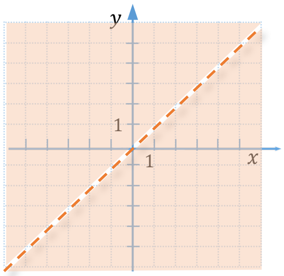
b. First, we graph the boundary line $y \le 3x - 1$, using the slope and y-intercept. Since the inequality $\le$ contains an equation, the line is marked as **solid**. This indicates that the points on the line belong to solutions of the inequality. To decide which half-plane to shade as the solution region, we observe that y is lower than or equal to the $3x - 1$, which tells us that the solution points lie below or on the boundary line. So, we shade the half-plane below the line.



c. As before, to graph $x \ge -3$, first, we graph the solid vertical line $x = -3$, and then we shade the half-plane consisting of points with x-coordinates larger or equal to -3. So the solution set is the half-plane to the right of the boundary line, including this line.



d. The solution set of the inequality $y \ne x$ consists of all points that do not satisfy the equation $y = x$. This means that we mark the boundary line as dashed and shade the rest of the points of the coordinate plane.



G.4 Exercises

For each inequality, determine if the given points belong to the solution set of the inequality.

1. $y \geq -4x + 3$; $(1, -1)$, $(1, 0)$

2. $2x - 3y < 6$; $(3, 0)$, $(2, -1)$

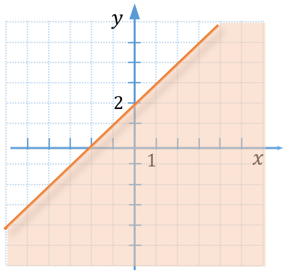
3. $y > -2$; $(0, 0)$, $(-1, -1)$

4. $x \geq -2$; $(-2, 1)$, $(-3, 1)$

5. Match the given inequalities with the graphs of their solution sets.

a. $y \geq x + 2$ **b.** $y < -x + 2$ **c.** $y \leq x + 2$ **d.** $y > -x + 2$

1I



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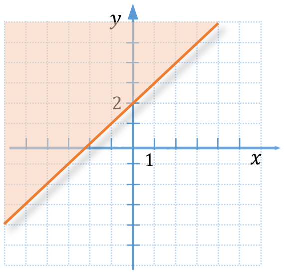
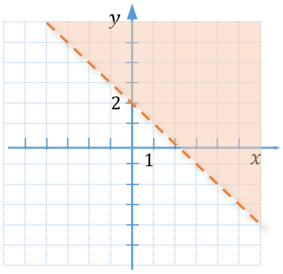


III



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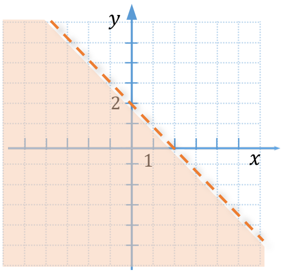


Graph each linear inequality in two variables.

6. $y \geq -\frac{1}{2}x + 3$

7. $y \leq \frac{1}{3}x - 2$

8. $y < 2x - 4$

9. $y > -x + 3$

10. $y \geq -3$

11. $y < 4.5$

12. $x > 1$

13. $x \leq -2.5$

14. $x + 3y > -3$

15. $5x - 3y \leq 15$

16. $y - 3x \geq 0$

17. $3x - 2y < -6$

18. $3x \leq 2y$

19. $3y \neq 4x$

20. $y \neq 2$

G5 Concept of Function, Domain, and Range

In mathematics, we often investigate relationships between two quantities. For example, we might be interested in the average daily temperature in Abbotsford, BC, over the last few years, the amount of water wasted by a leaking tap over a certain period of time, or particular connections among a group of bloggers. The relations can be described in many different ways: in words, by a formula, through graphs or arrow diagrams, or simply by listing the ordered pairs of elements that are in the relation. A group of relations, called *functions*, will be of special importance in further studies. In this section, we will define functions, examine various ways of determining whether a relation is a function, and study related concepts such as *domain* and *range*.

Relations, Domains, and Ranges

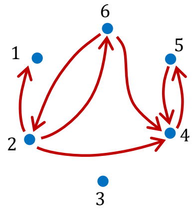
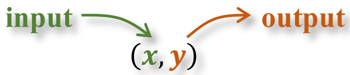


Figure 1

Consider a relation of knowing each other in a group of 6 people, represented by the arrow diagram shown in *Figure 1*. In this diagram, the points 1 through 6 represent the six people and an arrow from point x to point y tells us that the person x knows the person y. This correspondence could also be represented by listing the ordered pairs (x, y) whenever person x knows person y. So, our relation can be shown as the set of points

$$\{(2,1),(2,4),(2,6),(4,5),(5,4),(6,2),(6,4)\}$$

The x-coordinate of each pair (x, y) is called the **input**, and the y-coordinate is called the **output**.



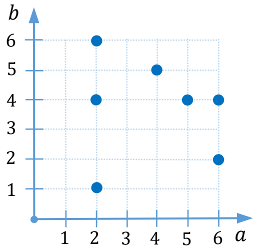


Figure 2a

The ordered pairs of numbers can be plotted in a system of coordinates, as in *Figure 2a*. The obtained graph shows that some inputs are in a relation with many outputs. For example, input 2 is in a relation with output 1, and 4, and 6. Also, the same output, 4, is assigned to many inputs. For example, the output 4 is assigned to the input 2, and 5, and 6.

The set of all the inputs of a relation is its **domain**. Thus, the domain of the above relation consists of all first coordinates

$$\{2, 4, 5, 6\}$$

The set of all the outputs of a relation is its **range**. Thus, the range of our relation consists of all second coordinates

$$\{1, 2, 4, 5, 6\}$$

The domain and range of a relation can be seen on its graph through the **perpendicular projection** of the graph **onto the horizontal axis**, for the **domain**, and **onto the vertical axis**, for the **range**. See *Figure 2b*.

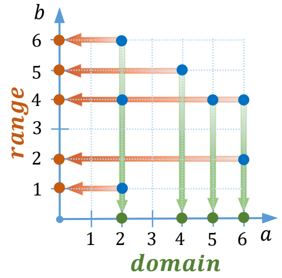
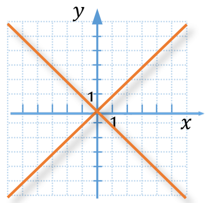


Figure 2b

In summary, we have the following definition of a relation and its domain and range:

Definition 5.1 ► A **relation** is any **set of ordered pairs**. Such a set establishes a **correspondence** between the **input** and **output** values. In particular, any subset of a coordinate plane represents a relation.
The **domain** of a relation consists of all **inputs** (**first coordinates**).
The **range** of a relation consists of all **outputs** (**second coordinates**).



Relations can also be given by an equation or an inequality. For example, the equation

$$|y| = |x|$$

describes the set of points in the xy-plane that lie on two diagonals, $y = x$ and $y = -x$. In this case, the domain and range for this relation are both the set of real numbers because the projection of the graph onto each axis covers the entire axis.

Functions, Domains, and Ranges

Relations that have exactly one output for every input are of special importance in mathematics. This is because as long as we know the rule of a correspondence defining the relation, the output can be uniquely determined for every input. Such relations are called **functions**. For example, the linear equation $y = 2x + 1$ defines a function, as for every input x, one can calculate the corresponding y-value in a unique way. Since both the input and the output can be any real number, the domain and range of this function are both the set of real numbers.

Definition 5.2 ► A **function** is a relation that assigns **exactly one** **output** value in the **range** to each **input** value of the **domain**.

If (x, y) is an ordered pair that belongs to a function, then x can be any arbitrarily chosen input value of the domain of this function, while y must be the uniquely determined value that is assigned to x by this function. That is why x is referred to as an **independent** variable while y is referred to as the **dependent** variable (because the y-value depends on the chosen x-value).

independent variable → (x, y) → **dependent variable**

How can we recognize if a relation is a function?

If the relation is given as a set of ordered pairs, it is enough to check if there are no two pairs with the same inputs. For example:

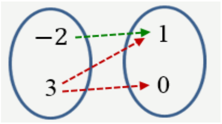
$\{(2,1), (2,4), (1,3)\}$
relation

The pairs (2,1) and (2,4) have the same inputs. So, there are **two y-values** assigned to the x-value 2, which makes it not a function.

$\{(2,1), (1,3), (4,1)\}$
function

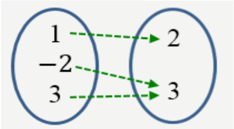
There are no pairs with the same inputs, so each x-value is associated with exactly one pair and consequently with exactly one y-value. This makes it a function.

If the relation is given by a diagram, we want to check if no point from the domain is assigned to two points in the range. For example:



relation

There are **two arrows** starting from 3. So, there are two y-values assigned to 3, which makes it <u>not</u> a function.



function

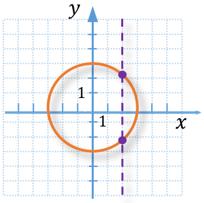
Only one arrow starts from each point of the domain, so each x-value is associated with exactly one y-value. Thus this is a function.

VERTICAL LINE TEST

If the relation is given by a <u>graph</u>, we use **The Vertical Line Test**:

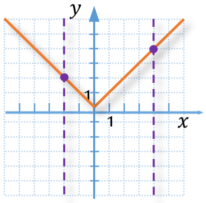
A relation is a **function** if **<u>no</u> vertical line** intersects the graph more than once.

For example:



relation

There is a vertical line that intersects the graph **twice**. So, there are two y-values assigned to an x-value, which makes it <u>not</u> a function.



function

Any vertical line intersects the graph only **once**. So, by The Vertical Line Test, this is a function.

If the relation is given by an <u>equation</u>, we check whether the y-value can be determined uniquely. For example:

$x^2 + y^2=1$

relation

Both points $(0,1)$ and $(0,-1)$ belong to the relation. So, there are **two y-values** assigned to 0, which makes it <u>not</u> a function.

$y = \sqrt{x}$

function

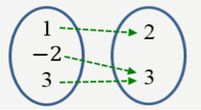
The y-value is uniquely defined as the square root of the x-value, for $x \geq 0$. So, this is a function.

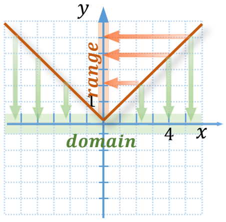
In general, to determine if a given relation is a function, we analyse the relation to see whether it assigns two different y-values to the same x-value. If it does, it is just a relation, <u>not</u> a function. If it doesn't, it is a function.

Since functions are a special type of relation, the **domain and range of a function** can be determined the same way as in the case of a relation.
Let us look at domains and ranges of the above examples of functions.

The domain of the function $\{(2,1),(1,3),(4,1)\}$ is the set of the first coordinates of the ordered pairs, which is $\{1,2,4\}$. The range of this function is the set of second coordinates of the ordered pairs, which is $\{1,3\}$.

The domain of the function defined by the diagram is the first set of points, particularly $\{1,-2,3\}$.
The range of this function is the second set of points, which is $\{2,3\}$.





The domain of the function given by the accompanying graph is the projection of the graph onto the x-axis, which is the set of all real numbers $\mathbb{R}$.
The range of this function is the projection of the graph onto the y-axis, which is the interval of points larger or equal to zero, $[0,\infty)$.

The domain of the function given by the equation $y=\sqrt{x}$ is the set of nonnegative real numbers, $[0,\infty)$, since the square root of a negative number is not real.
The range of this function is also the set of nonnegative real numbers, $[0,\infty)$, as the value of a square root is never negative.

Example 1 ►

Determining Whether a Relation is a Function and Finding Its Domain and Range

Decide whether each relation defines a function, and give the domain and range.

a. $y=\frac{1}{x-2}$

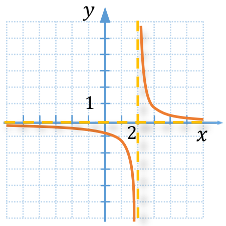
b. $y<2x+1$

c. $x=y^2$

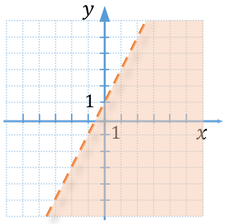
d. $y=\sqrt{2x-1}$

Solution ►

a. Since $\frac{1}{x-2}$ can be calculated uniquely for every x from its domain, the relation $y=\frac{1}{x-2}$ is a function.
The domain consists of all real numbers that make the denominator, $x-2$, different than zero. Since $x-2=0$ for $x=2$, then the domain, D, is the set of all real numbers except for 2. We write $D=\mathbb{R}\setminus\{2\}$.
Since a fraction with nonzero numerator cannot be equal to zero, the range of $y=\frac{1}{x-2}$ is the set of all real numbers except for 0. We write $range=\mathbb{R}\setminus\{0\}$.

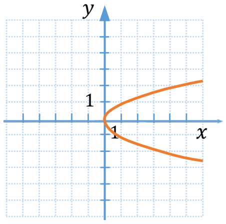


b. The inequality $y<2x+1$ is not a function as for every x-value there are many y-values that are lower than $2x+1$. Particularly, points $(0,0)$ and $(0,-1)$ satisfy the inequality and show that the y-value is not unique for $x=0$.



In general, because of the many possible y-values, no inequality defines a function.

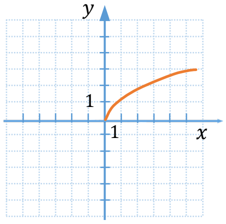
Since there are no restrictions on x-values, the domain of this relation is the set of all real numbers, $\mathbb{R}$. The range is also the set of all real numbers, $\mathbb{R}$, as observed in the accompanying graph.



c. Here, we can show two points, $(1,1)$ and $(1,-1)$, that satisfy the equation, which contradicts the requirement of a single y-value assigned to each x-value. So, this relation is not a function.

Since x is a square of a real number, it cannot be a negative number. So the domain consists of all nonnegative real numbers. We write, $D = [0, \infty)$. However, y can be any real number, so $range = \mathbb{R}$.

d. The equation $y = \sqrt{2x-1}$ represents a function, as for every x-value from the domain, the y-value can be calculated in a unique way.



The domain of this function consists of all real numbers that would make the radicand $2x - 1$ nonnegative. So, to find the domain, we solve the inequality:

$$2x - 1 \geq 0$$
$$2x \geq 1$$
$$x \geq \tfrac{1}{2}$$

Thus, $D = \left[\frac{1}{2}, \infty\right)$. As for the range, since the values of a square root are nonnegative, we have $range = [0, \infty)$

G.5 Exercises

Decide whether each relation defines a function, and give its ***domain*** *and* ***range****.*

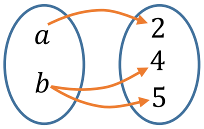
1. $\{(2,4), (0,2), (2,3)\}$

2. $\{(3,4), (1,2), (2,3)\}$

3. $\{(2,3), (3,4), (4,5), (5,2)\}$

4. $\{(1,1), (1,-1), (2,5), (2,-5)\}$

5.



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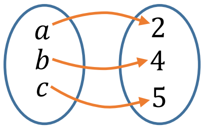

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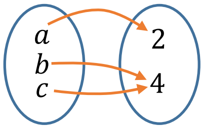

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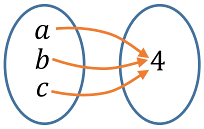


9.

x	y
0	1
0	−1
1	2
1	−2

10.

x	y
−1	4
0	2
1	0
2	−2

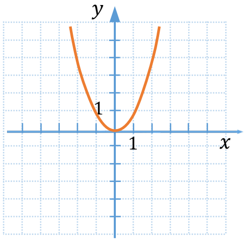
11.

x	y
3	1
6	2
9	1
12	2

12.

x	y
−2	3
−2	0
−2	−3
−2	−6

13.



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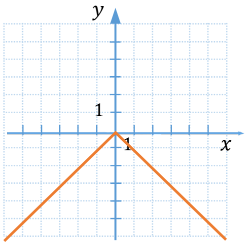

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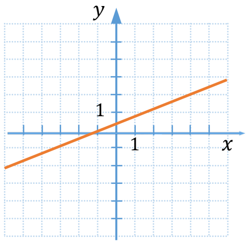

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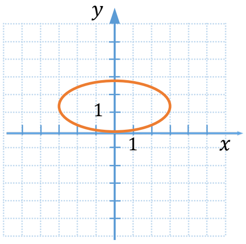

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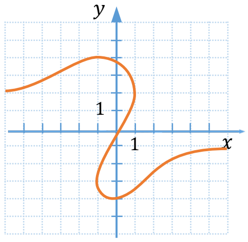

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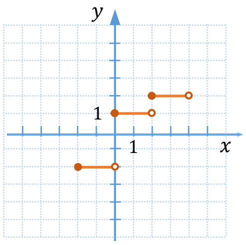

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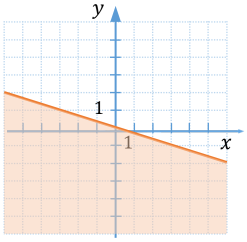

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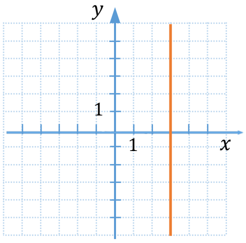


Find the ***domain*** *of each relation and decide whether the relation defines y as a function of x.*

21. $y = 3x + 2$

22. $y = 5 - 2x$

23. $y = |x| - 3$

24. $x = |y| + 1$

25. $y^2 = x^2$

26. $y^2 = x^4$

27. $x = y^4$

28. $y = x^3$

29. $y = -\sqrt{x}$

30. $y = \sqrt{2x - 5}$

31. $y = \frac{1}{x+5}$

32. $y = \frac{1}{2x-3}$

33. $y = \frac{x-3}{x+2}$

34. $y = \frac{1}{|2x-3|}$

35. $y \leq 2x$

36. $y - 3x \geq 0$

37. $y \neq 2$

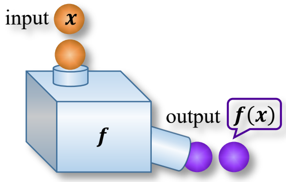
38. $x = -1$

39. $y = x^2 + 2x + 1$

40. $xy = -1$

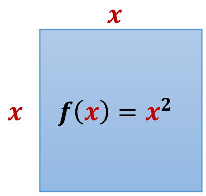
41. $x^2 + y^2 = 4$

G6 Function Notation and Evaluating Functions



A function is a correspondence that assigns a single value of the range to each value of the domain. Thus, a function can be seen as an input-output machine, where the input is taken independently from the domain, and the output is the corresponding value of the range. The rule that defines a function is often written as an equation, with the use of x and y for the independent and dependent variables, for instance, $y = 2x$ or $y = x^2$. To emphasize that y depends on x, we write $\boldsymbol{y = f(x)}$, where f is the name of the function. The expression $\boldsymbol{f(x)}$, read as "$\boldsymbol{f}$ ***of*** $\boldsymbol{x}$", represents the dependent variable assigned to the particular x. Such notation shows the dependence of the variables as well as allows for using different names for various functions. It is also handy when evaluating functions. In this section, we introduce and use *function notation*, and show how to evaluate functions at specific input-values.

Function Notation



Consider the equation $y = x^2$, which relates the length of a side of a square, x, and its area, y. In this equation, the y-value depends on the value x, and it is uniquely defined. So, we say that y is a function of x. Using function notation, we write

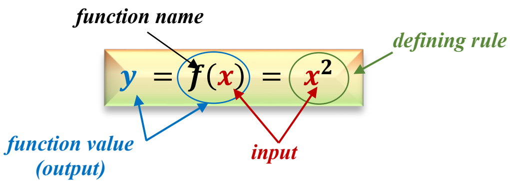
$$f(x) = x^2$$

The expression $f(x)$ is just another name for the dependent variable y, and it shouldn't be confused with a product of f and x. Even though $f(x)$ is really the same as y, we often write $f(x)$ rather than just y, because the notation $f(x)$ carries more information. Particularly, it tells us the name of the function so that it is easier to refer to the particular one when working with many functions. It also indicates the independent value for which the dependent value is calculated. For example, using function notation, we find the area of a square with a side length of 2 by evaluating $f(2) = 2^2 = 4$. So, 4 is the area of a square with a side length of 2.

The statement $f(2) = 4$ tells us that the pair $(2,4)$ belongs to function f, or equivalently, that 4 is assigned to the input of 2 by the function f. We could also say that function f attains the value 4 at 2.

If we calculate the value of function f for $x = 3$, we obtain $f(3) = 3^2 = 9$. So the pair $(3,9)$ also belongs to function f. This way, we may produce many ordered pairs that belong to f and consequently, make a graph of f.

Here is what each part of **function notation** represents:



Note: Functions are customarily denoted by a single letter, such as f, g, h, but also by abbreviations, such as sin, cos, or tan.

Function Values

Function notation is handy when evaluating functions for several input values. To evaluate a function given by an equation at a specific x-value from the domain, we substitute the x-value into the defining equation. For example, to evaluate $f(x)=\frac{1}{x-1}$ at $x=3$, we calculate

$$f(3)=\frac{1}{3-1}=\frac{1}{2}$$

So $f(3)=\frac{1}{2}$, which tells us that when $x=3$, the y-value is $\frac{1}{2}$, or equivalently, that the point $\left(3,\frac{1}{2}\right)$ belongs to the graph of the function f.

Notice that function f cannot be evaluated at $x=1$, as it would make the denominator $(x-1)$ equal to zero, which is not allowed. We say that $f(1)=DNE$ (read: *Does Not Exist*). Because of this, the domain of function f, denoted D_f, is $\mathbb{R}\setminus\{1\}$.

Graphing a function usually requires evaluating it for several x-values and then plotting the obtained points. For example, evaluating $f(x)=\frac{1}{x-1}$ for $x=\frac{3}{2}, 2, 5, \frac{1}{2}, 0, -1$, gives us

$$f\left(\frac{3}{2}\right)=\frac{1}{\frac{3}{2}-1}=\frac{1}{\frac{1}{2}}=2$$

$$f(2)=\frac{1}{2-1}=\frac{1}{1}=1$$

$$f(5)=\frac{1}{5-1}=\frac{1}{4}$$

$$f\left(\frac{1}{2}\right)=\frac{1}{\frac{1}{2}-1}=\frac{1}{-\frac{1}{2}}=-2$$

$$f(0)=\frac{1}{0-1}=-1$$

$$f(-1)=\frac{1}{-1-1}=-\frac{1}{2}$$

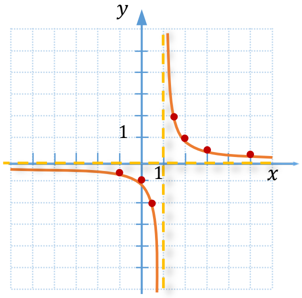


Figure 1

Thus, the points $\left(\frac{3}{2},2\right)$, $(2,1)$, $\left(3,\frac{1}{2}\right)$, $\left(5,\frac{1}{4}\right)$, $\left(\frac{1}{2},-2\right)$, $(0,-1)$, $\left(-1,-\frac{1}{2}\right)$ belong to the graph of f. After plotting them in a system of coordinates and predicting the pattern for other x-values, we produce the graph of function f, as in *Figure 1*.

Observe that the graph seems to be approaching the vertical line $x=1$ as well as the horizontal line $y=0$. These two lines are called ***asymptotes*** and are not a part of the graph of function f; however, they shape the graph. Asymptotes are customarily graphed by dashed lines.

Sometimes a function is given not by an equation but by a graph, a set of ordered pairs, a word description, etc. To evaluate such a function at a given input, we simply apply the function rule to the input.

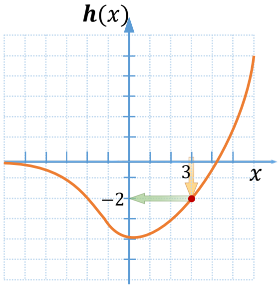


Figure 2a

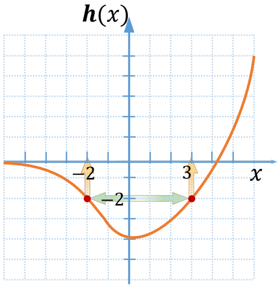


Figure 2b

For example, to find the value of function h, given by the graph in *Figure 2a*, for $x = 3$, we read the second coordinate of the intersection point of the vertical line $x = 3$ with the graph of h. Following the arrows in *Figure 2*, we conclude that $h(3) = -2$.

Notice that to find the x-value(s) for which $h(x) = -2$, we reverse the above process. This means: we read the first coordinate of the intersection point(s) of the horizontal line $y = -2$ with the graph of h. By following the reversed arrows in *Figure 2b*, we conclude that $h(x) = -2$ for $x = 3$ and for $x = -2$.

Example 1 ►

Evaluating Functions

Evaluate each function at $x = 2$ and write the answer using function notation.

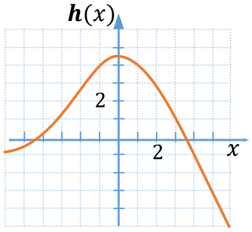
a. $f(x) = 3 - 2x$

b. function f squares the input

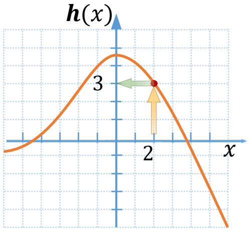
c.

x	$g(x)$
−1	2
2	5
3	−1

d.



Solution ►



a. Following the formula, we have $f(2) = 3 - 2(2) = 3 - 4 = \mathbf{-1}$

b. Following the word description, we have $f(2) = 2^2 = \mathbf{4}$

c. $g(2)$ is the value in the second column of the table that corresponds to 2 from the first column. Thus, $g(2) = \mathbf{5}$.

d. As shown in the graph, $h(2) = \mathbf{3}$.

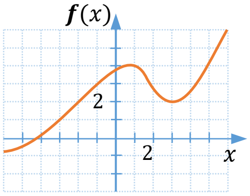
Example 2 ►

Finding from a Graph the x-value for a Given $f(x)$-value

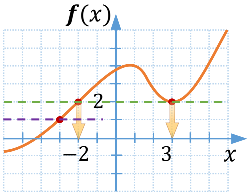
Given the graph, find all x-values for which

a. $f(x) = 1$

b. $f(x) = 2$



Solution ►



a. The purple line $y = 1$ cuts the graph at $x = -3$, so $f(x) = 1$ for $x = -3$.

b. The green line $y = 2$ cuts the graph at $x = -2$ and $x = 3$, so $f(x) = 2$ for $x \in \{-2, 3\}$.

Example 3 ► Evaluating Functions and Expressions Involving Function Values

Suppose $f(x) = \frac{1}{2}x - 1$ and $g(x) = x^2 - 5$. Evaluate each expression.

a. $f(4)$ **b.** $g(-2)$ **c.** $g(a)$ **d.** $f(2a)$

e. $g(a-1)$ **f.** $3f(-2)$ **g.** $g(2+h)$ **h.** $f(2+h) - f(2)$

Solution ►

a. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the value 4. So, $f(4) = \frac{1}{2}(4) - 1 = 2 - 1 = \mathbf{1}$.

b. Replace x in the equation $g(x) = x^2 - 5$ by the value -2, using parentheses around the -2. So, $g(-2) = (-2)^2 - 5 = 4 - 5 = \mathbf{-1}$.

c. Replace x in the equation $g(x) = x^2 - 5$ by the input a. So, $g(a) = \boldsymbol{a^2 - 5}$.

d. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the input $2a$. So, $f(2a) = \frac{1}{\cancel{2}}(\cancel{2}a) - 1 = \boldsymbol{a - 1}$.

$(a-1)^2$
$= (a-1)(a-1)$
$= a^2 - a - a + 1$
$= a^2 - 2a + 1$

e. Replace x in the equation $g(x) = x^2 - 5$ by the input $(a - 1)$, using parentheses around the input. So, $g(a-1) = (a-1)^2 - 5 = a^2 - 2a + 1 - 5 = \boldsymbol{a^2 - 2a - 4}$.

f. The expression $3f(-2)$ means three times the value of $f(-2)$, so we calculate $3f(-2) = 3 \cdot \left(\frac{1}{\cancel{2}}(-\cancel{2}) - 1\right) = 3(-1-1) = 3(-2) = \mathbf{-6}$.

$(2+h)^2$
$= (2+h)(2+h)$
$= 4 + 2h + 2h + h^2$
$= 4 + 4h + h^2$

g. Replace x in the equation $g(x) = x^2 - 5$ by the input $(2 + h)$, using parentheses around the input. So, $g(2+h) = (2+h)^2 - 5 = 4 + 4h + h^2 - 5 = \boldsymbol{h^2 + 4h - 1}$.

h. When evaluating $f(2+h) - f(2)$, focus on evaluating $f(2+h)$ first and then, to subtract the expression $f(2)$, use a bracket just after the subtraction sign. So,

$$f(2+h) - f(2) = \underbrace{\frac{1}{2}(2+h) - 1}_{f(2+h)} - \underbrace{\left[\frac{1}{2}(2) - 1\right]}_{f(2)} = 1 + \frac{1}{2}h - 1 - [1 - 1] = \boldsymbol{\frac{1}{2}h}$$

Note: To perform the perfect squares in the solution to *Example 3e* and *3g*, we follow the **perfect square formula** $\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$ or $\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2}$. One can check that this formula can be obtained as a result of applying the distributive law, often referred to as the *FOIL* method, when multiplying two binomials (see the examples in callouts in the left margin). However, we prefer to use the perfect square formula rather than the *FOIL* method, as it makes the calculation process more efficient. See Section P2 for more details.

Function Notation in Graphing and Application Problems

By *Definition 1.1* in *Section G1*, a linear equation is an equation of the form $Ax + By = C$. The graph of any linear equation is a line, and any nonvertical line satisfies the Vertical Line Test. Thus, any linear equation $Ax + By = C$ with $B \neq 0$ defines a linear function.

How can we write this function using function notation?

Since $y = f(x)$, we can replace the variable y in the equation $Ax + By = C$ with $f(x)$ and then solve for $f(x)$. So, we obtain

Alternatively, we can solve the original equation for y and then replace y with $f(x)$.

$$Ax + B \cdot f(x) = C$$

$$B \cdot f(x) = -Ax + C$$

$$f(x) = -\frac{A}{B}x + \frac{C}{B}$$

must assume that $B \neq 0$

Definition 6.1 ► Any function that can be written in the form

$$\boldsymbol{f(x) = mx + b},$$

where m and b are real numbers, is called a **linear function**. The value $\boldsymbol{m}$ represents the **slope** of the graph, and the point $(0, \boldsymbol{b})$ represents the $\boldsymbol{y}$**-intercept** of this function. The **domain** of any linear function is the set of all real numbers, $\mathbb{R}$.

In particular:

Definition 6.2 ► A linear function with slope $\boldsymbol{m = 0}$ takes the form

$$\boldsymbol{f(x) = b},$$

where b is a real number, and is called a **constant function**.

Note: Similarly as the domain of any linear function, the **domain** of a constant function is the set $\mathbb{R}$. However, the **range** of a constant function is the one element set $\{\boldsymbol{b}\}$, while the range of any nonconstant linear function is the set $\mathbb{R}$.

Generally, any equation in two variables, x and y, that defines a function can be written using function notation by solving the equation for y and then letting $y = f(x)$. For example, to rewrite the equation $-4x^2 + 2y = 5$ **explicitly** as a function f of x, we solve for y,

implicit form

$$2y = 4x^2 + 5$$

explicit form

$$y = 2x^2 + \frac{5}{2},$$

and then replace y by $f(x)$. So, $f(x) = 2x^2 + \frac{5}{2}$.

Using function notation, the graph of a function is defined as follows:

Definition 6.3 ► The graph of a function $\boldsymbol{f}$ of $\boldsymbol{x}$ is the set of ordered pairs $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$ for every input x form the domain $\boldsymbol{D_f}$ of the function. This can be stated as

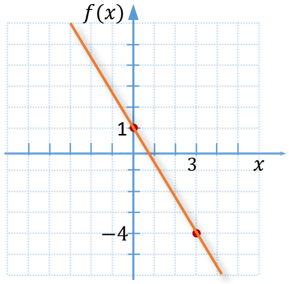
$$\boldsymbol{graph\ of\ f} = \{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) | \boldsymbol{x} \in \boldsymbol{D_f}\}$$

Example 4 ► **Function Notation in Writing and Graphing Functions**

Each of the given equations represents a function of x. Rewrite the formula in explicit form, using function notation. Then graph this function and state its domain and range.

a. $5x + 3 \cdot f(x) = 3$ **b.** $|x| - y = -3$

Solution ► **a.** After solving the equation for $f(x)$,



$$5x + 3 \cdot f(x) = 3$$
$$3 \cdot f(x) = -5x + 3$$
$$f(x) = -\tfrac{5}{3}x + 1,$$

we observe that the function f is linear. So, we can graph it using the slope-intercept method. The graph confirms that the domain and range of this function are both the set of all real numbers, $\mathbb{R}$.

b. After solving the equation for y,

$$|x| - y = 2$$
$$|x| - 2 = y,$$

we obtain the function $f(x) = |x| - 2$.

If we are not sure what the graph of this function looks like, we may evaluate $f(x)$ for several x-values, plot the obtained points, and observe the pattern. For example, let $x = -2, -1, 0, 1, 2$. We fill in the table of values,

$\boldsymbol{x}$	$\boldsymbol{\vert x\vert - 2 = f(x)}$	$\boldsymbol{(x, f(x))}$
−2	$\vert -2\vert - 2 = 0$	$(-2,0)$
−1	$\vert -1\vert - 2 = -1$	$(-1,-1)$
0	$\vert 0\vert - 2 = -2$	$(0,-2)$
1	$\vert 1\vert - 2 = -1$	$(1,-1)$
2	$\vert 2\vert - 2 = 0$	$(2,0)$

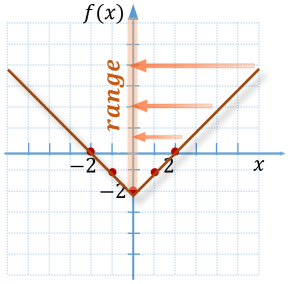


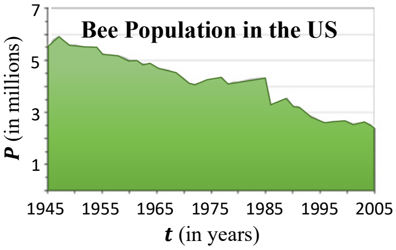
Figure 3

and plot the points listed in the third column. One may evaluate $f(x)$ for several more x-values, plot the additional points, and observe that these points form a V-shape with a vertex at $(0,-2)$. By connecting the points as in *Figure 3*, we obtain the graph of function $f(x) = |x| - 2$.

Since one can evaluate the function $f(x) = |x| - 2$ for any real x, the domain of f is the set $\mathbb{R}$. The range can be observed by projecting the graph perpendicularly onto the vertical axis. So, the range is the interval $[-2, \infty)$, as shown in *Figure 3*.

Example 5 ► A Function in Applied Situations

The bee population in the US was declining during the years 1945-2005, as shown in the accompanying graph.



a. Based on the graph what was the approximate value of $P(1960)$ and $P(2000)$ and what does it tell us about the bee population?

b. Estimate the average rate of change in the bee population over the years $1960 - 2000$, and interpret the result in the context of the problem.

c. Approximate the year(s) in which $P(t)$ was 4 million bees.

d. What is the general tendency of the function $P(t)$ over the years $1945 - 2005$?

e. Assuming that function P continue declining at the same rate, predict the year in which the bees in the US would become extinct.

Solution ►

a. One may read from the graph that $P(1960) \approx 5$ and $P(2000) \approx 2.6$ (see the orange line in *Figure 4a*). The first equation tells us that in 1960 there were approximately 5 million bees in the US. The second equation indicates that in the year 2000 there were approximately 2.6 million bees in the US.

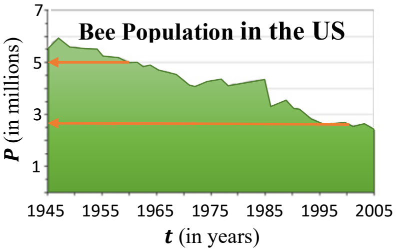


Figure 4a

b. The average rate of change is represented by the slope of a straight line between $(1960, 5)$ and $(2000, 2.6)$. Since the change in bee population over the years $1960 - 2000$ is $2.6 - 5 = -2.4$ million, and the change in time $1960 - 2000 = 40$ years, then the slope is $-\frac{2.4}{40} = -0.06$ million per year. This means that, in the US, the population of bees decreased an average of 60,000 each year between 1960 and 2000.

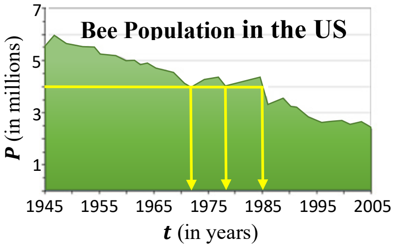


Figure 4b

c. As indicated by yellow arrows in *Figure 4b*, $P(t) = 4$ for $t \approx 1972$, $t \approx 1978$, and $t \approx 1985$.

d. The general tendency of function $P(t)$ over the years $1945 - 2005$ is declining.

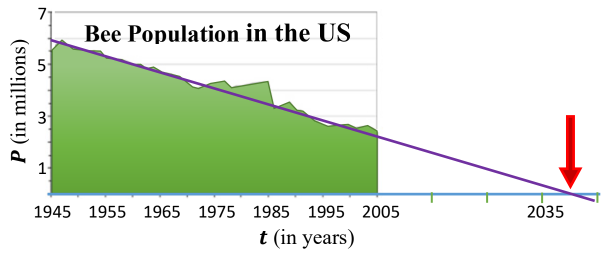
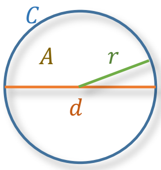


Figure 4c

e. Assuming the same declining tendency, to estimate the year in which the bee population in the US will disappear, we extend the t-axis and the approximate line of tendency (see the purple line in *Figure 4c*) to see where they intersect. After extending of the scale on the t-axis, we predict that the bee population will disappear around the year 2040.

Example 6 ► **Constructing Functions**



Consider a circle with area A, circumference C, radius r, and diameter d.

a. Write A as a function of r.
b. Write r as a function of d.
c. Write A as a function of d.
d. Write r as a function of C.
e. Write A as a function of C.

Solution ►

a. Using the formula for the area of a circle, $A = \pi r^2$, the function A of r is $\boldsymbol{A(r) = \pi r^2}$.

b. To express r as a function of d, we solve the formula $d = 2r$ for r. This gives us $r = \frac{d}{2}$. So, the function r of d is $\boldsymbol{r(d) = \frac{d}{2}}$.

c. To write A as a function of d, we start by connecting the formula for the area A in terms of r and the formula that expresses r in terms of d. Since

$$A = \pi r^2 \quad \text{and} \quad r = \frac{d}{2},$$

then using substitution, we have

$$A = \pi r^2 = \pi \cdot \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}.$$

Hence, our function A of d is $\boldsymbol{A(d) = \frac{1}{4}\pi d^2}$.

d. The relation between circumference C and radius r is $C = 2\pi r$. After solving this formula for r, we have $r = \frac{C}{2\pi}$. So, our function is $\boldsymbol{r(C) = \frac{C}{2\pi}}$.

e. To write A as a function of C, we use the formula $r = \frac{C}{2\pi}$ to replace r in the area formula $A = \pi r^2$ by the expression $\frac{C}{2\pi}$. This gives us

$$A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{\cancel{\pi}C^2}{4\pi^{\cancel{2}}} = \frac{C^2}{4\pi}.$$

So, our function is $\boldsymbol{A(C) = \frac{C^2}{4\pi}}$.

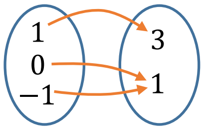
G.6 Exercises

*For each function, find **a)** $f(-1)$ and **b)** all x-values such that $f(x) = 1$.*

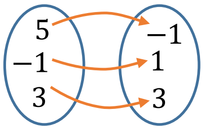
1. $\{(2,4), (-1,2), (3,1)\}$

2. $\{(-1,1), (1,2), (2,1)\}$

3.



4.



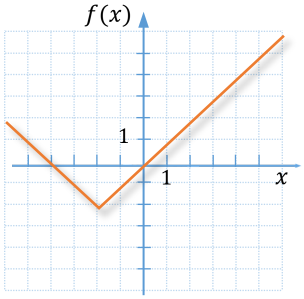
5.

x	$f(x)$
−1	4
0	2
2	1
4	−1

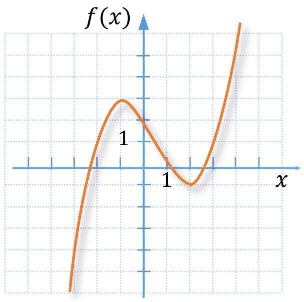
6.

x	$f(x)$
−3	1
−1	2
1	2
3	1

7.



8.



Let $f(x) = -3x + 5$ and $g(x) = -x^2 + 2x - 1$. Find the following.

9. $f(1)$

10. $g(0)$

11. $g(-1)$

12. $f(-2)$

13. $f(p)$

14. $g(a)$

15. $g(-x)$

16. $f(-x)$

17. $f(a + 1)$

18. $g(a + 2)$

19. $g(x - 1)$

20. $f(x - 2)$

21. $f(2 + h)$

22. $g(1 + h)$

23. $g(a + h)$

24. $f(a + h)$

25. $f(3) - g(3)$

26. $g(a) - f(a)$

27. $3g(x) + f(x)$

28. $f(x + h) - f(x)$

Fill in each blank.

29. The graph of the equation $2x + y = 6$ is a ____________. The point (1,___) lies on the graph of this line. Using function notation, the above equation can be written as $f(x) =$ ___________ . Since $f(1) =$ _____ , the point (___ ,___) lies on the graph of function f.

Graph each function. Give the domain and range.

30. $f(x) = -2x + 5$

31. $g(x) = \frac{1}{3}x + 2$

32. $h(x) = -3x$

33. $F(x) = x$

34. $G(x) = 0$

35. $H(x) = 2$

36. $x - h(x) = 4$

37. $-3x + f(x) = -5$

38. $2 \cdot g(x) - 2 = x$

39. $k(x) = |x - 3|$

40. $m(x) = 3 - |x|$

41. $q(x) = x^2$

42. $Q(x) = x^2 - 2x$

43. $p(x) = x^3 + 1$

44. $s(x) = \sqrt{x}$

Solve each problem.

45. A taxi driver charges \$1.50 per kilometer.

a. Complete the table by writting the charge $f(x)$ for a trip of x kilometers.

b. Find the linear function that calculates the charge $f(x) =$ ________ for a trip of x kilometers.

c. Graph $f(x)$ for the domain $\{0, 2, 4\}$.

x	$f(x)$
0	
2	
4	

46. Given the information about the linear function f, find the following:

a. $f(1)$

b. x-value such that $f(x) = -0.4$

c. slope of f

d. y-intercept of f

e. an equation for $f(x)$

x	$f(x)$
−2	3.2
−1	2.3
0	1.4
1	0.5
2	−0.4
3	−1.3

47. Suppose the cost of renting a car at Los Angeles International Airport consists of the initial fee of \$18.80 and \$24.60 per day. Let $C(d)$ represent the total cost of renting the car for d days.

a. Write a linear function that models this situation.

b. Find $C(4)$ and interpret your answer in the context of the problem.

c. Find the value of d satisfying the equation $C(d) = 191$ and interpret it in the context of this problem.

48. Suppose a house cleaning service charges \$20 per visit plus \$32 per hour.

a. Express the total charge, C, as a function of the number of hours worked, n.

b. Find $C(3)$ and interpret your answer in the context of this problem.

c. If Stacy was charged \$244 for a one-visit work, how long it took to clean her house?

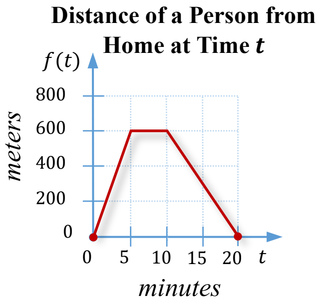
49. *Refer to the given graph of function f to answer the questions below.*

a. What is the range of possible values for the independent variable? What is the range of possible values for the independent variable?

b. For how long is the person going away from home? Coming closer to home?

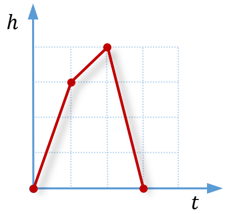
c. How far away from home is the person after 10 minutes?

d. Call this function f. What is $f(15)$ and what does this mean in the context of the problem?

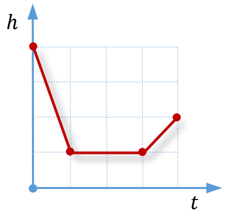

Distance of a Person from Home at Time *t*

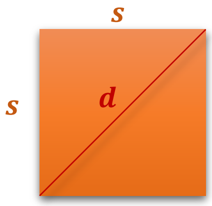
Questions 51 and 52 show graphs of the height of water in a bathtub. The t-axis represents time, and the h-axis represents height. Interpret the graph by describing the rate of change of the height of water in the bathtub.

50.



51.





52. Consider a square with area A, side s, perimeter P, and diagonal d.

- **a.** Write A as a function of s.
- **b.** Write s as a function of P.
- **c.** Write A as a function of P.
- **d.** Write A as a function of d.

(*Hint*: in part (**d**) apply the Pythagorean equation $a^2 + b^2 = c^2$, where c is the hypotenuse of a right angle triangle with arms a and b.)

Attributions

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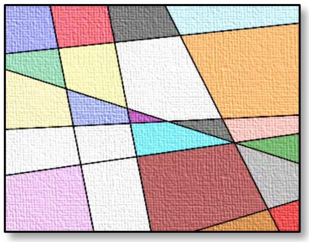
Systems of Linear Equations



As stated in *Section G1, Definition 1.1*, a linear equation in two variables is an equation of the form $Ax + By = C$, where A and B are not both zero. Such an equation has a line as its graph. Each point of this line is a solution to the equation in the sense that the coordinates of such a point satisfy the equation. So, there are infinitely many ordered pairs (x, y) satisfying the equation. Although analysis of the relation between x and y is instrumental in some problems, many application problems call for a particular, single solution. This occurs when, for example, x and y are required to satisfy an additional linear equation whose graph intersects the original line. In such a case, the solution to both equations is the point at which the lines intersect. Generally, to find unique values for **two** given **variables**, we need a system of **two equations** in these variables. In this section, we discuss several methods for solving systems of two linear equations.

E1 Systems of Linear Equations in Two Variables

Any collection of equations considered together is called a **system of equations**. For example, a system consisting of two equations, $x + y = 5$ and $4x - y = 10$, is written as

$$\begin{cases} x + y = 5 \\ 4x - y = 10 \end{cases}$$

Since the equations in the system are linear, the system is called a **linear system of equations**.

Definition 1.1 ► A **solution** of a system of two equations in two variables, x and y, is any ordered pair (x, y) satisfying both equations of the system.

A **solution set** of a system of two linear equations in two variables, x and y, is the set of all possible solutions (x, y).

Note: The two variables used in a system of two equations can be denoted by any two different letters. In such case, to construct an ordered pair, we follow an alphabetical order. For example, if the variables are p and q, the corresponding ordered pair is (p, q), as p appears in the alphabet before q. This also means that a corresponding system of coordinates has the horizontal axis denoted as p-axis and the vertical axis denoted as q-axis.

Example 1 ► **Deciding Whether an Ordered Pair Is a Solution**

Decide whether the ordered pair $(3, 2)$ is a solution of the given system.

a. $\begin{cases} x + y = 5 \\ 4x - y = 10 \end{cases}$

b. $\begin{cases} m + 2n = 7 \\ 3m - n = 6 \end{cases}$

Solution ► **a.** To check whether the pair $(3, 2)$ is a solution, we let $x = 3$ and $y = 2$ in both equations of the system and check whether these equations are true. Since both equations,

$$\begin{aligned} 3 + 2 &= 5 \\ 5 &= 5 \quad ✓ \end{aligned} \qquad \text{and} \qquad \begin{aligned} 4 \cdot 3 - 2 &= 10 \\ 10 &= 10, \quad ✓ \end{aligned}$$

are true, then the pair $(3, 2)$ is a solution to the system.

b. First, we notice that alphabetically, m is before n. So, we let $m = 3$ and $n = 2$ and substitute these values into both equations.

$$3 + 2 \cdot 2 = 7 \qquad\qquad 3 \cdot 3 - 2 = 6$$
$$7 = 7 \ ✓ \qquad \text{but} \qquad 7 = 6 \ ✗$$

Since the pair $(3, 2)$ is not a solution of the second equation, it is not a solution to the whole system.

Solving Systems of Linear Equations by Graphing

Solutions to a system of two linear equations are all the ordered pairs that satisfy both equations. If an ordered pair satisfies an equation, then such a pair belongs to the graph of this equation. This means that the solutions to a system of two linear equations are the points that belong to both graphs of these lines. So, to solve such system, we can graph each line and take the common points as solutions.

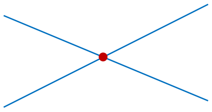


Figure 1a

How many solutions can a linear system of two equations have?

There are three possible arrangements of two lines in a plane. The lines can **intersect** each other, be **parallel**, or be the **same**.

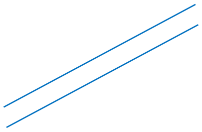


Figure 1b

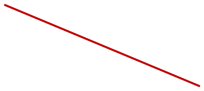


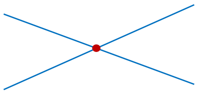
Figure 1c

1. If a system of equations corresponds to a pair of **intersecting** lines, it has exactly **one solution**. The solution set consists of the **intersection point**, as shown in *Figure 1a*.
2. If a system of equations corresponds to a pair of **parallel** lines, it has **no solutions**. The solution set is empty, as shown in *Figure 1b*.
3. If a system of equations corresponds to a pair of the **same** lines, it has **infinitely many solutions**. The solution set consists of all the **points of the line**, as shown in *Figure 1c*.

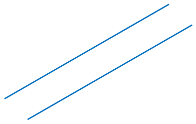
Definition 1.2 ► A linear system is called **consistent** if it has **at least one solution**. Otherwise, the system is **inconsistent**.

A linear system of two equations is called **independent** if the two lines are different. Otherwise, the system is **dependent**.

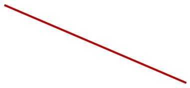
Here is the classification of systems corresponding to the following graphs:



consistent
independent



inconsistent
independent



consistent
dependent

Example 2 ► Solving Systems of Linear Equations by Graphing

Solve each system by graphing and classify it as *consistent or inconsistent* and *dependent or independent*.

a. $\begin{cases} 3p + q = 5 \\ p - 2q = 4 \end{cases}$ **b.** $\begin{cases} 3y - 2x = 6 \\ 4x - 6y = -12 \end{cases}$ **c.** $\begin{cases} f(x) = -\frac{1}{2}x + 3 \\ 2g(x) + x = -4 \end{cases}$

Solution ►

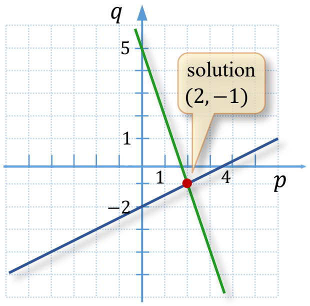
a. To graph the first equation, it is convenient to use the slope-intercept form,

$$q = -3p + 5.$$

To graph the second equation, it is convenient to use the p- and q-intercepts, $(4, 0)$ and $(0, -2)$.

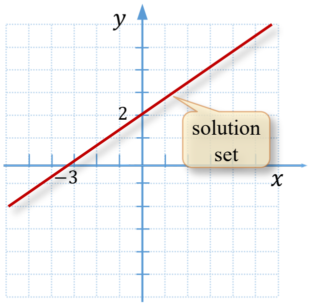
The first equation is graphed in green and the second – in blue. The intersection point is at $(\mathbf{2}, \mathbf{-1})$, which is the **only solution** of the system.

The system is **consistent**, as it has a solution, and **independent**, as the lines are different.



b. Notice that when using the x- and y-intercept method of graphing, both equations have x-intercepts equal to $(-3, 0)$ and y-intercepts equal to $(0, 2)$. So, both equations represent the same line. Therefore the solution set to this system consists of all points of the line $3y - 2x = 6$. We can record this set of points with the use of set-builder notation as $\{(\boldsymbol{x}, \boldsymbol{y}) | \mathbf{3}\boldsymbol{y} - \mathbf{2}\boldsymbol{x} = \mathbf{6}\}$, and state that the system has **infinitely many solutions**.

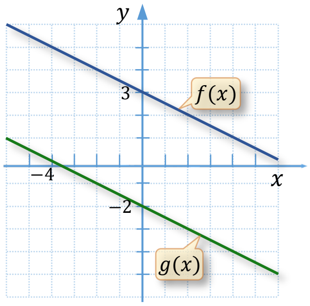
The system is **consistent**, as it has solutions, and **dependent**, as both lines are the same.



c. We plan to graph both functions, f and g, on the same grid. Function f is already given in the slope-intercept form, which is convenient for graphing. To graph function g, we can either use the x- and y-intercept method or solve the equation for $g(x)$ and use the slope-intercept method. So, we have

$$2g(x) = -x - 4$$

$$g(x) = -\frac{1}{2}x - 2$$



After graphing both functions, we observe that the two lines are parallel because they have the same slope. Having different y-intercepts, the lines do not have any common points. Therefore, the system has **no solutions**.

Such a system is **inconsistent**, as it has no solutions, and **independent**, as the lines are different.

Solving a system of equations by graphing, although useful, is not always a reliable method. For example, if the solution is an ordered pair of fractional numbers, we may have a hard time to read the exact values of these numbers from the graph. Luckily, a system of equations can be solved for exact values using algebra. Below, two algebraic methods for solving systems of two equations, referred to as substitution and elimination, are shown.

Solving Systems of Linear Equations by the Substitution Method

In the substitution method, as shown in *Example 3* below, we eliminate a variable from one equation by substituting an expression for that variable from the other equation. This method is particularly suitable for solving systems in which one variable is either already isolated or is easy enough to isolate.

Example 3 ► Solving Systems of Linear Equations by Substitution

Solve each system by substitution.

a. $\begin{cases} x = y + 1 \\ x + 2y = 4 \end{cases}$ **b.** $\begin{cases} 3a - 2b = 6 \\ 6a + 4b = -20 \end{cases}$

Solution ► **a.** Since x is already isolated in the first equation, $x = y + 1$, we replace x by $y + 1$ in the second equation, $x + 2y = 4$. Thus,

$$(y + 1) + 2y = 4$$

which after solving for y, gives us

$$3y = 3$$
$$\boldsymbol{y} = \mathbf{1}$$

Then, we substitute the value $y = 1$ back into the first equation, and solve for x. This gives us $\boldsymbol{x} = 1 + 1 = \mathbf{2}$.

One can check that $x = 2$ and $y = 1$ satisfy both of the original equations. So, the solution set of this system is $\{(\mathbf{2}, \mathbf{1})\}$.

b. To use the substitution method, we need to solve one of the equations for one of the variables, whichever is easier. Out of the coefficients by the variables, -2 seems to be the easiest coefficient to work with. So, let us solve the first equation, $3a - 2b = 6$, for b.

$$3a - 2b = 6$$
$$-2b = -3a + 6$$

substitution equation

$$b = \frac{3}{2}a - 3 \qquad (*)$$

Then, substitute the expression $\frac{3}{2}a - 3$ to the second equation, $6a + 4b = -20$, for b. So, we obtain

$$6b + 4\left(\frac{3}{2}a - 3\right) = -20,$$

which can be solved for a:

equation in one variable

$$6a + 6a - 12 = -20$$
$$12a = -8$$

$$a = -\frac{8}{12}$$
$$\boldsymbol{a = -\frac{2}{3}}$$

Then, we plug $a = -\frac{2}{3}$ back into the substitution equation $(*)$ $b = \frac{3}{2}a - 3$ to find the b-value. This gives us

$$\boldsymbol{b} = \frac{3}{2}\left(-\frac{2}{3}\right) - 3 = -1 - 3 = \boldsymbol{-4}$$

To check that the values $a = -\frac{2}{3}$ and $b = -4$ satisfy both equations of the system, we substitute them into each equation, and simplify each side. Since both equations,

$$3\left(-\frac{2}{3}\right) - 2(-4) = 6 \qquad \text{and} \qquad 6\left(-\frac{2}{3}\right) + 4(-4) = -20$$
$$-2 + 8 = 6 \qquad\qquad -4 - 16 = -20$$
$$6 = 6 \ \checkmark \qquad\qquad -20 = -20, \ \checkmark$$

are satisfied, the solution set of this system is $\left\{\left(-\frac{2}{3}, -4\right)\right\}$.

Summary of Solving Systems of Linear Equations by Substitution

Step 1 **Solve one of the equations for one of the variables.** Choose to solve for the variable with the easiest coefficient to work with. The obtained equation will be referred to as the **substitution equation** $(*)$.

Step 2 **Plug the substitution equation into the other equation.** The result should be an equation with just one variable.

Step 3 **Solve** the resulting equation to find the value of the variable.

Step 4 **Find the value of the other variable** by substituting the result from Step 3 into the substitution equation from Step 1.

Step 5 **Check** if the variable values satisfy both of the original equations. Then **state the solution set** by listing the ordered pair(s) of numbers.

Solving Systems of Linear Equations by the Elimination Method

Another algebraic method, the **elimination method**, involves combining the two equations in a system so that one variable is eliminated. This is done using the addition property of equations.

Recall: If $\boldsymbol{a = b}$ and $\boldsymbol{c = d}$, then $\boldsymbol{a + c = b + d}$.

Example 4 ► **Solving Systems of Linear Equations by Elimination**

Solve each system by elimination.

a. $\begin{cases} r + 2s = 3 \\ 3r - 2s = 5 \end{cases}$ **b.** $\begin{cases} 2x + 3y = 6 \\ 3x + 5y = -2 \end{cases}$

Solution ► **a.** Notice that the equations contain opposite terms, $2s$ and $-2s$. Therefore, if we add these equations, side by side, the s-variable will be eliminated. So, we obtain

$$+\begin{cases} r + 2s = 3 \\ 3r - 2s = 5 \end{cases}$$
$$\overline{\quad 4r = 8 \quad}$$
$$r = 2$$

Now, since the r-value is already known, we can substitute it to one of the equations of the system to find the s-value. Using the first equation, we obtain

$$2 + 2s = 3$$
$$2s = 1$$
$$s = \frac{1}{2}$$

One can check that the values $r = 2$ and $s = \frac{1}{2}$ make both equations of the original system true. Therefore, the pair $(\mathbf{2}, \frac{1}{2})$ is the solution of this system. We say that the solution set is $\left\{(\mathbf{2}, \frac{1}{2})\right\}$.

b. First, we choose which variable to eliminate. Suppose we plan to remove the x-variable. To do this, we need to transform the equations in such a way that the coefficients in the x-terms become opposite. This can be achieved by multiplying, for example, the first equation by 3 and the second equation, by -2.

$$\begin{cases} 2x + 3y = 6 & \text{multiply 1}^{\text{st}}\text{ eq by 3} \\ 3x + 5y = -2 & \text{multiply 2}^{\text{nd}}\text{ eq by } -2 \end{cases}$$

$$+\begin{cases} 6x + 9y = 18 \\ -6x - 10y = 4 \end{cases}$$

Then, we add the two equations, side by side,

$$-y = 22$$

and solve the resulting equation for y,

$$y = -22.$$

To find the x-value, we substitute $y = -22$ to one of the original equations. Using the first equation, we obtain

$$2x + 3(-22) = 6$$
$$2x - 66 = 6$$
$$2x = 72$$
$$x = 36$$

One can check that the values $x = 36$ and $y = -22$ make both equations of the original system true. Therefore, the solution of this system is the pair $(\mathbf{36}, \mathbf{-22})$. We say that the solution set is $\{(\mathbf{36}, \mathbf{-22})\}$.

Summary of Solving Systems of Linear Equations by Elimination

- **Write both equations in standard form $Ax + By = C$.** Keep A and B as integers by clearing any fractions, if needed.
- **Choose a variable to eliminate.**
- **Make the chosen variable's terms opposites** by multiplying one or both equations by appropriate numbers if necessary.
- **Eliminate a variable by adding the respective sides of the equations** and then solve for the remaining variable.
- **Find the value of the other variable** by substituting the result from Step 4 into either of the original equations and solve for the other variable.
- **Check** if the variable values satisfy both of the original equations. Then **state the solution set** by listing the ordered pair(s) of numbers.

Comparing Methods of Solving Systems of Equations

When deciding which method to use, consider the suggestions in the table below.

Method	Strengths	Weaknesses
Graphical	• **Visualization.** The solutions can be "seen" and approximated.	• **Inaccuracy.** When solutions involve numbers that are not integers, they can only be approximated. • **Grid limitations.** Solutions may not appear on the part of the graph drawn.
Substitution	• **Exact solutions.** • Most convenient to use when a variable has a **coefficient of 1**.	• **Computations.** Often requires extensive computations with fractions.
Elimination	• **Exact solutions.** • Most convenient to use when all **coefficients** by variables are **different than 1**.	• **Preparation.** The method requires that the coefficients by one of the variables are opposite.

Solving Systems of Linear Equations in Special Cases

As it was shown in solving linear systems of equations by graphing, some systems have no solution or infinitely many solutions. The next example demonstrates how to solve such systems algebraically.

Example 5 ► **Solving Inconsistent or Dependent Systems of Linear Equations**

Solve each system algebraically.

a. $\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$ **b.** $\begin{cases} 2x - y = 3 \\ 6x - 3y = 9 \end{cases}$

Solution ► **a.** When trying to eliminate one of the variables, we might want to multiply the first equation by 2. This, however, causes both variables to be eliminated, resulting in

$$\begin{array}{r} + \begin{cases} 2x + 6y = 8 \\ -2x - 6y = 3 \end{cases} \\ \hline 0 = 11, \end{array}$$

parallel lines

which is *never true*. This means that there is no ordered pair (x, y) that would make this equation true. Therefore, there is **no solution** to this system. The solution set is $\emptyset$. The system is **inconsistent**, so the equations must describe **parallel lines**.

b. When trying to eliminate one of the variables, we might want to multiply the first equation by -3. This, however, causes both variables to be eliminated and we obtain

$$\begin{array}{r} + \begin{cases} -6x + 3y = -9 \\ \ \ 6x - 3y = 9 \end{cases} \\ \hline 0 = 0, \end{array}$$

same line

which is *always true*. This means that any x-value together with its corresponding y-value satisfy the system. Therefore, there are **infinitely many solutions** to this system. These solutions are all points of one of the equations. Therefore, the solution set can be recorded in set-builder notation, as

Read: **the set of all ordered pairs (x, y), such that $2x - y = 3$**

$$\{(\boldsymbol{x}, \boldsymbol{y}) | \mathbf{2}\boldsymbol{x} - \boldsymbol{y} = \mathbf{3}\}$$

Since the equations of the system are equivalent, they represent the same line. So, the system is **dependent**.

Summary of Special Cases of Linear Systems

If both variables are eliminated when solving a linear system of two equations, then the solution sets are determined as follows.

Case 1 If the resulting statement is **true**, there are **infinitely many solutions**. The system is ***consistent***, and the equations are ***dependent***.

Case 2 If the resulting statement is **false**, there is **no solution**. The system is ***inconsistent***, and the equations are ***independent***.

Another way of determining whether a system of two linear equations is inconsistent or dependent is by examining slopes and y-intercepts in the two equations.

Example 6 ► Using Slope-Intercept Form to Determine the Number of Solutions and the Type of System

For each system, determine the number of solutions and classify the system without actually solving it.

a. $\begin{cases} \frac{1}{2}x = \frac{1}{8}y + \frac{1}{4} \\ 4x - y = -2 \end{cases}$

b. $\begin{cases} 2x + 5y = 6 \\ 0.4x + y = 1.2 \end{cases}$

Solution ►

a. First, let us clear the fractions in the first equation by multiplying it by 8,

$$\begin{cases} 4x = y + 2 \\ 4x - y = -2 \end{cases},$$

and then solve each equation for y.

parallel lines

$$\begin{cases} 4x - 2 = y \\ 4x + 2 = y \end{cases}$$

Then, observe that the slopes in both equations are the same and equal to 4, but the y-intercepts are different, -2 and 2. The same slopes tell us that the corresponding lines are **parallel** while different y-intercepts tell us that the two lines are **different**. So, the system has **no solution**, which means it is **inconsistent**, and the lines are **independent**.

b. We will start by solving each equation for y. So, we have

$$\begin{cases} 2x + 5y = 6 \\ 0.4x + y = 1.2 \end{cases}$$

$$\begin{cases} 5y = -2x + 6 \\ y = -0.4x + 1.2 \end{cases}$$

same line

$$\begin{cases} y = -\frac{2}{5}x + \frac{6}{5} \\ y = -0.4x + 1.2 \end{cases}$$

Notice that $-\frac{2}{5} = -0.4$ and $\frac{6}{5} = 1.2$. Since the resulting equations have the same slopes and the same y-intercepts, they represent the same line. Therefore, the system has **infinitely many solutions**, which means it is **consistent**, and the lines are **dependent**.

E.1 Exercises

1. Describe the graph of a system of equations that has no solution.

Decide whether the given ordered pair is a solution of the given system.

2. $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$; $(5, 2)$

3. $\begin{cases} x + y = 1 \\ 2x - 3y = -8 \end{cases}$; $(-1, 2)$

4. $\begin{cases} p + 3q = 1 \\ 5p - q = -9 \end{cases}$; $(-2, 1)$

5. $\begin{cases} 2a + b = 3 \\ a - 2b = -9 \end{cases}$; $(-1, 5)$

Solve each system of equations ***graphically***. *Then, classify the system as* ***consistent*** *or* ***inconsistent*** *and* ***dependent*** *or* ***independent***.

6. $\begin{cases} 3x + y = 5 \\ x - 2y = 4 \end{cases}$

7. $\begin{cases} 3x + 4y = 8 \\ x + 2y = 6 \end{cases}$

8. $\begin{cases} f(x) = x - 1 \\ g(x) = -2x + 5 \end{cases}$

9. $\begin{cases} f(x) = -\frac{1}{4}x + 1 \\ g(x) = \frac{1}{2}x - 2 \end{cases}$

10. $\begin{cases} y - x = 5 \\ 2x - 2y = 10 \end{cases}$

11. $\begin{cases} 6x - 2y = 2 \\ 9x - 3y = -1 \end{cases}$

12. $\begin{cases} y = 3 - x \\ 2x + 2y = 6 \end{cases}$

13. $\begin{cases} 2x - 3y = 6 \\ 3y - 2x = -6 \end{cases}$

14. $\begin{cases} 2u + v = 3 \\ 2u = v + 7 \end{cases}$

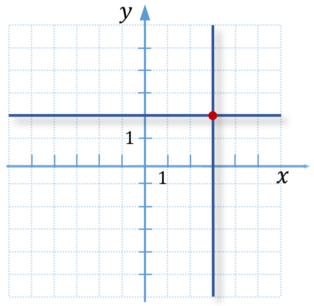
15. $\begin{cases} 2b = 6 - a \\ 3a - 2b = 6 \end{cases}$

16. $\begin{cases} f(x) = 2 \\ x = -3 \end{cases}$

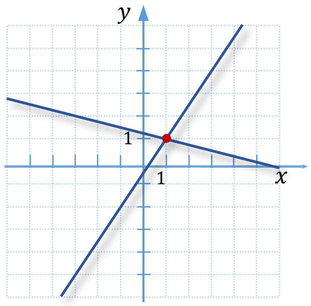
17. $\begin{cases} f(x) = x \\ g(x) = -1.5 \end{cases}$

18. Classify each system **I** to **VI** as *consistent* or *inconsistent* and the equations as *dependent* or *independent*. Then, match it with the corresponding system of equations **A** to **F**.

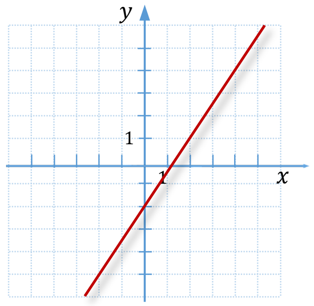
I



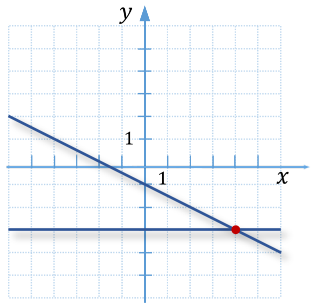
II



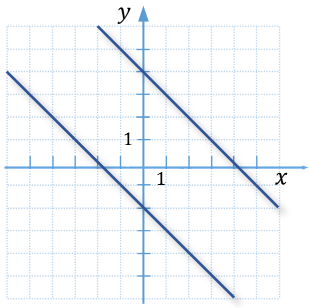
III



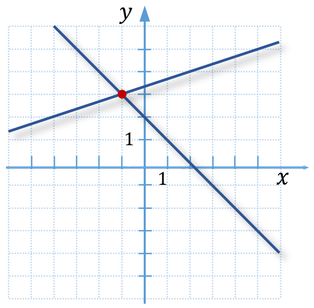
IV



V



VI



A $\begin{cases} 3y - x = 10 \\ x = -y + 2 \end{cases}$

B $\begin{cases} 9x - 6y = 12 \\ y = \frac{3}{2}x - 2 \end{cases}$

C $\begin{cases} 2y - 3x = -1 \\ x + 4y = 5 \end{cases}$

D $\begin{cases} x + y = 4 \\ y = -x - 2 \end{cases}$

E $\begin{cases} \frac{1}{2}x + y = -1 \\ y = -3 \end{cases}$

F $\begin{cases} x = 3 \\ y = 2 \end{cases}$

*Solve each system by **substitution**. If the system describes **parallel lines** or the **same line**, say so.*

19. $\begin{cases} y = 2x + 1 \\ 3x - 4y = 1 \end{cases}$

20. $\begin{cases} 5x - 6y = 23 \\ x = 6 - 3y \end{cases}$

21. $\begin{cases} x + 2y = 3 \\ 2x + y = 5 \end{cases}$

22. $\begin{cases} 2y = 1 - 4x \\ 2x + y = 0 \end{cases}$

23. $\begin{cases} y = 4 - 2x \\ y + 2x = 6 \end{cases}$

24. $\begin{cases} y - 2x = 3 \\ 4x - 2y = -6 \end{cases}$

25. $\begin{cases} 4s - 2t = 18 \\ 3s + 5t = 20 \end{cases}$

26. $\begin{cases} 4p + 2q = 8 \\ 5p - 7q = 1 \end{cases}$

27. $\begin{cases} \frac{x}{2} + \frac{y}{2} = 5 \\ \frac{3x}{2} - \frac{2y}{3} = 2 \end{cases}$

28. $\begin{cases} \frac{x}{4} + \frac{y}{3} = 0 \\ \frac{x}{8} - \frac{y}{6} = 2 \end{cases}$

29. $\begin{cases} 1.5a - 0.5b = 8.5 \\ 3a + 1.5b = 6 \end{cases}$

30. $\begin{cases} 0.3u - 2.4v = -2.1 \\ 0.04u + 0.03v = 0.7 \end{cases}$

*Solve each system by **elimination**. If the system describes **parallel lines** or the **same line**, say so.*

31. $\begin{cases} x + y = 20 \\ x - y = 4 \end{cases}$

32. $\begin{cases} 6x + 5y = -7 \\ -6x - 11y = 1 \end{cases}$

33. $\begin{cases} x - y = 5 \\ 3x + 2y = 10 \end{cases}$

34. $\begin{cases} x - 4y = -3 \\ -3x + 5y = 2 \end{cases}$

35. $\begin{cases} 2x + 3y = 1 \\ 3x - 5y = -8 \end{cases}$

36. $\begin{cases} -2x + 5y = 14 \\ 7x + 6y = -2 \end{cases}$

37. $\begin{cases} 2x + 3y = 1 \\ 4x + 6y = 2 \end{cases}$

38. $\begin{cases} 6x - 10y = -4 \\ 5y - 3x = 7 \end{cases}$

39. $\begin{cases} 0.3x - 0.2y = 4 \\ 0.2x + 0.3y = 1 \end{cases}$

40. $\begin{cases} \frac{2}{3}x + \frac{1}{7}y = -11 \\ \frac{1}{7}x - \frac{1}{3}y = -10 \end{cases}$

41. $\begin{cases} 3a + 2b = 3 \\ 9a - 8b = -2 \end{cases}$

42. $\begin{cases} 5m - 9n = 7 \\ 7n - 3m = -5 \end{cases}$

43. Can a linear system of two equations has exactly two solutions? Justify your answer.

*Write each equation in **slope-intercept form** and then tell how many solutions the system has. Do not actually solve.*

44. $\begin{cases} -x + 2y = 8 \\ 4x - 8y = 1 \end{cases}$

45. $\begin{cases} 6x = -9y + 3 \\ 2x = -3y + 1 \end{cases}$

46. $\begin{cases} y - x = 6 \\ x + y = 6 \end{cases}$

Solve each system by the method of your choice.

47. $\begin{cases} 3x + y = -7 \\ x - y = -5 \end{cases}$

48. $\begin{cases} 3x - 2y = 0 \\ 9x + 8y = 7 \end{cases}$

49. $\begin{cases} 3x - 5y = 7 \\ 2x + 3y = 30 \end{cases}$

50. $\begin{cases} 2x + 3y = 10 \\ -3x + y = 18 \end{cases}$

51. $\begin{cases} \frac{1}{6}x + \frac{1}{3}y = 8 \\ \frac{1}{4}x + \frac{1}{2}y = 30 \end{cases}$

52. $\begin{cases} \frac{1}{2}x - \frac{1}{8}y = -\frac{1}{2} \\ 4x - y = -2 \end{cases}$

53. $\begin{cases} a + 4b = 2 \\ 5a - b = 3 \end{cases}$

54. $\begin{cases} 3a - b = 7 \\ 2a + 2b = 5 \end{cases}$

55. $\begin{cases} 6 \cdot f(x) = 2x \\ -7x + 15 \cdot g(x) = 10 \end{cases}$

*Solve the system of linear equations. Assume that **a** and **b** represent nonzero constants.*

56. $\begin{cases} x + ay = 1 \\ 2x + 2ay = 4 \end{cases}$

57. $\begin{cases} -ax + y = 4 \\ ax + y = 4 \end{cases}$

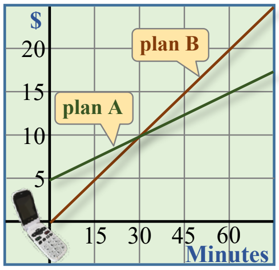
58. $\begin{cases} -ax + y = 2 \\ ax + y = 4 \end{cases}$

59. $\begin{cases} ax + by = 2 \\ -ax + 2by = 1 \end{cases}$

60. $\begin{cases} 2ax - y = 3 \\ y = 5ax \end{cases}$

61. $\begin{cases} 3ax + 2y = 1 \\ -ax + y = 2 \end{cases}$

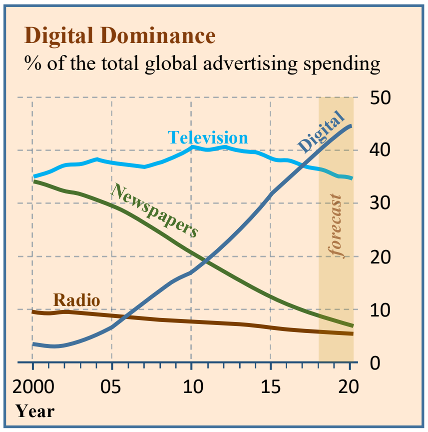
Refer to the accompanying graph to answer questions 72-73.



62. According to the graph, for how many long-distance minutes the charge would be the same in plan A as in plan B? Give an ordered pair of the form (minutes, dollars) to represent this situation.

63. For what range of long-distance minutes would plan B be cheaper?

Refer to the accompanying graph to answer questions 74-77.


Digital Dominance

% of the total global advertising spending

64. According to the predictions shown in the graph, what percent of the global advertising spending will be allocated to digital advertising in 2020? Give an ordered pair of the form (year, percent) to represent this information.

65. Estimate the year in which spending on digital advertising matches spending on television advertising.

66. Since when did spending on digital advertising exceed spending on advertising in newspapers? What was this spending as a percentage of global advertising spending at that time?

67. Since when did spending on digital advertising exceed spending on radio advertising? What was this spending as a percentage of global advertising spending at that time?

E2 Applications of Systems of Linear Equations in Two Variables

Systems of equations are frequently used in solving applied problems. Although many problems with two unknowns can be solved with the use of a single equation with one variable, it is often easier to translate the information given in an application problem with two unknowns into two equations in two variables.

Here are some guidelines to follow when solving applied problems with two variables.

Solving Applied Problems with the Use of System of Equations

- **Read** the problem, several times if necessary. When reading, watch the given information and what the problem asks for. Recognize the type of problem, such as geometry, total value, motion, solution, percent, investment, etc.
- **Assign variables** for the unknown quantities. Use meaningful letters, if possible.
- **Organize** the given information. Draw appropriate tables or diagrams; list relevant formulas.
- **Write equations** by following a relevant formula(s) or a common sense pattern.
- **Solve** the system of equations.
- **Check** if the solution is reasonable in the context of the problem.
- **State the answer** to the problem.

Below we show examples of several types of applied problems that can be solved with the aid of systems of equations.

Number Relations Problems

Example 1 ► **Finding Unknown Numbers**

The difference between twice a number and a second number is 3. The sum of the two numbers is 18. Find the two numbers.

Solution ► Let a be the first number and b be the second number. The first sentence of the problem translates into the equation

$$2a - b = 3.$$

The second sentence translates to

$$a + b = 18.$$

Now, we can solve the system of the above equations, using the elimination method. Since

$$+\begin{cases} 2a - b = 3 \\ \underline{a + b = 18} \end{cases}$$

$$3a = 21$$

$$a = \mathbf{7},$$

then $b = 18 - a = 18 - 7 = \mathbf{11}$. Therefore, the two numbers are **7** and **11**.

Observation: *A single equation in two variables gives us infinitely many solutions. For example, some of the solutions of the equation* $a + b = 16$ *are* $(0, 16), (1, 15), (2, 14)$, *and so on. Generally, any ordered pair of the type* $(a, 16 - a)$ *is the solution to this equation. So, when working with two variables, to find a specific solution we are in need of a second equation (not equivalent to the first) that relates these variables. This is why problems with two unknowns are solved with the use of systems of two equations.*

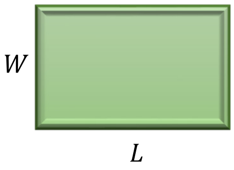
Geometry Problems

When working with geometry problems, we often use formulas for perimeter, area, or volume of basic figures. Sometimes, we rely on particular properties or theorems, such as *the sum of angles in a triangle is* 180° or *the ratios of corresponding sides of similar triangles are equal*.

Example 2 ► Finding Dimensions of a Rectangle

Pat plans a rectangular vegetable garden. The width of the rectangle is to be 5 meters shorter than the length. If the perimeter is planned to be 34 meters, what will the dimensions of the garden be?

Solution ►



The problem refers to the perimeter of a rectangular garden. Suppose L and W represent the length and width of the rectangle. Then the perimeter is represented by the expression $2L + 2W$. Since the perimeter of the garden should equal 34 meters, we set up the first equation

$$2L + 2W = 34 \qquad (1)$$

The second equation comes directly from translating the second sentence of the problem, which tells us that the width is to be 5 meters shorter than the length. So, we write

$$W = L - 5 \qquad (2)$$

Now, we can solve the system of the above equations, using the substitution method. After substituting equation (2) into equation (1), we obtain

$$2L + 2(L - 5) = 34$$
$$2L + 2L - 10 = 34$$
$$4L = 44$$
$$L = \mathbf{11}$$

So, $W = L - 5 = 11 - 5 = \mathbf{6}$.

Therefore, the garden is **11 meters** long and **6 meters** wide.

Number-Value Problems

Problems that refer to the number of different types of items and the value of these items are often solved by setting two equations. Either one equation compares the number of items, and the other compares the value of these items, like in coin types of problems, or both equations compare the values of different arrangements of these items.

Example 3 ► Finding the Number of Each Type of Items

A restoration company purchased 45 paintbrushes, some at $7.99 each and some at $9.49 each. If the total charge before tax was $379.05, how many of each type were purchased?

Solution ► Let x represent the number of brushes at $7.99 each, and let y represent the number of brushes at $9.49 each. Then the value of x brushes at $7.99 each is $7.99x$. Similarly, the value of y brushes at $9.49 each is $9.49y$. To organize the given information, we suggest to create and complete the following table.

	brushes at $7.99 each +	brushes at $9.49 each	= Total
number of brushes	x	y	45
value of brushes (in $)	$7.99x$	$9.49y$	379.05

Since we work with two variables, we need two different equations in these variables. The first equation comes from comparing the number of brushes, as in the middle row. The second equation comes from comparing the values of these brushes, as in the last row. So, we have the system

$$\begin{cases} x + y = 45 \\ 7.99x + 9.49y = 379.05 \end{cases}$$

to solve. This can be solved by substitution. From the first equation, we have $y = 45 - x$, which after substituting to the second equation gives us

$$\begin{aligned} 7.99x + 9.49(45 - x) &= 379.05 \\ 799x + 949(45 - x) &= 37905 \\ 799x + 42705 - 949x &= 37905 \\ -150x &= -4800 \\ x &= \mathbf{32} \end{aligned}$$

Then, $y = 45 - x = 45 - 32 = \mathbf{13}$.

Therefore, the restoration company purchased **32** brushes at $7.99 each and **13** brushes at $9.49 each.

Example 4 ► Finding the Unit Cost of Each Type of Items

The cost of 48 ft of red oak and 72 ft of fibreboard is $271.20. At the same prices, 32 ft of red oak and 60 ft of fibreboard cost $200. Find the unit price of red oak and fibreboard.

Solution ► Let r be the unit price of red oak and let f be the unit price of fibreboard. Then the value of 48 ft of red oak is represented by $48r$, and the value of 72 ft of fibreboard is represented by $72f$. Using the total cost of $271.20, we write the first equation

$$48r + 72f = 271.20$$

Similarly, using the total cost of $200 for 32 ft of red oak and 60 ft of fibreboard, we write the second equation

$$32r + 60f = 200$$

We will solve the system of the above equations via the elimination method.

$$\begin{cases} 48r + 72f = 271.20 \\ 32r + 60f = 200 \end{cases}$$

$$+\underline{\begin{cases} -96r - 144f = -542.40 \\ 96r + 180f = 600 \end{cases}} \quad 36f = 57.40$$

$$f = \mathbf{1.60}$$

After substituting $f = 1.60$ into the second equation, we obtain

$$\begin{aligned} 32r + 60 \cdot 1.60 &= 200 \\ 32r + 96 &= 200 \\ 32r &= 104 \\ r &= \mathbf{3.25} \end{aligned}$$

So, red oak costs **3.25 $/ft**, and fibreboard costs **1.60 $/ft**.

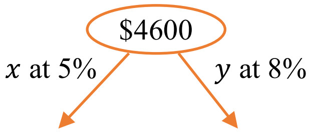
Investment Problems

Investment problems usually involve calculation of simple annual interest according to the formula $\boldsymbol{I = Prt}$, where I represents the amount of interest, P represents the principal which is the amount of investment, r represents the annual interest rate in decimal form, and t represents the time in years.

Example 5 ► **Finding the Amount of Each Loan**

Marven takes two student loans totaling $4600 to cover the cost of his yearly tuition. One loan is at 5% simple interest, and the other is at 8% simple interest. If his total annual interest charge is $278, find the amount of each loan.

Solution ► Suppose the amount of loan taken at 5% is x, and the amount of loan taken at 8% is y. The situation can be visualized by the diagram



one year interest: $0.05x \quad + \quad 0.08y \quad = 278$

So, the system of equations to solve is

$$\begin{cases} x + y = 4600 \\ 0.05x + 0.08y = 278 \end{cases}$$

Using the substitution method, we solve the first equation for x,

$$x = 4600 - y, \qquad (*)$$

substitute it into the second equation,

decimal elimination is optional

$$0.05(4600 - y) + 0.08y = 278,$$

and after elimination of decimals via multiplication by 100,

$$5(4600 - y) + 8y = 27800,$$

solve it for y:

$$23000 - 5y + 8y = 27800$$
$$3y = 4800$$
$$y = \mathbf{1600}$$

Then, after plugging in the y-value to the substitution equation $(*)$, we obtain

$$x = 4600 - 1600 = \mathbf{3000}$$

So, the amount of loan taken at 5% is $\mathbf{\$3000}$, and the amount of loan taken at 8% is $\mathbf{\$1600}$.

Mixture – Solution Problems

In mixture or solution problems, we typically mix two or more mixtures or solutions with different concentrations of a particular substance that we will refer to as the content. For example, if we are interested in the salt concentration in salty water, the salt is referred to as the content. When solving mixture problems, it is helpful to organize data in a table such as the one shown below.

	% ·	amount =	content
type I			
type II			
final mixture			

Example 6 ► **Solving a Mixture Problem**

Olivia wants to prepare 3 kg mixture of nuts and dried fruits that contains 30% of cranberries by mixing a blend that is 10% cranberry with a blend that is 40% cranberry. How much of each type of blend should she use to obtain the desired mixture?

Solution ► Suppose x is the amount of the 10% blend and y is the amount of the 40% blend. We complete the table

	% ·	amount (kg) =	cranberries (kg)

10% blend	0.1	x	$0.1x$
40% blend	0.4	y	$0.4y$
30% blend	0.3	3	0.9

The last column is completed by multiplying the "%" and "amount" columns.

The first equation comes from combining the weight of blends as shown in the "amount" column. The second equation comes from combining the weight of cranberries, as indicated in the last column. So, we solve

$$\begin{cases} x + y = 3 \\ 0.1x + 0.4y = 0.9 \end{cases}$$

Using the substitution method, we solve the first equation for x,

$$x = 3 - y, \qquad (*)$$

then substitute it into the second equation and solve for y:

$$\begin{aligned} 0.1(3 - y) + 0.4y &= 0.9 \\ 3 - y + 4y &= 9 \\ 3 + 3y &= 9 \\ 3y &= 6 \\ y &= \mathbf{2} \end{aligned}$$

Then, using the substitution equation $(*)$, we find the value of x:

$$x = 3 - 2 = \mathbf{1}$$

So, to obtain the desired blend, Olivia should mix **1 kg** of 10% and **2 kg** of 40% blend.

Motion Problems

In motion problems, we follow the formula $\boldsymbol{Rate} \cdot \boldsymbol{Time} = \boldsymbol{Distance}$. Drawing a diagram and completing a table based on the $\boldsymbol{R} \cdot \boldsymbol{T} = \boldsymbol{D}$ formula is usually helpful. In some motion problems, in addition to the rate of the moving object itself, we need to consider the rate of a moving medium such as water current or wind. The overall rate of a moving object is typically either the sum or the difference between the object's own rate and the rate of the moving medium.

Example 7 ► Finding Rates in a Motion Problem

A motorcycle travels 280 km in the same time that a car travels 245 km. If the motorcycle moves 14 kilometers per hour faster than the car, find the speed of each vehicle.

Solution ► Using meaningful letters, let m and c represent the speed of the motorcycle and the car, respectively. Since the speed of the motorcycle, m, is 14 km/h faster than the speed of the car, c, we can write the first equation:

$$m = c + 14$$

The second equation comes from comparing the travel time of each vehicle, as indicated in the table below.

	R ·	T	= D
motorcycle	m	$\frac{280}{m}$	280
car	c	$\frac{245}{c}$	245

To find the expression for time, we follow the formula $T = \frac{D}{R}$, which comes from solving $R \cdot T = D$ for T.

So, we need to solve the system

$$\begin{cases} m = c + 14 \\ \frac{280}{m} = \frac{245}{c} \end{cases}$$

Cross-multiplication can **only** be applied to a **proportion** (an equation with a single fraction on each side.)

Notice that multiplication by the **LCD** would give the same result.

After substituting the first equation into the second, we obtain

$$\frac{280}{c+14} = \frac{245}{c},$$

which can be solved by cross-multiplying

$$280c = 245(c + 14)$$
$$280c = 245c + 3430$$
$$35c = 3430$$
$$c = \mathbf{98}$$

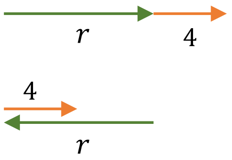
Then, we use this value to find $m = c + 14 = 98 + 14 = \mathbf{112}$.

So, the speed of the motorcycle is **112 km/h**, and the speed of the car is **98 km/h**.

Example 8 ► **Solving a Motion Problem with a Current**

A motorboat trip upstream a river takes 6 hours, while the return trip takes only 2 hours. Assuming the constant current of 4 mph, find the speed of the boat in still water.

Solution ►



Let r be the speed of the boat in still water.
Then the speed of the boat moving downstream is 4 mph faster because of the current going in the same direction as the boat. So, it is represented by $r + 4$.
The speed of the boat moving against the current is 4 mph slower. So, it is represented by $r - 4$.
Also, let d represent the distance covered by the boat going in one direction.

To organize the information, we can complete the table below.

	R ·	T	= D
downstream	$r + 4$	2	d
upstream	$r - 4$	6	d

The two equations come from following the formula $R \cdot T = D$, as indicated in each row.

$$\begin{cases}(r+4)\cdot 2 = d \\ (r-4)\cdot 6 = d\end{cases}$$

Since the left sides of both equations represent the same distance d, we can equal them and solve for r:

$$(r+4)\cdot 2 = (r-4)\cdot 6$$
$$2r + 8 = 6r - 24$$
$$32 = 4r$$
$$\mathbf{8} = r$$

So, the speed of the boat in still water was **8 mph**.

E.2 Exercises

1. If a barrel contains 50 liters of 16% alcohol wine, what is the volume of pure alcohol there?

2. If \$3000 is invested into bonds paying 4.5% simple annual interest, how much interest is expected in a year?

3. If a kilogram of baked chicken breast costs \$9.25, how much would n kilograms of this meat cost?

4. A ticket to a concert costs \$18. Write an expression representing the revenue from selling n such tickets.

5. A canoe that moves at r km/h in still water encounters c km/h current. Write an expression that would represent the speed of this canoe moving **a.** with the current (*downstream*), **b.** against the current (upstream).

6. A helicopter moving at r km/h in still air encounters a wind that blows at w km/h. Write an expression representing the rate of the helicopter travelling **a.** with the wind (*tailwind*), **b.** against the wind (*headwind*).

Solve each problem using two variables.

7. The larger out of two complementary angles is 6° more than three times the smaller one. Find the measure of each angle. (*Recall:* complementary angles add to 90°)

8. The larger out of two supplementary angles is 5° more than four times the smaller one. Find the measure of each angle. (*Recall:* supplementary angles add to 180°)

9. The sum of the height and the base of a triangle is 231 centimeters. The height is half the base. Find the base and height.

10. A tennis court is 13 meters longer than it is wide. What are the dimensions of the court if its perimeter is 72 meters.

11. A marathon is a run that covers about 42 km. In 2017, Mary Keitany won the "Women Only" marathon in London. During this marathon, at a particular moment, she was five times as far from the start of the course as she was from its end. At that time, how many kilometers had she already run?

12. Jane asked her students to find the two numbers that she had in mind. She told them that the smaller number is 2 more than one-third of the larger number and that three times the larger number is 1 less than eight times the smaller one. Find the numbers.

13. During the 2014 Winter Olympics in Sochi, Russia won a total of 33 medals. There were 4 more gold medals than bronze ones. If the number of silver medals was the average of the number of gold and the number of bronze medals, how many of each type of medal did Russia earn?

14. A 156 cm long piece of wire is cut into two pieces. Then, each piece is bent to make an equilateral triangle. If the side of one triangle is twice as long as the side of the other triangle, how should the wire be cut?

15. A bistro cafe sells espresso and cappuccino cups of coffee. One day, the cafe has sold 452 cups of coffee. How many cups of each type of coffee the shop has sold that day if the number of cappuccino cups was 3 times as large as the number of espresso cups of coffee sold?

16. During an outdoor festival, a retail booth was selling solid-colour scarfs for \$8.75 each and printed scarfs for \$11.49 each. If \$478.60 was collected for selling 50 scarfs, how many of each type were sold?

17. The Mission Folk Music Festival attracts local community every summer since 1988. In recent years, a one-day admission to this festival costs \$45 for an adult ticket and \$35 for a youth ticket. If the total of \$24,395 were collected from the sale of 605 tickets, how many of each type of tickets were sold?

18. Ellena's total GED score in Mathematics and Science was 328. If she scored 24 points higher in Mathematics than in Science, what were her GED scores in each subject?

19. One day, at his food stand, Tom sold 12 egg salad sandwiches and 18 meat sandwiches, totaling \$101.70. The next day he sold 23 egg salad sandwiches and 9 meat sandwiches, totaling \$93.18. How much did each type of sandwich cost?

20. At lunchtime, a group of conference members ordered three cappuccinos and four expressos for a total of \$20.07. Another group ordered two cappuccinos and three expressos for a total of \$14.43. How much did each type of coffee cost?

21. New York City and Paris are one of the most expensive cities to live in. Based on the average weekly cost of living in each city (not including accommodation), 2 weeks in New York and 3 weeks in Paris cost \$1852, while 4 weeks in New York and 2 weeks in Paris cost \$2344. Find the average weekly cost of living in each city?

22. One of the local storage facilities rents two types of storage lockers, a small one with 180 ft^2 in area, and a large one with the area of 600 ft^2. In total, the facility has 42 storage lockers that provide 15,120 ft^2 of the overall storage area. How many of each type of storage lockers does the facility have?

23. Ryan took two student loans for a total of \$4800. One of these loans was borrowed at 3.25% simple interest and the other one at 2.75%. If after one year Ryan's overall interest charge for both of the loans was \$143.50, what was the amount of each loan?

24. An investor made two investments totaling $36,000. In one year, these investments generated $1650 in simple interest. If the interest rate for the two investments were 5% and 3.75%, how much was invested at each rate?

25. A stockbroker invested some money in a low-risk fund and twice as much in a high-risk fund. In a year, the low-risk fund earned 3.7%, and the high-risk fund lost 8.2%. If the two investments resulted in the overall loss of $111.20, how much was invested in each fund?

26. Patricia's bank offers her two types of investments, one at 4.5% and the other one at 6.25% simple interest. Patricia invested $1500 more at 6.25% than at 4.5%. How much was invested at each rate if the total interest accumulated after one year was $462.25?

27. How many liters of 4% brine and 20% brine should be mixed to obtain 12 liters of 8% brine?

28. Sam has $11 in dimes and quarters. If he counted 71 coins in all, how many of each type of coin are there?

29. Cottage cheese contains 12% of protein and 6% of carbs while vanilla yogurt is 5% protein and 15% carbs. How many grams of each product should be used to serve a meal that contains 10 grams of protein and 10 grams of carbs?

30. Kidney beans contain 24% protein while lima beans contain just 8% protein. How many dekagrams of each type of bean should be used to prepare 60 dekagrams of a bean-mix that is 12% protein?

31. Cezary purchased a shirt costing $42.75 with a $50 bill. The cashier gave him the change in quarters and loonies. If the change consisted of 14 coins, how many of each kind were there?

32. When travelling with the current, a speedboat covers 24 km in half an hour. It takes 40 minutes for the boat to cover the same distance against the current. Find the rate of the boat in still water and the rate of the current.

33. A houseboat travelling with the current went 45 km in 3 hours. It took 2 hours longer to travel the same distance against the current. Find the rate of the houseboat in still water and the rate of the current.

34. When flying with the wind, a passenger plane covers a distance of 1760 km in 2 hours. When flying against the same wind, the plane covers 2400 km in 3 hours. Determine the rate of the plane in still air and the rate of the wind.

35. Flying with a tailwind, a pilot of a small plane could cover the distance of 1500 km between two cities in 5 hours. Flying with the headwind of the same intensity, he would need 6 hours to cover the same distance. Find the rate of the plane in still air and the rate of the wind.

36. Robert kayaked 10 km downstream a river in 2 hours. When returning, Robert could kayak only 6 km in the same amount of time. What was his rate of kayaking in still water and what was the rate of the current?

37. A small private plane flying into a wind covered 1080 km in 4 hours. When flying back, with a tailwind of the same intensity, the plane needed only 3 hours to cover the same distance. What are the rate of the plane in still air and the rate of the wind?

38. Flight time against a headwind for a trip of 2300 kilometers is 4 hours. If the headwind were half as great, the same flight would take 10 minutes less time. Find the rate of the wind and the rate of the plane in still air.

39. Teresa was late for her conference presentation after driving at an average speed of 60 km/h. If she had driven 4 km/h faster, her travelling time would be half an hour shorter. How far was the conference?

40. Two vehicles leave a gas station at the same time and travel in the same direction. One travels at 96 km/h and the other at 108 km/h. The drivers of the two vehicles can communicate with each other with a short-distance radio device as long as they are within 10 km range. When will they lose this contact?

41. The windshield fluid tank in Izabella's car contains 5 L of 7% antifreeze. To the nearest tenths of a litre, how much of this mixture should be drained and replaced with pure antifreeze so that the mixture becomes 20% antifreeze?

42. Mike has two gallons of stain that is 10% brown and 90% neutral and a gallon of pure brown stain. To stain a deck, he needs 2 gallons of a stain that is 40% brown and 60% neutral. How much of each type of stain should he use to prepare the desired mix?

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Polynomials and Polynomial Functions

One of the simplest types of algebraic expressions is a polynomial. Polynomials are formed only by addition and multiplication of variables and constants. Since both addition and multiplication produce unique values for any given inputs, polynomials are in fact functions. Some of the simplest polynomial functions are linear functions, such as $P(x) = 2x + 1$, and quadratic functions, such as $Q(x) = x^2 + x - 6$. Due to their comparably simple form, polynomial functions appear in a wide variety of areas of mathematics and science, economics, business, and many other areas of life. Polynomial functions are often used to model various natural phenomena, such as the shape of a mountain, the distribution of temperature, the trajectory of projectiles, etc. The shape and properties of polynomial functions are helpful when constructing such structures as roller coasters or bridges, solving optimization problems, or even analysing stock market prices.

In this chapter, we will introduce polynomial terminology, perform operations on polynomials, and evaluate and compose polynomial functions.

P1 Addition and Subtraction of Polynomials

Terminology of Polynomials

Recall that products of constants, variables, or expressions are called **terms** (see *Section R3, Definition 3.1*). **Terms** that are **products** of only **numbers** and **variables** are called **monomials**. Examples of monomials are $-2x$, xy^2, $\frac{2}{3}x^3$, etc.

Definition 1.1 ► A **polynomial** is a sum of monomials.

A **polynomial** in a single variable is the sum of terms of the form ax^n, where a is a **numerical coefficient**, x is the variable, and n is a whole number.

An ***n*-th degree polynomial** in x-variable has the form

$$\boldsymbol{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0},$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$.

Note: A polynomial can always be considered as a sum of monomial terms even though there are negative signs when writing it.
For example, polynomial $x^2 - 3x - 1$ can be seen as the sum of signed terms

$$x^2 + -3x + -1$$

Definition 1.2 ► The **degree of a monomial** is the sum of exponents of all its variables.

For example, the degree of $5x^3y$ is 4, as the sum of the exponent of x^3, which is 3 and the exponent of y, which is 1. To record this fact, we write $\deg(5x^3y) = 4$.

The **degree of a polynomial** is the highest degree out of all its terms.

For example, the degree of $2x^2y^3 + 3x^4 - 5x^3y + 7$ is 5 because $\deg(2x^2y^3) = 5$ and the degrees of the remaining terms are not greater than 5.

Polynomials that are sums of two terms, such as $x^2 - 1$, are called **binomials**.
Polynomials that are sums of three terms, such as $x^2 + 5x - 6$ are called **trinomials**.

The **leading term** of a polynomial is the highest degree term.
The **leading coefficient** is the numerical coefficient of the leading term.
So, the leading term of the polynomial $1 - x - x^2$ is $-x^2$, even though it is not the first term. The leading coefficient of the above polynomial is -1, as $-x^2$ can be seen as $(-1)x^2$.

A first degree term is often referred to as a **linear term**. A second degree term can be referred to as a **quadratic term**. A zero degree term is often called a **constant** or a **free term**.

Below are the parts of an n-th degree polynomial in a single variable x:

$$\text{leading coefficient} \longrightarrow \underbrace{a_n x^n}_{\text{leading term}} + a_{n-1}x^{n-1} + \cdots + \underbrace{a_2 x^2}_{\text{quadratic term}} + \underbrace{a_1 x}_{\text{linear term}} + \underbrace{a_0}_{\text{constant (free) term}}$$

Note: Single variable polynomials are usually arranged in descending powers of the variable. Polynomials in more than one variable are arranged in decreasing degrees of terms. If two terms are of the same degree, they are arranged with respect to the descending powers of the variable that appers first in alphabetical order.

For example, polynomial $x^2 + x - 3x^4 - 1$ is customarily arranged as follows

$$-3x^4 + x^2 + x - 1,$$

while polynomial $3x^3y^2 + 2y^6 - y^2 + 4 - x^2y^3 + 2xy$ is usually arranged as below.

$$\underbrace{2y^6}_{\text{6th degree term}} \underbrace{+3x^3y^2 - x^2y^3}_{\text{5th degree terms arranged with respect to } x} \underbrace{+2xy - y^2}_{\text{2nd degree terms arranged with respect to } x} \underbrace{+4}_{\text{zero degree term}}$$

Example 1 ► Writing Polynomials in Descending Order and Identifying Parts of a Polynomial

Suppose $P = x - 6x^3 - x^6 + 4x^4 + 2$ and $Q = 2y - 3xyz - 5x^2 + xy^2$. For each polynomial:

a. Write the polynomial in descending order.
b. State the degree of the polynomial and the number of its terms.
c. Identify the leading term, the leading coefficient, the coefficient of the linear term, the coefficient of the quadratic term, and the free term of the polynomial.

Solution ►

a. After arranging the terms in descending powers of x, polynomial P becomes

$$-x^6 + 4x^4 - 6x^3 + x + 2,$$

while polynomial Q becomes

$$xy^2 - 3xyz - 5x^2 + 2y.$$

Notice that the first two terms, xy^2 and $-3xyz$, are both of the same degree. So, to decide which one should be written first, we look at powers of x. Since these powers are again the same, we look at powers of y. This time, the power of y in xy^2 is higher than the power of y in $-3xyz$. So, the term xy^2 should be written first.

b. The polynomial P has **5 terms**. The highest power of x in P is 6, so the **degree** of the polynomial P is **6**.
The polynomial Q has **4 terms**. The highest degree terms in Q are xy^2 and $-3xyz$, both third degree. So the **degree** of the polynomial Q is **3**.

$x = 1 \cdot x$
$-x = (-1)x$

c. The leading term of the polynomial $P = -x^6 + 4x^4 - 6x^3 + x - 2$ is $-x^6$, so the **leading coefficient** equals $-\mathbf{1}$.
The linear term of P is x, so the **coefficient of the linear term** equals **1**.
P doesn't have any quadratic term so the coefficient of the quadratic term equals **0**.
The **free term** of P equals $-\mathbf{2}$.

The leading term of the polynomial $Q = xy^2 - 3xyz - 5x^2 + 2y$ is xy^2, so the **leading coefficient** is equal to **1**.
The linear term of Q is $2y$, so the **coefficient of the linear term** equals **2**.
The quadratic term of Q is $-5x^2$, so the **coefficient of the quadratic term** equals $-\mathbf{5}$.
The polynomial Q does not have a free term, so the **free term** equals **0**.

Example 2 ► **Classifying Polynomials**

Describe each polynomial as a *constant*, *linear*, *quadratic*, or *n-th degree* polynomial. Decide whether it is a *monomial*, *binomial*, or *trinomial*, if applicable.

a. $x^2 - 9$
b. $-3x^7y$
c. $x^2 + 2x - 15$
d. π
e. $4x^5 - x^3 + x - 7$
f. $x^4 + 1$

Solution ►

a. $x^2 - 9$ is a second degree polynomial with two terms, so it is a **quadratic binomial**.

b. $-3x^7y$ is an **8-th degree monomial**.

c. $x^2 + 2x - 15$ is a second degree polynomial with three terms, so it is a **quadratic trinomial**.

d. π is a 0-degree term, so it is a **constant monomial**.

e. $4x^5 - x^3 + x - 7$ is a **5-th degree polynomial**.

f. $x^4 + 1$ is a **4-th degree binomial**.

Polynomials as Functions and Evaluation of Polynomials

Each term of a polynomial in one variable is a product of a number and a power of the variable. The polynomial itself is either one term or a sum of several terms. Since taking a power of a given value, multiplying, and adding given values produce unique answers, polynomials are also functions. While f, g, or h are the most commonly used letters to represent functions, other letters can also be used. To represent polynomial functions, we customarily use capital letters, such as P, Q, R, etc.

Any polynomial function P of degree n, has the form

$$\boldsymbol{P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$, and $n \in \mathbb{W}$.

Since polynomials are functions, they can be evaluated for different x-values.

Example 3 ► Evaluating Polynomials

Given $P(x) = 3x^3 - x^2 + 4$, evaluate the following expressions:

a.	$P(0)$	**b.**	$P(-1)$
c.	$2 \cdot P(1)$	**d.**	$P(a)$

Solution ►

a. $P(0) = 3 \cdot 0^3 - 0^2 + 4 = \mathbf{4}$

b. $P(-1) = 3 \cdot (-1)^3 - (-1)^2 + 4 = 3 \cdot (-1) - 1 + 4 = -3 - 1 + 4 = \mathbf{0}$

When evaluating at negative x-values, it is essential to use brackets in place of the variable before substituting the desired value.

c. $2 \cdot P(1) = 2 \cdot \underbrace{(3 \cdot 1^3 - 1^2 + 4)}_{this\ is\ P(1)} = 2 \cdot (3 - 1 + 4) = 2 \cdot 6 = \mathbf{12}$

d. To find the value of $P(a)$, we replace the variable x in $P(x)$ with a. So, this time the final answer,

$$P(a) = \mathbf{3a^3 - a^2 + 4},$$

is an expression in terms of a rather than a specific number.

Since polynomials can be evaluated at any real x-value, then the **domain** (*see Section G3, Definition 5.1*) of any polynomial is the set $\mathbb{R}$ of all real numbers.

Addition and Subtraction of Polynomials

Recall that terms with the same variable part are referred to as **like terms** (see *Section R3, Definition 3.1*). Like terms can be **combined** by adding their coefficients. For example,

$$\underbrace{2x^2y - 5x^2y = (2-5)x^2y}_{\substack{by\ distributive\ property \\ (factoring)}} = -3x^2y$$

In practice, this step is not necessary to write.

Unlike terms, such as $2x^2$ and $3x$, **cannot be combined**.

Example 4 ► Simplifying Polynomial Expressions

Simplify each polynomial expression.

a. $5x - 4x^2 + 2x + 7x^2$ **b.** $8p - (2 - 3p) + (3p - 6)$

Solution ► **a.** To simplify $5x - 4x^2 + 2x + 7x^2$, we combine like terms, starting from the highest degree terms. It is suggested to underline the groups of like terms, using different type of underlining for each group, so that it is easier to see all the like terms and not to miss any of them. So,

$$\underline{\underline{5x}}\ \underline{-4x^2}\ \underline{\underline{+2x}}\ \underline{+7x^2} = \mathbf{3x^2 + 7x}$$

Remember that the sign in front of a term belongs to this term.

b. To simplify $8p - (2 - 3p) + (3p - 6)$, first we remove the brackets using the distributive property of multiplication and then we combine like terms. So, we have

$-(2 - 3p)$
$= (-1)(2 - 3p)$

$$8p - (2 - 3p) + (3p - 6)$$

$$= \underline{8p}\ - 2\ \underline{+3p}\ \underline{+3p} - 6$$

$$= \mathbf{14p - 8}$$

Example 5 ► Adding or Subtracting Polynomials

Perform the indicated operations.

a. $(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$
b. $(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$
c. $[9p - (3p - 2)] - [4p - (3 - 7p) + p]$

Solution ► **a.** To add polynomials, combine their like terms. So,

remove any bracket preceeded by a "+" sign

$$(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$$

$$= 6a^5 - 4a^3 \underline{\underline{+3a}}\ \underset{\sim}{-1} + 2a^4 + a^2 \underline{\underline{-5a}}\ \underset{\sim}{+9}$$

$$= \mathbf{6a^5 + 2a^4 - 4a^3 + a^2 - 2a + 8}$$

b. To subtract a polynomial, add its opposite. In practice, remove any bracket preceeded by a negative sign by reversing the signs of all the terms of the polynomial inside the bracket. So,

$$(4y^3 - 3y^2 + y + 6) - (\,y^3 + 3y - 2)$$

$$= \underline{4y^3} - 3y^2 \underline{\underline{+y}}\ \underset{\sim}{+6}\ \underline{-y^3}\ \underline{\underline{-3y}}\ \underset{\sim}{+2}$$

$$= \mathbf{3y^3 - 3y^2 - 2y + 8}$$

To remove a bracket preceeded by a "−" sign, reverse each sign inside the bracket.

c. First, perform the operations within the square brackets and then subtract the resulting polynomials. So,

$$
\begin{aligned}
&[9p-(3p-2)]-[4p-(3-7p)+p] \\
&=[9p-3p+2]-[4p-3+7p+p] \\
&=[6p+2]-[12p-3] \\
&=6p+2-12p+3 \\
&=\boldsymbol{-6p+5}
\end{aligned}
$$

collect like terms before removing the next set of brackets

Addition and Subtraction of Polynomial Functions

Similarly as for polynomials, addition and subtraction can also be defined for general functions.

Definition 1.3 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **sum function $\boldsymbol{f+g}$** is defined as

$$(\boldsymbol{f}+\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})$$

and the **difference function $\boldsymbol{f-g}$** is defined as

$$(\boldsymbol{f}-\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the sum or difference function is the intersection $\boldsymbol{D_f \cap D_g}$ of the domains of the two functions.

A frequently used application of a sum or difference of polynomial functions comes from the business area. The fact that profit P equals revenue R minus cost C can be recorded using function notation as

$$P(x)=(R-C)(x)=R(x)-C(x),$$

where x is the number of items produced and sold. Then, if $R(x)=6.5x$ and $C(x)=3.5x+900$, the profit function becomes

$$\boldsymbol{P(x)}=R(x)-C(x)=6.5x-(3.5x+900)=6.5x-3.5x-900=\boldsymbol{3x-900}.$$

Example 6 ► **Adding or Subtracting Polynomial Functions**

Suppose $P(x)=x^2-6x+4$ and $Q(x)=2x^2-1$. Find the following:

a. $(P+Q)(x)$ and $(P+Q)(2)$
b. $(P-Q)(x)$ and $(P-Q)(-1)$
c. $(P+Q)(k)$
d. $(P-Q)(2a)$

Solution ► **a.** Using the definition of the sum of functions, we have

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{x}) = P(x)+Q(x) = \underbrace{x^2-6x+4}_{P(x)} + \underbrace{2x^2-1}_{Q(x)} = \boldsymbol{3x^2-6x+3}$$

Therefore, $(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{2}) = 3\cdot 2^2 - 6\cdot 2 + 3 = 12 - 12 + 3 = \boldsymbol{3}$.

Alternatively, $(P+Q)(2)$ can be calculated without refering to the function $(P+Q)(x)$, as shown below.

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{2}) = P(2)+Q(2) = \underbrace{2^2-6\cdot 2+4}_{P(2)} + \underbrace{2\cdot 2^2-1}_{Q(2)}$$

$$= 4-12+4+8-1 = \boldsymbol{3}.$$

b. Using the definition of the difference of functions, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{x}) = P(x)-Q(x) = \underbrace{x^2-6x+4}_{P(x)} - \underbrace{(2x^2-1)}_{Q(x)}$$

$$= x^2-6x+4-2x^2+1 = \boldsymbol{-x^2-6x+5}$$

To evaluate $(P-Q)(-1)$, we will take advantage of the difference function calculated above. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{-1}) = -(-1)^2 - 6(-1) + 5 = -1+6+5 = \boldsymbol{10}.$$

c. By *Definition 1.3*,

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = P(k)+Q(k) = k^2-6k+4+2k^2-1 = \boldsymbol{3k^2-6k+3}$$

Alternatively, we could use the sum function already calculated in the solution to *Example 6a*. Then, the result is instant: $(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = \boldsymbol{3k^2-6k+3}$.

d. To find $(P-Q)(2a)$, we will use the difference function calculated in the solution to *Example 6b*. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{2a}) = -(2a)^2 - 6(2a) + 5 = \boldsymbol{-4a^2-12a+5}.$$

P.1 Exercises

Determine whether the expression is a monomial.

1. $-\pi x^3y^2$

2. $5x^{-4}$

3. $5\sqrt{x}$

4. $\sqrt{2}x^4$

Identify the degree and coefficient.

5. xy^3

6. $-x^2y$

7. $\sqrt{2}xy$

8. $-3\pi x^2y^5$

Arrange each polynomial in descending order of powers of the variable. Then, identify the degree and the leading coefficient of the polynomial.

9. $5 - x + 3x^2 - \frac{2}{5}x^3$

10. $7x + 4x^4 - \frac{4}{3}x^3$

11. $8x^4 + 2x^3 - 3x + x^5$

12. $4y^3 - 8y^5 + y^7$

13. $q^2 + 3q^4 - 2q + 1$

14. $3m^2 - m^4 + 2m^3$

State the degree of each polynomial and identify it as a monomial, binomial, trinomial, or n-th degree polynomial if $n > 2$.

15. $7n - 5$

16. $4z^2 - 11z + 2$

17. 25

18. $-6p^4q + 3p^3q^2 - 2pq^3 - p^4$

19. $-mn^6$

20. $16k^2 - 9p^2$

Let $P(x) = -2x^2 + x - 5$ *and* $Q(x) = 2x - 3$. *Evaluate each expression.*

21. $P(-1)$

22. $P(0)$

23. $2P(1)$

24. $P(a)$

25. $Q(-1)$

26. $Q(5)$

27. $Q(a)$

28. $Q(3a)$

29. $3Q(-2)$

30. $3P(a)$

31. $3Q(a)$

32. $Q(a+1)$

Simplify each polynomial expression.

33. $5x + 4y - 6x + 9y$

34. $4x^2 + 2x - 6x^2 - 6$

35. $6xy + 4x - 2xy - x$

36. $3x^2y + 5xy^2 - 3x^2y - xy^2$

37. $9p^3 + p^2 - 3p^3 + p - 4p^2 + 2$

38. $n^4 - 2n^3 + n^2 - 3n^4 + n^3$

39. $4 - (2 + 3m) + 6m + 9$

40. $2a - (5\mathrm{a} - 3) - (7a - 2)$

41. $6 + 3x - (2x + 1) - (2x + 9)$

42. $4y - 8 - (-3 + y) - (11y + 5)$

Perform the indicated operations.

43. $(x^2 - 5y^2 - 9z^2) + (-6x^2 + 9y^2 - 2z^2)$

44. $(7x^2y - 3xy^2 + 4xy) + (-2x^2y - xy^2 + xy)$

45. $(-3x^2 + 2x - 9) - (x^2 + 5x - 4)$

46. $(8y^2 - 4y^3 - 3y) - (3y^2 - 9y - 7y^3)$

47. $(3r^6 + 5) + (-7r^2 + 2r^6 - r^5)$

48. $(5x^{2a} - 3x^a + 2) + (-x^{2a} + 2x^a - 6)$

49. $(-5a^4 + 8a^2 - 9) - (6a^3 - a^2 + 2)$

50. $(3x^{3a} - x^a + 7) - (-2x^{3a} + 5x^{2a} - 1)$

51. $(10xy - 4x^2y^2 - 3y^3) - (-9x^2y^2 + 4y^3 - 7xy)$

52. Subtract $(-4x + 2z^2 + 3m)$ from the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$.

53. Subtract the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$ from $(-4x + 2z^2 + 3m)$.

54. $[2p - (3p - 6)] - \big[\big(5p - (8 - 9p)\big) + 4p\big]$

55. $-[3z^2 + 5z - (2z^2 - 6z)] + \big[\big(8z^2 - (5z - z^2)\big) + 2z^2\big]$

56. $5k - (5k - [2k - (4k - 8k)]) + 11k - (9k - 12k)$

For each pair of functions, find ***a)*** $(f+g)(x)$ *and* ***b)*** $(f-g)(x)$.

57. $f(x) = 5x - 6, \quad g(x) = -2 + 3x$

58. $f(x) = x^2 + 7x - 2, \quad g(x) = 6x + 5$

59. $f(x) = 3x^2 - 5x, \quad g(x) = -5x^2 + 2x + 1$

60. $f(x) = 2x^n - 3x - 1, \quad g(x) = 5x^n + x - 6$

61. $f(x) = 2x^{2n} - 3x^n + 3, \quad g(x) = -8x^{2n} + x^n - 4$

Let $\boldsymbol{P(x) = x^2 - 4}$, $\boldsymbol{Q(x) = 2x + 5}$, *and* $\boldsymbol{R(x) = x - 2}$. *Find each of the following.*

62. $(P+R)(-1)$

63. $(P-Q)(-2)$

64. $(Q-R)(3)$

65. $(R-Q)(0)$

66. $(R-Q)(k)$

67. $(P+Q)(a)$

68. $(Q-R)(a+1)$

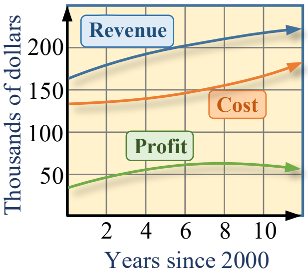
69. $(P+R)(2k)$

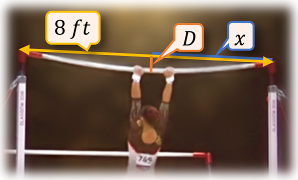
Solve each problem.

70. Suppose that during the years 2000-2012 the revenue R and the cost C of a particular business are modelled by the polynomials

$$R(t) = -0.296t^2 + 9.72t + 164 \text{ and } C(t) = 0.154t^2 + 2.15t + 135,$$

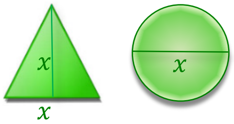
where t represents the number of years since 2000 and both $R(t)$ and $C(t)$ are in thousands of dollars. Write a polynomial that models the profit $P(t)$ of this business during the years 2000-2012.





71. Suppose that the deflection D of an 8 feet-long gymnastic bar can be approximated by the polynomial function $D(x) = 0.037x^4 - 0.59x^3 + 2.35x^2$, where x is the distance in feet from one end of the bar and D is in centimeters. To the nearest tenths of a centimeter, determine the maximum deflection for this bar, assuming that it occurs at the middle of the bar.

72. Write a polynomial that can be used to calculate the sum of areas of a triangle with the base and height of length x and a circle with diameter x. Determine the total area of the two shapes for $x = 5$ centimeters. *Round the answer to the nearest centimeter square.*



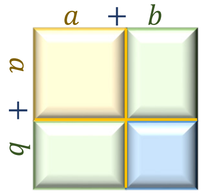
73. Suppose the cost in dollars of sewing n dresses is given by $C(n) = 32n + 1500$. If the dresses can be sold for \$56 each, complete the following.

a. Write a function $R(n)$ that gives the revenue for selling n dresses.

b. Write a formula $P(n)$ for the profit. *Recall that profit is defined as the difference between revenue and cost.*

c. Evaluate $P(100)$ and interpret the answer.

P2 Multiplication of Polynomials



As shown in the previous section, addition and subtraction of polynomials results in another polynomial. This means that the **set of polynomials** is **closed under** the operation of **addition** and **subtraction**. In this section, we will show that the set of polynomials is also closed under the operation of **multiplication**, meaning that a product of polynomials is also a polynomial.

Properties of Exponents

Since multiplication of polynomials involves multiplication of powers, let us review properties of exponents first.

Recall:

power exponent

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$$

base

For example, $x^4 = x \cdot x \cdot x \cdot x$ and we read it "x to the fourth power" or shorter "x to the fourth". If $n = 2$, the power x^2 is customarily read "x squared". If $n = 3$, the power x^3 is often read "x cubed".

Let $a \in \mathbb{R}$, and $m, n \in \mathbb{W}$. The table below shows basic exponential rules with some examples justifying each rule.

Power Rules for Exponents

General Rule	Description	Example
$a^m \cdot a^n = a^{m+n}$	To **multiply powers** of the same bases, keep the base and **add** the **exponents**.	$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x)$ $= x^{2+3} = x^5$
$\frac{a^m}{a^n} = a^{m-n}$	To **divide powers** of the same bases, keep the base and **subtract** the **exponents**.	$\frac{x^5}{x^2} = \frac{(x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x})}{(\cancel{x} \cdot \cancel{x})}$ $= x^{5-2} = x^3$
$(a^m)^n = a^{mn}$	To raise a **power to a power, multiply the exponents.**	$(x^2)^3 = (x \cdot x)(x \cdot x)(x \cdot x)$ $= x^{2 \cdot 3} = x^6$
$(ab)^n = a^n b^n$	To raise a **product to a power**, raise **each factor to that power**.	$(2x)^3 = 2^3 x^3$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	To raise a **quotient to a power**, raise the **numerator and the denominator to that power**.	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2}$
$a^0 = 1$ for $a \neq 0$ 0^0 is undefined	A **nonzero number** raised **to the power of zero** equals **one**.	$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1$

Example 1 ► Simplifying Exponential Expressions

Simplify.

a. $(-3xy^2)^4$ **b.** $(5p^3q)(-2pq^2)$

c. $\left(\frac{-2x^5}{x^2y}\right)^3$ **d.** $x^{2a}x^a$

Solution ► **a.** To simplify $(-3xy^2)^4$, we apply the fourth power to each factor in the bracket. So,

$$(-3xy^2)^4 = \underbrace{(-3)^4}_{\substack{\text{even power}\\ \text{of a negative}\\ \text{is a positive}}} \cdot x^4 \cdot \underbrace{(y^2)^4}_{\substack{\text{multiply}\\ \text{exponents}}} = \mathbf{3^4x^4y^8}$$

$p = p^1$

b. To simplify $(5p^3q)(-2pq^2)$, we multiply numbers, powers of p, and powers of q. So,

$$(5p^3q)(-2pq^2) = (-2) \cdot 5 \cdot \underbrace{p^3 \cdot p}_{\substack{\text{add}\\ \text{exponents}}} \cdot \underbrace{q \cdot q^2}_{\substack{\text{add}\\ \text{exponents}}} = \mathbf{-10p^4q^3}$$

c. To simplify $\left(\frac{-2x^5}{x^2y}\right)^3$, first we reduce the common factors and then we raise every factor of the numerator and denominator to the third power. So, we obtain

$$\left(\frac{-2x^{\overset{3}{\cancel{5}}}}{\cancel{x^2}y}\right)^3 = \left(\frac{-2x^3}{y}\right)^3 = \frac{(-2)^3(x^3)^3}{y^3} = \frac{\mathbf{-8x^9}}{\mathbf{y^3}}$$

d. When multiplying powers with the same bases, we add exponents, so $x^{2a}x^a = \mathbf{x^{3a}}$

Multiplication of Polynomials

Multiplication of polynomials involves finding products of monomials. To multiply monomials, we use the commutative property of multiplication and the product rule of powers.

Example 2 ► Multiplying Monomials

Find each product.

a. $(3x^4)(5x^3)$ **b.** $(5b)(-2a^2b^3)$ **c.** $-4x^2(3xy)(-x^2y)$

Solution ► **a.** $(3x^4)(5x^3) = 3 \cdot \underbrace{x^4 \cdot 5}_{\substack{\text{commutative}\\ \text{property}}} \cdot x^3 = 3 \cdot 5 \cdot \underbrace{x^4 \cdot x^3}_{\substack{\text{product}\\ \text{rule of powers}}} = \mathbf{15x^7}$

The intermediate steps are not necessary to write. The final answer is immediate if we follow the order: **sign**, **number**, **power** of each variable.

To find the product of monomials, find the following:

- the final **sign**,
- the **number**,
- the **power**.

b. $(5b)(-2a^2b^3) = 5(-2)a^2bb^3 = \mathbf{-10a^2b^4}$

c. $-4x^2(3xy)(-x^2y) = \underbrace{(-4) \cdot 3 \cdot (-1)}_{\substack{\text{multiply}\\ \text{coefficients}}} \underbrace{x^2xx^2}_{\substack{\text{apply product}\\ \text{rule of powers}}} \underbrace{yy} = \mathbf{12x^5y^2}$

To multiply polynomials by a monomial, we use the distributive property of multiplication.

Example 3 ► **Multiplying Polynomials by a Monomial**

Find each product.

a. $-2x(3x^2 - x + 7)$ **b.** $(5b - ab^3)(3ab^2)$

Solution ►

a. To find the product $-2x(3x^2 - x + 7)$, we distribute the monomial $-2x$ to each term inside the bracket. So, we have

$$-2x(3x^2 - x + 7) = \underbrace{-2x(3x^2) - 2x(-x) - 2x(7)}_{\textit{this step can be done mentally}} = \boldsymbol{-6x^3 + 2x^2 - 14x}$$

b. $$(5b - ab^3)(3ab^2) = \underbrace{5b(3ab^2) - ab^3(3ab^2)}_{\textit{this step can be done mentally}}$$

$$= 15ab^3 - 3a^2b^5 = \boldsymbol{-3a^2b^5 + 15ab^3}$$

arranged in decreasing order of powers

When multiplying polynomials by polynomials we **multiply each term of the first polynomial by each term of the second polynomial**. This process can be illustrated with finding areas of a rectangle whose sides represent each polynomial. For example, we multiply $(2x + 3)(x^2 - 3x + 1)$ as shown below

	x^2	$-3x$	$+1$
$2x$	$2x^3$	$-6x^2$	$2x$
$+3$	$3x^2$	$-9x$	3

So, $$\begin{array}{rrrrr} (2x+3)(x^2-3x+1) = & 2x^3 & -6x^2 & +2x & \\ & & +3x^2 & -9x & +3 \\ \hline = & \mathbf{2x^3} & \mathbf{-3x^2} & \mathbf{-7x} & \mathbf{+3} \end{array}$$

line up like terms to combine them

Example 4 ► **Multiplying Polynomials by Polynomials**

Find each product.

a. $(3y^2 - 4y - 2)(5y - 7)$ **b.** $4a^2(2a - 3)(3a^2 + a - 1)$

Solution ►

a. To find the product $(3y^2 - 4y - 2)(5y - 7)$, we can distribute the terms of the second bracket over the first bracket and then collect the like terms. So, we have

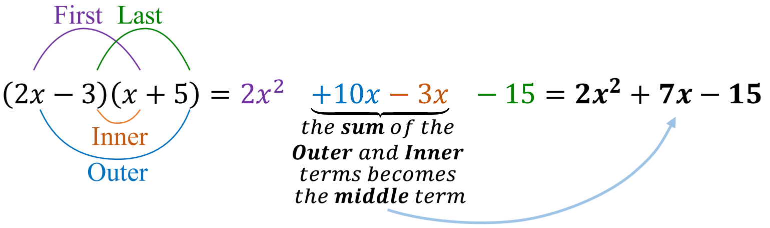
$$\begin{array}{rrrrr} (3y^2-4y-2)(5y-7) = & 15y^3 & -20y^2 & -10y & \\ & & -21y^2 & +28y & +14 \\ \hline = & \mathbf{15y^3} & \mathbf{-41y^2} & \mathbf{+18y} & \mathbf{+14} \end{array}$$

b. To find the product $4a^2(2a - 3)(3a^2 + a - 1)$, we will multiply the two brackets first, and then multiply the resulting product by $4a^2$. So,

$$4a^2(2a-3)(3a^2+a-1) = 4a^2\underbrace{\left(6a^3 \underline{+2a^2} \underline{\underline{-2a}} \underline{-9a^2} \underline{\underline{-3a}} +3\right)}_{\text{collect like terms before removing the bracket}}$$

$$= 4a^2(6a^3-7a^2-5a+3) = \mathbf{24a^5 - 28a^4 - 20a^3 + 12a^2}$$

In multiplication of binomials, it might be convenient to keep track of the multiplying terms by following the **FOIL** mnemonic, which stands for multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials. Here is how it works:



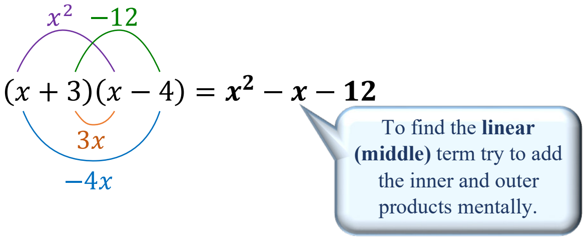
$$(2x-3)(x+5) = 2x^2 \underbrace{+10x - 3x}_{\substack{\text{the sum of the}\\ \text{Outer and Inner}\\ \text{terms becomes}\\ \text{the middle term}}} - 15 = \mathbf{2x^2 + 7x - 15}$$

Example 5 ► **Using the FOIL Method in Binomial Multiplication**

Find each product.

a. $(x+3)(x-4)$ **b.** $(5x-6)(2x+3)$

Solution ► **a.** To find the product $(x+3)(x-4)$, we may follow the **FOIL** method



$$(x+3)(x-4) = \mathbf{x^2 - x - 12}$$

To find the **linear (middle)** term try to add the inner and outer products mentally.

b. Observe that the linear term of the product $(5x-6)(2x+3)$ is equal to the sum of $-12x$ and $15x$, which is $3x$. So, we have

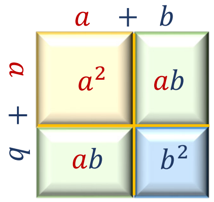
$(5x-6)(2x+3) = \mathbf{10x^2 + 3x - 18}$

Special Products

Suppose we want to find the product $(a+b)(a+b)$. This can be done via the FOIL method

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b}) = a^2 + ab + ab + b^2 = \mathbf{a^2 + 2ab + b^2},$$

or via the geometric visualization:



Observe that since the products of the inner and outer terms are both equal to ab, we can use a shortcut and write the middle term of the final product as $2ab$. We encorage the reader to come up with similar observations regarding the product $(a-b)(a-b)$. This regularity in multiplication of a binomial by itself leads us to the **perfect square formula**:

$$\boxed{\boldsymbol{(a \pm b)^2 = a^2 \pm 2ab + b^2}}$$

In the above notation, the " ± " sign is used to record two formulas at once, the perfect square of a sum and the perfect square of a difference. It tells us to either use a " + " in both places, or a " − " in both places. The ***a*** and ***b*** can actually represent any expression. For example, to expand $(2x-y^2)^2$, we can apply the perfect square formula by treating $2x$ as a and y^2 as b. Here is the calculation.

$$(2x-y^2)^2 = (2x)^2 - 2(2x)y^2 + (y^2)^2 = \boldsymbol{4x^2 - 4xy^2 + y^4}$$

Conjugate binomials have the **same first terms** and **opposite second terms**.

Another interesting pattern can be observed when multiplying two **conjugate** brackets, such as $(a+b)$ and $(a-b)$. Using the FOIL method,

$$\boldsymbol{(a+b)(a-b)} = a^2 + \cancel{ab} - \cancel{ab} - b^2 = \boldsymbol{a^2 - b^2},$$

we observe that the products of the inner and outer terms are opposite. So, they add to zero and the product of conjugate brackets becomes the difference of squares of the two terms. This regularity in multiplication of conjugate brackets leads us to the **difference of squares formula**.

$$\boxed{\boldsymbol{(a+b)(a-b) = a^2 - b^2}}$$

Again, ***a*** and ***b*** can represent any expression. For example, to find the product $(3x+0.1y^2)(3x-0.1y^2)$, we can apply the difference of squares formula by treating $3x$ as a and $0.1y^2$ as b. Here is the calculation.

$$\boldsymbol{(3x+0.1y^3)(3x-0.1y^3)} = (3x)^2 - (0.1y^3)^2 = \boldsymbol{9x^2 - 0.01y^6}$$

We encourage the use of the above formulas whenever applicable, as it allows for more efficient calculations and helps to observe patterns useful in future factoring.

Example 6 ►

Using Special Product Formulas in Polynomial Multiplication

Find each product. Apply special products formulas, if applicable.

a. $(5x+3y)^2$

b. $(x+y-5)(x+y+5)$

Solution ►

a. Applying the perfect square formula, we have

$$(5x+3y)^2 = (5x)^2 + 2(5x)3y + (3y)^2 = \boldsymbol{25x^2 + 30xy + 9y^2}$$

b. The product $(x+y-5)(x+y+5)$ can be found by multiplying each term of the first polynomial by each term of the second polynomial, using the distributive property. However, we can find the product $(x+y-5)(x+y+5)$ in a more efficient way by applying the difference of squares formula. Treating the expression $x+y$ as the first term a and the 5 as the second term b in the formula $(a+b)(a-b)=a^2-b^2$, we obtain

$$\begin{aligned}(x+y-5)(x+y+5) &= (x+y)^2-5^2\\ &= \underbrace{\boldsymbol{x^2+2xy+y^2}}_{\substack{\text{here we apply}\\ \text{the perfect square}\\ \text{formula}}}\ \boldsymbol{-\,25}\end{aligned}$$

Caution: *The perfect square formula shows that* $(\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{2}} \neq \boldsymbol{a}^{\boldsymbol{2}}+\boldsymbol{b}^{\boldsymbol{2}}$.
The difference of squares formula shows that $(\boldsymbol{a}-\boldsymbol{b})^{\boldsymbol{2}} \neq \boldsymbol{a}^{\boldsymbol{2}}-\boldsymbol{b}^{\boldsymbol{2}}$.
More generally, $(\boldsymbol{a}\pm\boldsymbol{b})^{\boldsymbol{n}} \neq \boldsymbol{a}^{\boldsymbol{n}}\pm\boldsymbol{b}^{\boldsymbol{n}}$ *for any natural* $n \neq 1$.

Product Functions

The operation of multiplication can be defined not only for polynomials but also for general functions.

Definition 2.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **product function**, denoted $\boldsymbol{f\cdot g}$ or $\boldsymbol{fg}$, is defined as

$$(\boldsymbol{f\cdot g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the product function is the intersection $\boldsymbol{D_f\cap D_g}$ of the domains of the two functions.

Example 7 ► **Multiplying Polynomial Functions**

Suppose $P(x)=x^2-4x$ and $Q(x)=3x+2$. Find the following:

a. $(PQ)(x)$, $(PQ)(-2)$, and $P(-2)Q(-2)$
b. $(QQ)(x)$ and $(QQ)(1)$
c. $2(PQ)(k)$

Solution ► **a.** Using the definition of the product function, we have

$$\begin{aligned}(PQ)(x) = P(x)\cdot Q(x) = (x^2-4x)(3x+2) &= 3x^3+2x^2-12x^2-8x\\ &= \boldsymbol{3x^3-10x^2-8x}\end{aligned}$$

To find $(PQ)(-2)$, we substitute $x=-2$ to the above polynomial function. So,

$$\begin{aligned}(PQ)(-2) = 3(-2)^3-10(-2)^2-8(-2) &= 3\cdot(-8)-10\cdot 4+16\\ &= -24-40+16 = \boldsymbol{-48}\end{aligned}$$

To find $P(-2)Q(-2)$, we calculate

$$\begin{aligned}P(-2)Q(-2) = \left((-2)^2-4(-2)\right)(3(-2)+2) = (4+8)(-6+2) &= 12\cdot(-4)\\ &= \boldsymbol{-48}\end{aligned}$$

Observe that both expressions result in the same value. This was expected, as by the definition, $(PQ)(-2) = P(-2) \cdot Q(-2)$.

b. Using the definition of the product function as well as the perfect square formula, we have

$$(QQ)(x) = Q(x) \cdot Q(x) = [Q(x)]^2 = (3x+2)^2 = \mathbf{9x^2 + 12x + 4}$$

Therefore, $(QQ)(1) = 9 \cdot 1^2 + 12 \cdot 1 + 4 = 9 + 12 + 4 = \mathbf{25}$.

c. Since $(PQ)(x) = 3x^3 - 10x^2 - 8x$, as shown in the solution to *Example 7a*, then $(PQ)(k) = 3k^3 - 10k^2 - 8k$. Therefore,

$$2(PQ)(k) = 2[3k^3 - 10k^2 - 8k] = \mathbf{6k^3 - 20k^2 - 16k}$$

P.2 Exercises

1. Decide whether each expression has been simplified correctly. If not, correct it.

a. $x^2 \cdot x^4 = x^8$

b. $-2x^2 = 4x^2$

c. $(5x)^3 = 5^3x^3$

d. $-\left(\frac{x}{5}\right)^2 = -\frac{x^2}{25}$

e. $(a^2)^3 = a^5$

f. $4^5 \cdot 4^2 = 16^7$

g. $\frac{6^5}{3^2} = 2^3$

h. $xy^0 = 1$

i. $(-x^2y)^3 = -x^6y^3$

Simplify each expression.

2. $3x^2 \cdot 5x^3$

3. $-2y^3 \cdot 4y^5$

4. $3x^3(-5x^4)$

5. $2x^2y^5(7xy^3)$

6. $(6t^4s)(-3t^3s^5)$

7. $(-3x^2y)^3$

8. $\frac{12x^3y}{4xy^2}$

9. $\frac{15x^5y^2}{-3x^2y^4}$

10. $(-2x^5y^3)^2$

11. $\left(\frac{4a^2}{b}\right)^3$

12. $\left(\frac{-3m^4}{n^3}\right)^2$

13. $\left(\frac{-5p^2q}{pq^4}\right)^3$

14. $3a^2(-5a^5)(-2a)^0$

15. $-3a^3b(-4a^2b^4)(ab)^0$

16. $\frac{(-2p)^2pq^3}{6p^2q^4}$

17. $\frac{(-8xy)^2y^3}{4x^5y^4}$

18. $\left(\frac{-3x^4y^6}{18x^6y^7}\right)^3$

19. $((-2x^3y)^2)^3$

20. $((-a^2b^4)^3)^5$

21. x^nx^{n-1}

22. $3a^{2n}a^{1-n}$

23. $(5^a)^{2b}$

24. $(-7^{3x})^{4y}$

25. $\frac{-12x^{a+1}}{6x^{a-1}}$

26. $\frac{25x^{a+b}}{-5x^{a-b}}$

27. $\left(x^{a+b}\right)^{a-b}$

28. $(x^2y)^n$

Find each product.

29. $8x^2y^3(-2x^5y)$
30. $5a^3b^5(-3a^2b^4)$
31. $2x(-3x+5)$
32. $4y(1-6y)$
33. $-3x^4y(4x-3y)$
34. $-6a^3b(2b+5a)$
35. $5k^2(3k^2-2k+4)$
36. $6p^3(2p^2+5p-3)$
37. $(x+6)(x-5)$
38. $(x-7)(x+3)$
39. $(2x+3)(3x-2)$
40. $3p(5p+1)(3p+2)$
41. $2u^2(u-3)(3u+5)$
42. $(2t+3)(t^2-4t-2)$
43. $(2x-3)(3x^2+x-5)$
44. $(a^2-2b^2)(a^2-3b^2)$
45. $(2m^2-n^2)(3m^2-5n^2)$
46. $(x+5)(x-5)$
47. $(a+2b)(a-2b)$
48. $(x+4)(x+4)$
49. $(a-2b)(a-2b)$
50. $(x-4)(x^2+4x+16)$
51. $(y+3)(y^2-3y+9)$
52. $(x^2+x-2)(x^2-2x+3)$
53. $(2x^2+y^2-2xy)(x^2-2y^2-xy)$

True or False? If it is false, show a counterexample by choosing values for a and b that would not satisfy the equation.

54. $(a+b)^2=a^2+b^2$
55. $a^2-b^2=(a-b)(a+b)$
56. $(a-b)^2=a^2+b^2$
57. $(a+b)^2=a^2+2ab+b^2$
58. $(a-b)^2=a^2+ab+b^2$
59. $(a-b)^3=a^3-b^3$

Find each product. Use the ***difference of squares*** *or the* ***perfect square*** *formula, if applicable.*

60. $(2p+3)(2p-3)$
61. $(5x-4)(5x+4)$
62. $\left(b-\frac{1}{3}\right)\left(b+\frac{1}{3}\right)$
63. $\left(\frac{1}{2}x-3y\right)\left(\frac{1}{2}x+3y\right)$
64. $(2xy+5y^3)(2xy-5y^3)$
65. $(x^2+7y^3)(x^2-7y^3)$
66. $(1.1x+0.5y)(1.1x-0.5y)$
67. $(0.8a+0.2b)(0.8a+0.2b)$
68. $(x+6)^2$
69. $(x-3)^2$
70. $(4x+3y)^2$
71. $(5x-6y)^2$
72. $\left(3a+\frac{1}{2}\right)^2$
73. $\left(2n-\frac{1}{3}\right)^2$
74. $(a^3b^2-1)^2$
75. $(x^4y^2+3)^2$
76. $(3a^2+4b^3)^2$
77. $(2x^2-3y^3)^2$
78. $3y(5xy^3+2)(5xy^3-2)$
79. $2a(2a^2+5ab)(2a^2+5ab)$
80. $3x(x^2y-xy^3)^2$
81. $(-xy+x^2)(xy+x^2)$
82. $(4p^2+3pq)(-3pq+4p^2)$
83. $(x+1)(x-1)(x^2+1)$
84. $(2x-y)(2x+y)(4x^2+y^2)$
85. $(a-b)(a+b)(a^2-b^2)$
86. $(a+b+1)(a+b-1)$
87. $(2x+3y-5)(2x+3y+5)$
88. $(3m+2n)(3m-2n)(9m^2-4n^2)$
89. $\left((2k-3)+h\right)^2$
90. $\left((4x+y)-5\right)^2$
91. $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}+y^{2b}\right)$
92. $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}-y^{2b}\right)$

Use the difference of squares formula, $(a+b)(a-b)=a^2-b^2$, to find each product.

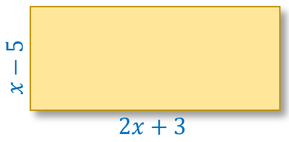
93. $101 \cdot 99$

94. $198 \cdot 202$

95. $505 \cdot 495$

Find the area of each figure. Express it as a polynomial in descending powers of the variable x.

96.



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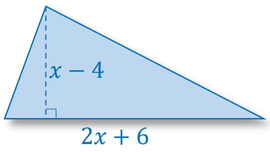

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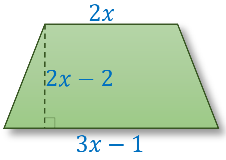


*For each pair of functions, **f** and **g**, find the **product** function* $(\boldsymbol{fg})(\boldsymbol{x})$.

99. $f(x)=5x-6, \quad g(x)=-2+3x$

100. $f(x)=x^2+7x-2, \quad g(x)=6x+5$

101. $f(x)=3x^2-5x, \quad g(x)=9+x-x^2$

102. $f(x)=x^n-4, \quad g(x)=x^n+1$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x}$, *and* $\boldsymbol{R(x)=x-2}$. *Find each of the following.*

103. $(PR)(x)$

104. $(PQ)(x)$

105. $(PQ)(a)$

106. $(PR)(-1)$

107. $(PQ)(3)$

108. $(PR)(0)$

109. $(QR)(x)$

110. $(QR)\left(\frac{1}{2}\right)$

111. $(QR)(a+1)$

112. $P(a-1)$

113. $P(2a+3)$

114. $P(1+h)-P(1)$

Solve each problem.

115. Squares with x centimeters long sides are cut out from each corner of a rectangular piece of cardboard measuring 50 cm by 70 cm. Then the flaps of the remaining cardboard are folded up to construct a box. Find the volume $V(x)$ of the box in terms of the length x.

116. A rectangular flower-bed has a perimeter of 60 meters. If the rectangle is w meters wide, write a polynomial that can be used to determine the area $A(w)$ of the flower-bed in terms of w.

P3 Division of Polynomials

In this section we will discuss dividing polynomials. The result of division of polynomials is not always a polynomial. For example, $x + 1$ divided by x becomes

$$\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x},$$

which is not a polynomial. Thus, the set of polynomials is not closed under the operation of division. However, we can perform division with remainders, mirroring the algorithm of division of natural numbers. We begin with dividing a polynomial by a monomial and then by another polynomial.

Division of Polynomials by Monomials

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial, and then simplify each quotient. In other words, we use the reverse process of addition of fractions, as illustrated below.

$$\boldsymbol{\frac{a+b}{d} = \frac{a}{d} + \frac{b}{d}}$$

Example 1 ► **Dividing Polynomials by Monomials**

Divide and simplify.

a. $(6x^3 + 15x^2 - 2x) \div (3x)$ **b.** $\frac{xy^2 - 8x^2y + 6x^3y^2}{-2xy^2}$

Solution ►

a. $$(6x^3 + 15x^2 - 2x) \div (3x) = \frac{6x^3 + 15x^2 - 2x}{3x} = \frac{\overset{2}{\cancel{6}}x^{\overset{2}{\cancel{3}}}}{\cancel{3}\cancel{x}} + \frac{\overset{5}{\cancel{15}}x^{\cancel{2}}}{\cancel{3}\cancel{x}} - \frac{2\cancel{x}}{3\cancel{x}} = \mathbf{2x^2 + 5x - \frac{2}{3}}$$

b. $$\frac{xy^2 - 8x^2y + 6x^3y^2}{-2xy^2} = -\frac{\cancel{x}\cancel{y^2}}{2\cancel{x}\cancel{y^2}} + \frac{\overset{4}{\cancel{8}}x^{\cancel{2}}\cancel{y}}{\cancel{2}\cancel{x}y^2} - \frac{\overset{3}{\cancel{6}}x^{\overset{2}{\cancel{3}}}\cancel{y^2}}{\cancel{2}\cancel{x}\cancel{y^2}} = \mathbf{-\frac{1}{2} + \frac{4x}{y} - 3x^2}$$

Division of Polynomials by Polynomials

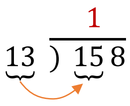
To divide a polynomial by another polynomial, we follow an algorithm similar to the long division algorithm used in arithmetic. For example, observe the steps taken in the long division algorithm when dividing 158 by 13 and the corresponding steps when dividing $x^2 + 5x + 8$ by $x + 3$.

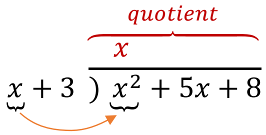
Step 1: Place the dividend under the long division symbol and the divisor in front of this symbol.

$$13\,\overline{)\,158} \qquad\qquad \underbrace{x+3}_{divisor}\,\overline{)\,\underbrace{x^2+5x+8}_{dividend}}$$

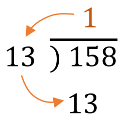
Remember: Both polynomials should be written in **decreasing order of powers**. Also, any **missing terms** after the leading term should be displayed with a **zero coefficient**. This will ensure that the terms in each column are of the same degree.

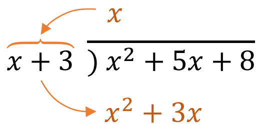
Step 2: Divide the first term of the dividend by the first term of the divisor and record the quotient above the division symbol.



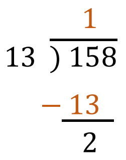


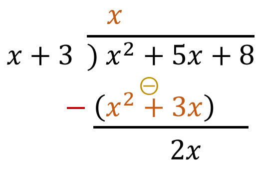
Step 3: Multiply the quotient from *Step 2* by the divisor and write the product under the dividend, lining up the columns with the same degree terms.



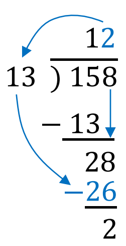


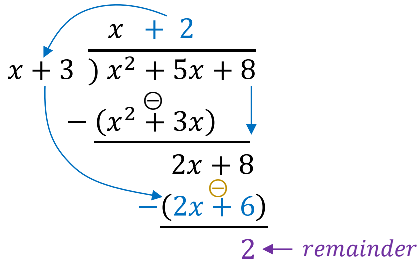
Step 4: Underline and subtract by adding opposite terms in each column. We suggest recording the new sign in a circle, so that it is clear what is being added.





Step 5: Drop the next term (or digit) and repeat the algorithm until the degree of the remainder is lower than the degree of the divisor.





In the example of long division of numbers, we have $158 = 13 \cdot 12 + 2$.

So, the quotient can be written as $\frac{158}{13} = \mathbf{12} + \frac{\mathbf{2}}{\mathbf{13}}$.

In the example of long division of polynomials, we have

$$x^2 + 5x + 8 = (x + 3) \cdot (x \;\; + \;\; 2) + 2.$$

So, the quotient can be written as $\frac{x^2+5x+8}{x+3} = \boldsymbol{x + 2} + \frac{\mathbf{2}}{\boldsymbol{x+3}}$.

Generally, if P, D, Q, and R are polynomials, such that $P(x) = D(x) \cdot Q(x) + R(x)$, then the ratio of polynomials P and D can be written as

$$\boxed{\frac{\boldsymbol{P(x)}}{\boldsymbol{D(x)}} = \boldsymbol{Q(x)} + \frac{\boldsymbol{R(x)}}{\boldsymbol{D(x)}},}$$

where $Q(x)$ is the quotient polynomial, and $R(x)$ is the remainder from the division of $P(x)$ by the divisor $D(x)$.

Observe: The degree of the remainder must be lower than the degree of the divisor, as otherwise, we could apply the division algorithm one more time.

Example 2 ► Dividing Polynomials by Polynomials

Divide.

a. $(3x^3 - 2x^2 + 5) \div (x^2 - 3)$ **b.** $\frac{2p^3+2p+3p^2}{5+2p}$

Solution ►

a. When writing the polynomials in the long division format, we use a zero placeholder term in place of the missing linear terms in both the dividend and the divisor. So, we have

$$\begin{array}{r} \mathbf{3x \ - \ 2} \\ x^2 + 0x - 3\ \overline{)\, 3x^3 - 2x^2 + 0x + 5} \\ -\underline{(3x^3 + 0x^2 \overset{\oplus}{-} 9x)} \\ -2x^2 + 9x + 5 \\ -\underline{(-2x^2 - 0x \overset{\ominus}{+} 6)} \\ \mathbf{9x - 1} \end{array}$$

Thus, $(3x^3 - 2x^2 + 5) \div (x^2 - 3) = 3x - 2 + \frac{9x-1}{x^2-3}$.

b. To perform this division, we arrange both polynomials in decreasing order of powers, and replace the constant term in the dividend with a zero. So, we have

$$\begin{array}{r} \boldsymbol{p^2 - p + \frac{7}{2}} \\ 2p + 5\ \overline{)\, 2p^3 + 3p^2 + 2p + 0} \\ -\underline{(2p^3 \overset{\ominus}{+} 5p^2)} \\ -2p^2 + 2p \\ -\underline{(-2p^2 \overset{\oplus}{-} 5p)} \\ 7p + 0 \\ -\underline{\left(7p \overset{\ominus}{+} \frac{35}{2}\right)} \\ \boldsymbol{-\frac{35}{2}} \end{array}$$

Thus, $\frac{2p^3+2p+3p^2}{5+2p} = p^2 - p + \frac{7}{2} + \frac{-\frac{35}{2}}{2p+5} = \boldsymbol{p^2 - p + \frac{7}{2} - \frac{35}{4p+10}}$.

Observe in the above answer that $\frac{-\frac{35}{2}}{2p+5}$ is written in a simpler form, $-\frac{35}{4p+10}$. This is because $\frac{-\frac{35}{2}}{2p+5} = -\frac{35}{2} \cdot \frac{1}{2p+5} = -\frac{35}{4p+10}$.

Quotient Functions

Similarly as in the case of polynomials, we can define quotients of functions.

Definition 3.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **quotient function**, denoted $\frac{f}{g}$, is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The **domain** of the quotient function is the intersection of the domains of the two functions, D_f and D_g, excluding the x-values for which $g(x) = 0$. So,

$$\boldsymbol{D_{\frac{f}{g}} = D_f \cap D_g \setminus \{x | g(x) = 0\}}$$

Example 3 ► **Dividing Polynomial Functions**

Suppose $P(x) = 2x^2 - x - 6$ and $Q(x) = x - 2$. Find the following:

a. $\left(\frac{P}{Q}\right)(x)$ and $\left(\frac{P}{Q}\right)(-3)$,

b. $\left(\frac{P}{Q}\right)(2)$ and $\left(\frac{P}{Q}\right)(2a)$,

c. domain of $\frac{P}{Q}$.

Notice that this equation holds only for $x \neq 2$.

Solution ► **a.** By *Definition 3.1*, $\left(\frac{P}{Q}\right)(x) = \frac{P(x)}{Q(x)} = \frac{2x^2-x-6}{x-2} = \mathbf{2x + 3}$

So, $\left(\frac{P}{Q}\right)(-3) = 2(-3) + 3 = \mathbf{-3}$. One can verify that the same value is found by evaluating $\frac{P(-3)}{Q(-3)}$.

b. Since the equation $\frac{(2x+3)(x-2)}{x-2} = 2x + 3$ is true only for $x \neq 2$, the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ cannot be used to evaluate $\left(\frac{P}{Q}\right)(2)$. However, by *Definition 3.1*, we have

$\left(\frac{P}{Q}\right)(2)$ is undefined, so 2 is not in the domain of $\frac{P}{Q}$

$$\left(\frac{P}{Q}\right)(2) = \frac{P(2)}{Q(2)} = \frac{2(2)^2 - (2) - 6}{(2) - 2} = \frac{8 - 2 - 6}{0} = \frac{0}{0} = \boldsymbol{undefined}$$

To evaluate $\left(\frac{P}{Q}\right)(2a)$, we first notice that if $a \neq 1$, then $2a \neq 2$. So, we can use the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ and evaluate $\left(\frac{P}{Q}\right)(2a) = 2(2a) + 3 = \mathbf{4a + 3}$ **for all $\boldsymbol{a \neq 1}$**.

c. The domain of any polynomial is the set of all real numbers. So, the domain of $\frac{P}{Q}$ is the set of all real numbers except for the x-values for which the denominator $Q(x) =$

$x-2$ is equal to zero. Since the solution to the equation $x-2=0$ is $x=2$, then the value 2 must be excluded from the set of all real numbers. Therefore, $\boldsymbol{D}_{\frac{\boldsymbol{P}}{\boldsymbol{Q}}} = \mathbb{R} \setminus \{\mathbf{2}\}$.

P.3 Exercises

1. *True or False?* The quotient in a division of a six-degree polynomial by a second-degree polynomial is a third-degree polynomial. Justify your answer.

2. *True or False?* The remainder in a division of a polynomial by a second-degree polynomial is a first-degree polynomial. Justify your answer.

Divide.

3. $\frac{20x^3-15x^2+5x}{5x}$

4. $\frac{27y^4+18y^2-9y}{9y}$

5. $\frac{8x^2y^2-24xy}{4xy}$

6. $\frac{5c^3d+10c^2d^2-15cd^3}{5cd}$

7. $\frac{9a^5-15a^4+12a^3}{-3a^2}$

8. $\frac{20x^3y^2+44x^2y^3-24x^2y}{-4x^2y}$

9. $\frac{64x^3-72x^2+12x}{8x^3}$

10. $\frac{4m^2n^2-21mn^3+18mn^2}{14m^2n^3}$

11. $\frac{12ab^2c+10a^2bc+18abc^2}{6a^2bc}$

Divide.

12. $(x^2+3x-18) \div (x+6)$

13. $(3y^2+17y+10) \div (3y+2)$

14. $(x^2-11x+16) \div (x+8)$

15. $(t^2-7t-9) \div (t-3)$

16. $\frac{6y^3-y^2-10}{3y+4}$

17. $\frac{4a^3+6a^2+14}{2a+4}$

18. $\frac{4x^3+8x^2-11x+3}{4x+1}$

19. $\frac{10z^3-26z^2+17z-13}{5z-3}$

20. $\frac{2x^3+4x^2-x+2}{x^2+2x-1}$

21. $\frac{3x^3-2x^2+5x-4}{x^2-x+3}$

22. $\frac{4k^4+6k^3+3k-1}{2k^2+1}$

23. $\frac{9k^4+12k^3-4k-1}{3k^2-1}$

24. $\frac{2p^3+7p^2+9p+3}{2p+2}$

25. $\frac{5t^2+19t+7}{4t+12}$

26. $\frac{x^4-4x^3+5x^2-3x+2}{x^2+3}$

27. $\frac{p^3-1}{p-1}$

28. $\frac{x^3+1}{x+1}$

29. $\frac{y^4+16}{y+2}$

30. $\frac{x^5-32}{x-2}$

For each pair of polynomials, $P(x)$ and $D(x)$, find such polynomials $Q(x)$ and $R(x)$ that $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{Q}(\boldsymbol{x}) \cdot \boldsymbol{D}(\boldsymbol{x}) + \boldsymbol{R}(\boldsymbol{x})$.

31. $P(x) = 4x^3-4x^2+13x-2$ and $D(x) = 2x-1$

32. $P(x) = 3x^3-2x^2+3x-5$ and $D(x) = 3x-2$

For each pair of functions, $\boldsymbol{f}$ *and* $\boldsymbol{g}$*, find the quotient function* $\left(\frac{f}{g}\right)(x)$ *and state its* ***domain****.*

33. $f(x) = 6x^2 - 4x,\ \ g(x) = 2x$

34. $f(x) = 6x^2 + 9x,\ \ g(x) = -3x$

35. $f(x) = x^2 - 36,\ \ g(x) = x + 6$

36. $f(x) = x^2 - 25,\ \ g(x) = x - 5$

37. $f(x) = 2x^2 - x - 3,\ \ g(x) = 2x - 3$

38. $f(x) = 3x^2 + x - 4,\ \ g(x) = 3x + 4$

39. $f(x) = 8x^3 + 125,\ \ g(x) = 2x + 5$

40. $f(x) = 64x^3 - 27,\ \ g(x) = 4x - 3$

Let $\boldsymbol{P(x) = x^2 - 4}$*,* $\boldsymbol{Q(x) = 2x}$*, and* $\boldsymbol{R(x) = x - 2}$*. Find each of the following. If the value can't be evaluated, say DNE (does not exist).*

41. $\left(\frac{R}{Q}\right)(x)$

42. $\left(\frac{P}{R}\right)(x)$

43. $\left(\frac{R}{P}\right)(x)$

44. $\left(\frac{R}{Q}\right)(2)$

45. $\left(\frac{R}{Q}\right)(0)$

46. $\left(\frac{P}{R}\right)(3)$

47. $\left(\frac{R}{P}\right)(-2)$

48. $\left(\frac{R}{P}\right)(2)$

49. $\left(\frac{P}{R}\right)(a)$, for $a \neq 2$

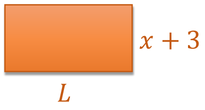
50. $\left(\frac{R}{Q}\right)\left(\frac{3}{2}\right)$

51. $\frac{1}{2}\left(\frac{Q}{R}\right)(x)$

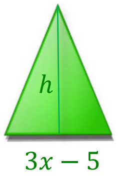
52. $\left(\frac{Q}{R}\right)(a - 1)$

Solve each problem.

53. The area A of a rectangle is $3x^2 + 7x - 6$ and its width W is $x + 3$.
a. Find a polynomial that represents the length L of the rectangle.
b. Find the length of the rectangle if the width is 7 meters.



54. The area A of a triangle is $6x^2 - 13x + 5$. Find the height h of the triangle whose base is $3x - 5$. What is the height of such a triangle if its base is 7 centimeters?



P4

Graphs of Basic Polynomial Functions

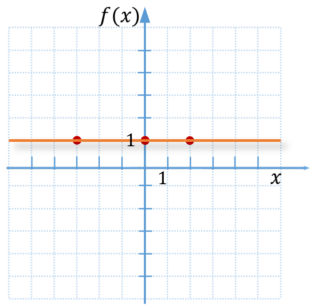
In this section, we will examine graphs of basic polynomial functions, such as constant, linear, quadratic, and cubic functions.

Graphs of Basic Polynomial Functions

Since polynomials are functions, they can be evaluated for different x-values and graphed in a system of coordinates. How do polynomial functions look like? Below, we graph several basic polynomial functions up to the third degree, and observe their shape, domain, and range.

CONSTANT

Let us start with a **constant function**, which is defined by a zero degree polynomial, such as $f(x) = 1$. In this example, for any real x-value, the corresponding y-value is constantly equal to 1. So, the graph of this function is a **horizontal line** with the y-intercept at 1.

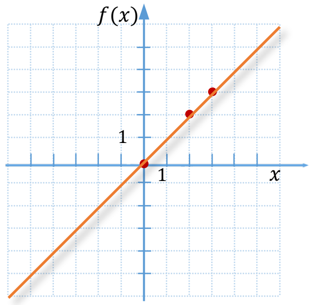


Domain: $\mathbb{R}$
Range: $\{1\}$

Generally, the graph of a **constant function**, $\boldsymbol{f(x) = c}$, is a horizontal line with the y-intercept at c. The domain of this function is $\mathbb{R}$ and the range is $\{c\}$.

LINEAR

The basic first degree polynomial function is the **identity function** given by the formula $\boldsymbol{f(x) = x}$. Since both coordinates of any point satisfying this equation are the same, the graph of the identity function is the diagonal line, as shown below.



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of any first degree polynomial function, $\boldsymbol{f(x) = mx + b}$ with $m \neq 0$, is a slanted line. So, the domain and range of such function is $\mathbb{R}$.

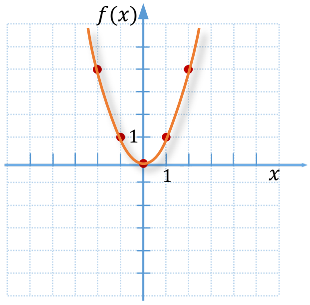
QUADRATIC

The basic second degree polynomial function is the **squaring function** given by the formula $\boldsymbol{f(x)=x^2}$. The shape of the graph of this function is refered to as the **basic parabola**. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

$\boldsymbol{x}$	$\boldsymbol{f(x)=x^2}$
-2	4
-1	1
0	0
1	1
2	4

vertex

symmetry about the y-axis



Domain: $\mathbb{R}$
Range: $[0,\infty)$

Generally, the graph of any second degree polynomial function, $\boldsymbol{f(x)=ax^2+bx+c}$ with $a\neq 0$, is a **parabola**. The domain of such function is $\mathbb{R}$ and the range depends on how the parabola is directed, with arms up or down.

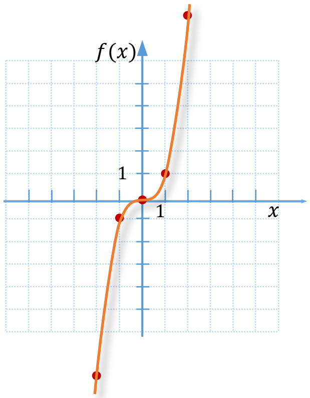
CUBIC

The basic third degree polynomial function is the **cubic function**, given by the formula $\boldsymbol{f(x)=x^3}$. The graph of this function has a shape of a 'snake'. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

$\boldsymbol{x}$	$\boldsymbol{f(x)=x^3}$
-2	-8
-1	-1
0	0
1	1
2	8

center

symmetry about the origin



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of a third degree polynomial function, $\boldsymbol{f(x)=ax^3+bx^2+cx+d}$ with $a\neq 0$, has a shape of a 'snake' with different size waves in the middle. The domain and range of such function is $\mathbb{R}$.

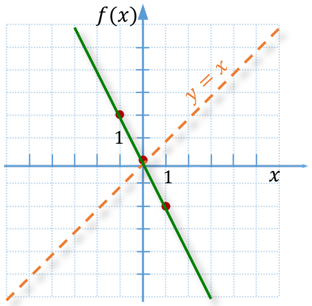
Example 1 ► Graphing Polynomial Functions

Graph each function using a table of values. Give the domain and range of each function by observing its graph. Then, on the same grid, graph the corresponding basic polynomial function. Observe and name the transformation(s) that can be applied to the basic shape in order to obtain the desired function.

a. $f(x) = -2x$ **b.** $f(x) = (x+2)^2$ **c.** $f(x) = x^3 - 2$

Solution ► **a.** The graph of $f(x) = -2x$ is a line passing through the origin and falling from left to right, as shown below in solid green.

x	$f(x) = -2x$
-1	2
0	0
1	-2



Domain of f: ℝ
Range of f: ℝ

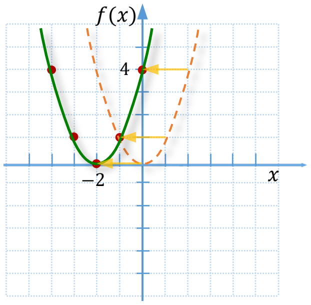
Observe that to obtain the green line, we multiply y-coordinates of the orange line by a factor of -2. Such a transformation is called a **dilation** in the **y-axis** by a factor of **-2**. This dilation can also be achieved by applying a **symmetry in the x-axis** first, and then **stretching** the resulting graph **in the y-axis** by a factor of **2**.

b. The graph of $f(x) = (x+2)^2$ is a parabola with a vertex at $(-2, 0)$, and its arms are directed upwards as shown below in solid green.

x	$f(x) = (x+2)^2$
-4	4
-3	1
-2	0
-1	1
0	4

vertex

symmetry about the line $x = -2$



Domain: ℝ
Range: $[0, \infty)$

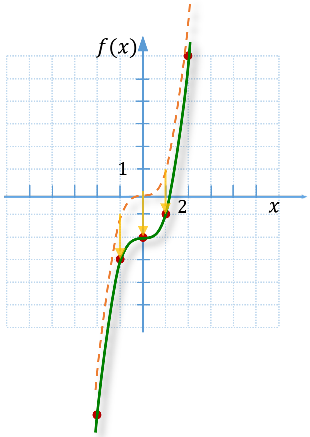
Observe that to obtain the solid green shape, it is enough to move the graph of the **basic parabola** by two units to the left. This transformation is called a **horizontal translation** by two units to the left. The translation to the left reflects the fact that the vertex of the parabola $f(x) = (x+2)^2$ is located at $x + 2 = 0$, which is equivalent to $x = -2$.

c. The graph of $f(x) = x^3 - 2$ has the shape of a basic cubic function with a center at $(0, -2)$.

x	$f(x) = x^3 - 2$
-2	-10
-1	-3
0	-2
1	-1
2	6

center

symmetry about $(0, -2)$



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Observe that the solid green graph can be obtained by shifting the graph of the **basic cubic function** by two units down. This transformation is called a **vertical translation** by two units down.

P.4 Exercises

1. *True or False?* The graph of $x^2 + 3$ is the same shape as a basic parabola with a vertex at $(3,0)$.

Graph *each function and state its* ***domain*** *and* ***range****.*

2. $f(x) = -2x + 3$

3. $f(x) = 3x - 4$

4. $f(x) = -x^2 + 4$

5. $f(x) = x^2 - 2$

6. $f(x) = \frac{1}{2}x^2$

7. $f(x) = -2x^2 + 1$

8. $f(x) = (x + 1)^2 - 2$

9. $f(x) = -x^3 + 1$

10. $f(x) = (x - 3)^3$

Guess the ***transformations*** *needed to apply to the graph of a basic parabola $f(x) = x^2$ to obtain the graph of the given function $g(x)$. Then* ***graph*** *both $f(x)$ and $g(x)$ on the same grid and confirm the the original guess.*

11. $g(x) = -x^2$

12. $g(x) = x^2 - 3$

13. $g(x) = x^2 + 2$

14. $g(x) = (x + 2)^2$

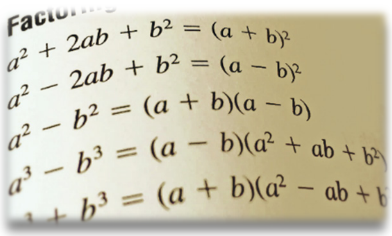
15. $g(x) = (x - 3)^2$

16. $g(x) = (x + 2)^2 - 1$

Attributions

p.**169** Roller Coaster in a Park by Priscilla Du Preez / Unsplash Licence

Factoring



Factoring is the reverse process of multiplication. Factoring polynomials in algebra has similar role as factoring numbers in arithmetic. Any number can be expressed as a product of prime numbers. For example, $6 = 2 \cdot 3$. Similarly, any polynomial can be expressed as a product of **prime** polynomials, which are polynomials that cannot be factored any further. For example, $x^2 + 5x + 6 = (x + 2)(x + 3)$. Just as factoring numbers helps in simplifying or adding fractions, factoring polynomials is very useful in simplifying or adding algebraic fractions. In addition, it helps identify zeros of polynomials, which in turn allows for solving higher degree polynomial equations.

In this chapter, we will examine the most commonly used factoring strategies with particular attention to special factoring. Then, we will apply these strategies in solving polynomial equations.

F1 Greatest Common Factor and Factoring by Grouping

Prime Factors

When working with integers, we are often interested in their factors, particularly prime factors. Likewise, we might be interested in factors of polynomials.

Definition 1.1 ► To **factor** a polynomial means to write the polynomial as a **product** of 'simpler' polynomials. For example,

$$5x + 10 = 5(x + 2), \quad \text{or} \quad x^2 - 9 = (x + 3)(x - 3).$$

In the above definition, 'simpler' means polynomials of **lower degrees** or polynomials with coefficients that **do not contain common factors** other than 1 or -1. If possible, we would like to see the polynomial factors, other than monomials, having **integral coefficients** and a **positive leading term**.

When is a polynomial factorization complete?

In the case of natural numbers, the complete factorization means a factorization into prime numbers, which are numbers divisible only by their own selves and 1. We would expect that similar situation is possible for polynomials. So, which polynomials should we consider as prime?

Observe that a polynomial such as $-4x + 12$ can be written as a product in many different ways, for instance

$$-(4x + 12), \quad 2(-2x + 6), \quad 4(-x + 3), \quad -4(x - 3), \quad -12\left(\frac{1}{3}x + 1\right), \text{ etc.}$$

Since the terms of $4x + 12$ and $-2x + 6$ still contain common factors different than 1 or -1, these polynomials are not considered to be factored completely, which means that they should not be called prime. The next two factorizations, $4(-x + 3)$ and $-4(x - 3)$ are both complete, so both polynomials $-x + 3$ and $x - 3$ should be considered as prime. But what about the last factorization, $-12\left(\frac{1}{3}x + 1\right)$? Since the remaining binomial $\frac{1}{3}x + 1$ does not have integral coefficients, such a factorization is not always desirable.

Here are some examples of **prime polynomials**:

- any monomials such as $-2x^2$, πr^2, or $\frac{1}{3}xy$;
- any linear polynomials with integral coefficients that have no common factors other than 1 or -1, such as $x - 1$ or $2x + 5$;
- some quadratic polynomials with integral coefficients that cannot be factored into any lower degree polynomials with integral coefficients, such as $x^2 + 1$ or $x^2 + x + 1$.

For the purposes of this course, we will assume the following definition of a prime polynomial.

Definition 1.2 ► A polynomial with integral coefficients is called **prime** if one of the following conditions is true

- it is a **monomial**, or
- the only **common factors** of its terms are **1 or -1** and it **cannot be factored into any lower degree polynomials** with integral coefficients.

Definition 1.3 ► A **factorization** of a polynomial with integral coefficients is **complete** if all of its **factors** are **prime**.

Here is an example of a polynomial factored completely:

$$-6x^3 - 10x^2 + 4x = -2x(3x - 1)(x + 2)$$

In the next few sections, we will study several factoring strategies that will be helpful in finding complete factorizations of various polynomials.

Greatest Common Factor

The first strategy of factoring is to factor out the **greatest common factor** (**GCF**).

Definition 1.4 ► The **greatest common factor** (**GCF**) of two or more terms is the largest expression that is a factor of all these terms.

In the above definition, the "largest expression" refers to the expression with the most factors, disregarding their signs.

To find the greatest common factor, we take the product of the least powers of each type of common factor out of all the terms. For example, suppose we wish to find the GCF of the terms

$$6x^2y^3,\ -18x^5y, \text{ and } 24x^4y^2.$$

First, we look for the GCF of 6, 18, and 24, which is 6. Then, we take the lowest power out of x^2, x^5, and x^4, which is x^2. Finally, we take the lowest power out of y^3, y, and y^2, which is y. Therefore,

$$\text{GCF}(6x^2y^3,\ -18x^5y,\ 24x^4y^2) = 6x^2y$$

This GCF can be used to factor the polynomial $6x^2y^3 - 18x^5y + 24x^4y^2$ by first seeing it as

$$6x^2y \cdot y^2 - 6x^2y \cdot 3x^3 + 6x^2y \cdot 4x^2y,$$

and then, using the **reverse distributing property**, ‘pulling’ the $6x^2y$ out of the bracket to obtain

$$6x^2y(y^2 - 3x^3 + 4x^2y).$$

Note 1: Notice that since 1 and -1 are factors of any expression, the GCF is defined up to the sign. Usually, we choose the positive GCF, but sometimes it may be convenient to choose the negative GCF. For example, we can claim that

$$\mathrm{GCF}(-2x, -4y) = 2 \quad \text{or} \quad \mathrm{GCF}(-2x, -4y) = -2,$$

depending on what expression we wish to leave after factoring the GCF out:

$$-2x - 4y = \underbrace{2}_{\substack{\text{positive}\\ \text{GCF}}} \underbrace{(-x - 2y)}_{\substack{\text{negative}\\ \text{leading}\\ \text{term}}} \text{ or } -2x - 4y = \underbrace{-2}_{\substack{\text{negative}\\ \text{GCF}}} \underbrace{(x + 2y)}_{\substack{\text{positive}\\ \text{leading}\\ \text{term}}}$$

Note 2: If the GCF of the terms of a polynomial is equal to 1, we often say that these terms do not have any common factors. What we actually mean is that the terms do not have a common factor other than 1, as factoring 1 out does not help in breaking the original polynomial into a product of simpler polynomials. See *Definition 1.1*.

Example 1 ► Finding the Greatest Common Factor

Find the greatest common factor for the given expressions.

a. $6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2$ **b.** $4\pi(y-x),\ 8\pi(x-y)$
c. $ab^2,\ a^2b,\ b,\ a$ **d.** $3x^{-1}y^{-3},\ x^{-2}y^{-2}z$

Solution ►

a. Since $\mathrm{GCF}(6, 3, 9) = 3$, the lowest power out of x^4, x^3, and x is x, and the lowest power out of $(x+1)^3$, $(x+1)$, and $(x+1)^2$ is $(x+1)$, then

$$\mathrm{GCF}(6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2) = \boldsymbol{3x(x+1)}$$

b. Since $y - x$ is opposite to $x - y$, then $y - x$ can be written as $-(x - y)$. So 4, π, and $(x - y)$ is common for both expressions. Thus,

$$\mathrm{GCF}\big(4\pi(y-x),\ 8\pi(x-y)\big) = \boldsymbol{4\pi(x-y)}$$

Note: The greatest common factor is unique up to its sign. Notice that in the above example, we could write $x - y$ as $-(y - x)$ and choose the GCF to be $4\pi(y - x)$.

c. The terms $ab^2,\ a^2b,\ b,$ and a have no common factor other than 1, so

$$\mathrm{GCF}(ab^2,\ a^2b,\ b,\ a) = \mathbf{1}$$

d. The lowest power out of x^{-1} and x^{-2} is x^{-2}, and the lowest power out of y^{-3} and y^{-2} is y^{-3}. Therefore,

$$\mathrm{GCF}(3x^{-1}y^{-3},\ x^{-2}y^{-2}z) = \boldsymbol{x^{-2}y^{-3}}$$

Example 2 ► Factoring out the Greatest Common Factor

Factor each expression by taking the greatest common factor out. Simplify the factors, if possible.

a. $54x^2y^2 + 60xy^3$

b. $ab - a^2b(a-1)$

c. $-x(x-5) + x^2(5-x) - (x-5)^2$

d. $x^{-1} + 2x^{-2} - x^{-3}$

Solution ►

a. To find the greatest common factor of 54 and 60, we can use the method of dividing by any common factor, as presented below.

all common factors are listed in this column

$$\begin{array}{c|cc} 2 & 54, & 60 \\ 3 & 27, & 30 \\ \hline & 9, & 10 \end{array}$$

no more common factors for 9 and 10

So, $\mathrm{GCF}(54, 60) = 2 \cdot 3 = 6$.

Since $\mathrm{GCF}(54x^2y^2,\ 60xy^3) = 6xy^2$, we factor the $6xy^2$ out by dividing each term of the polynomial $54x^2y^2 + 60xy^3$ by $6xy^2$, as below.

$$54x^2y^2 + 60xy^3$$
$$= \boldsymbol{6xy^2(9x + 10y)}$$

$\frac{54x^2y^2}{6xy^2} = 9x$

$\frac{60xy^3}{6xy^2} = 10y$

Note: Since factoring is the reverse process of multiplication, it can be checked by finding the product of the factors. If the product gives us the original polynomial, the factorization is correct.

b. First, notice that the polynomial has two terms, ab and $-a^2b(a-1)$. The greatest common factor for these two terms is ab, so we have

$$ab - a^2b(a-1) = ab(\mathbf{1} - a(a-1))$$

remember to leave **1** for the first term

$$= ab(1 - a^2 + a)$$

$$= ab(-a^2 + a + 1)$$

simplify and arrange in decreasing powers

$$= \boldsymbol{-ab(a^2 - a - 1)}$$

take the "–" out

Note: Both factorizations, $ab(-a^2+a+1)$ and $-ab(a^2-a-1)$ are correct. However, we customarily leave the polynomial in the bracket with a positive leading coefficient.

c. Observe that if we write the middle term $x^2(5-x)$ as $-x^2(x-5)$ by factoring the negative out of the $(5-x)$, then $(5-x)$ is the common factor of all the terms of the equivalent polynomial

$$-x(x-5)-x^2(x-5)-(x-5)^2.$$

Then notice that if we take $-(x-5)$ as the GCF, then the leading term of the remaining polynomial will be positive. So, we factor

$$-x(x-5)+x^2(5-x)-(x-5)^2$$

$$=-x(x-5)-x^2(x-5)-(x-5)^2$$

$$=-(x-5)\left(x+x^2+(x-5)\right)$$

simplify and arrange in decreasing powers

$$=-(\boldsymbol{x}-\mathbf{5})\left(\boldsymbol{x}^{\mathbf{2}}+\mathbf{2}\boldsymbol{x}-\mathbf{5}\right)$$

d. The $\mathrm{GCF}(x^{-1},\ 2x^{-2},\ -x^{-3})=x^{-3}$, as -3 is the lowest exponent of the common factor x. So, we factor out x^{-3} as below.

$$x^{-1}+2x^{-2}-x^{-3}$$

$$=\boldsymbol{x}^{-\mathbf{3}}\left(\boldsymbol{x}^{\mathbf{2}}+\mathbf{2}\boldsymbol{x}-\mathbf{1}\right)$$

the exponent 2 is found by subtracting -3 from -1

the exponent 1 is found by subtracting -3 from -2

To check if the factorization is correct, we multiply

$$x^{-3}\left(x^2+2x-1\right)$$

add exponents

$$=x^{-3}x^2+2x^{-3}x-1x^{-3}$$

$$=x^{-1}+2x^{-2}-x^{-3}$$

Since the product gives us the original polynomial, the factorization is correct.

Factoring by Grouping

When referring to a common factor, we have in mind a common factor other than 1.

Consider the polynomial $x^2+x+xy+y$. It consists of four terms that do not have any common factors. Yet, it can still be factored if we group the first two and the last two terms. The first group of two terms contains the common factor of x and the second group of two terms contains the common factor of y. Observe what happens when we factor each group.

$$\underbrace{x^2+x}+\underbrace{xy+y}$$

$$=x(x+1)+y(x+1)$$

$$=(x+1)(x+y)$$

now $(x+1)$ is the common factor of the entire polynomial

This method is called **factoring by grouping**, in particular, two-by-two grouping.

Warning: After factoring each group, make sure to write the "+" or "−" between the terms. Failing to write these signs leads to the false impression that the polynomial is already factored. For example, if in the second line of the above calculations we would fail to write the middle "+", the expression would look like a product $x(x+1)\,y(x+1)$, which is not the case. Also, since the expression $x(x+1)+y(x+1)$ is a sum, not a product, we should not stop at this step. We need to factor out the common bracket $(x+1)$ to leave it as a product.

A two-by-two grouping leads to a factorization only if **the binomials**, after factoring out the common factors in each group, **are the same**. Sometimes a rearrangement of terms is necessary to achieve this goal.

For example, the attempt to factor $x^3-15+5x^2-3x$ by grouping the first and the last two terms,

$$\underbrace{x^3-15}+\underbrace{5x^2-3x}$$

$$=(x^3-15)+x(5x-3)$$

does not lead us to a common binomial that could be factored out.

However, rearranging terms allows us to factor the original polynomial in the following ways:

$$\begin{aligned}&x^3-15+5x^2-3x\\&=\underbrace{x^3+5x^2}+\underbrace{-3x-15}\\&=x^2(x+5)-3(x+5)\\&=(x+5)(x^2-3)\end{aligned}$$

or

$$\begin{aligned}&x^3-15+5x^2-3x\\&=\underbrace{x^3-3x}+\underbrace{5x^2-15}\\&=x(x^2-3)+5(x^2-3)\\&=(x^2-3)(x+5)\end{aligned}$$

Factoring by grouping applies to polynomials with more than three terms. However, not all such polynomials can be factored by grouping. For example, if we attempt to factor x^3+x^2+2x-2 by grouping, we obtain

$$\underbrace{x^3+x^2}+\underbrace{2x-2}$$

$$=x^2(x+1)+2(x-1).$$

Unfortunately, the expressions $x+1$ and $x-1$ are not the same, so there is no common factor to factor out. One can also check that no other rearrangments of terms allows us for factoring out a common binomial. So, this polynomial cannot be factored by grouping.

Example 3 ► **Factoring by Grouping**

Factor each polynomial by grouping, if possible. Remember to check for the GCF first.

a. $2x^3 - 6x^2 + x - 3$ **b.** $5x - 5y - ax + ay$

c. $2x^2y - 8 - 2x^2 + 8y$ **d.** $x^2 - x + y + 1$

Solution ► **a.** Since there is no common factor for all four terms, we will attempt the two-by-two grouping method.

$$\underbrace{2x^3 - 6x^2} + \underbrace{x - 3}$$

$$= 2x^2(x-3) + 1(x-3)$$

$$= \boldsymbol{(x-3)(2x^2+1)}$$

write the **1** for the second term

b. As before, there is no common factor for all four terms. The two-by-two grouping method works only if the remaining binomials after factoring each group are exactly the same. We can achieve this goal by factoring $-a$, rather than a, out of the last two terms. So,

$$\underbrace{5x - 5y}\ \underbrace{-ax + ay}$$

$$= 5(x-y) - a(x-y)$$

$$= \boldsymbol{(x-y)(5 - a)}$$

reverse signs when 'pulling' a "–" out

c. Notice that 2 is the GCF of all terms, so we factor it out first.

$$2x^2y - 8 - 2x^2 + 8y$$

$$= 2(x^2y - 4 - x^2 + 4y)$$

Then, observe that grouping the first and last two terms of the remaining polynomial does not help, as the two groups do not have any common factors. However, exchanging for example the second with the fourth term will help, as shown below.

$$= 2(\underbrace{x^2y + 4y}\ \underbrace{-x^2 - 4})$$

$$= 2[y(x^2+4) - (x^2+4)]$$

$$= \boldsymbol{2(x^2+4)(y-1)}$$

the square bracket is essential here because of the factor of 2

reverse signs when 'pulling' a "–" out

now, there is no need for the square bracket as multiplication is associative

d. The polynomial $x^2 - x + y + 1$ does not have any common factors for all four terms. Also, only the first two terms have a common factor. Unfortunately, when attempting to factor using the two-by-two grouping method, we obtain

$$x^2 - x + y + 1$$

$$= x(x-1) + (y+1),$$

which cannot be factored, as the expressions $x - 1$ and $y + 1$ are different.

One can also check that no other arrangement of terms allows for factoring of this polynomial by grouping. So, this polynomial cannot be factored by grouping.

Example 4 ► **Factoring in Solving Formulas**

Solve $ab = 3a + 5$ for a.

Solution ► First, we move the terms containing the variable a to one side of the equation,

$$ab = 3a + 5$$
$$ab - 3a = 5,$$

and then factor a out

$$a(b-3) = 5.$$

So, after dividing by $b-3$, we obtain $\boldsymbol{a = \frac{5}{b-3}}$.

F.1 Exercises

*In problems **1-2**, state whether the given sentence is **true** or **false**.*

1. The polynomial $6x + 8y$ is ***prime***.

2. The **GCF** of the terms of the polynomial $3(x-2) + x(2-x)$ is $(x-2)(2-x)$.

3. Observe the two factorizations of the polynomial $\frac{1}{2}x - \frac{3}{4}y$ performed by different students:

Student A: $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{2}\left(x - \frac{3}{2}y\right)$ *Student B:* $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{4}(2x - 3y)$

Are the two factorizations correct? Which one is preferable, and why?

Find the GCF with a positive coefficient for the given expressions.

4. $8xy,\ 10xz,\ -14xy$

5. $21a^3b^6,\ -35a^7b^5,\ 28a^5b^8$

6. $4x(x-1),\ 3x^2(x-1)$

7. $-x(x-3)^2,\ x^2(x-3)(x+2)$

8. $9(a-5),\ 12(5-a)$

9. $(x-2y)(x-1),\ (2y-x)(x+1)$

10. $-3x^{-2}y^{-3},\ 6x^{-3}y^{-5}$

11. $x^{-2}(x+2)^{-2},\ -x^{-4}(x+2)^{-1}$

Factor out the greatest common factor. Leave the remaining polynomial with a positive leading coeficient. Simplify the factors, if possible.

12. $9x^2 - 81x$

13. $8k^3 + 24k$

14. $6p^3 - 3p^2 - 9p^4$

15. $6a^3 - 36a^4 + 18a^2$

16. $-10r^2s^2 + 15r^4s^2$

17. $5x^2y^3 - 10x^3y^2$

18. $a(x-2) + b(x-2)$

19. $a(y^2-3) - 2(y^2-3)$

20. $(x-2)(x+3) + (x-2)(x+5)$

21. $(n-2)(n+3) + (n-2)(n-3)$

22. $y(x-1)+5(1-x)$ **23.** $(4x-y)-4x(y-4x)$

24. $4(3-x)^2-(3-x)^3+3(3-x)$ **25.** $2(p-3)+4(p-3)^2-(p-3)^3$

Factor out the least power of each variable.

26. $3x^{-3}+x^{-2}$ **27.** $k^{-2}+2k^{-4}$ **28.** $x^{-4}-2x^{-3}+7x^{-2}$

29. $3p^{-5}+p^{-3}-2p^{-2}$ **30.** $3x^{-3}y-x^{-2}y^2$ **31.** $-5x^{-2}y^{-3}+2x^{-1}y^{-2}$

Factor by grouping, if possible.

32. $20+5x+12y+3xy$ **33.** $2a^3+a^2-14a-7$ **34.** $ac-ad+bc-bd$

35. $2xy-x^2y+6-3x$ **36.** $3x^2+4xy-6xy-8y^2$ **37.** $x^3-xy+y^2-x^2y$

38. $3p^2+9pq-pq-3q^2$ **39.** $3x^2-x^2y-yz^2+3z^2$ **40.** $2x^3-x^2+4x-2$

41. $x^2y^2+ab-ay^2-bx^2$ **42.** $xy+ab+by+ax$ **43.** $x^2y-xy+x+y$

44. $xy-6y+3x-18$ **45.** x^ny-3x^n+y-5 **46.** $a^nx^n+2a^n+x^n+2$

Factor completely. Remember to check for the GCF first.

47. $5x-5ax+5abc-5bc$ **48.** $6rs-14s+6r-14$

49. $x^4(x-1)+x^3(x-1)-x^2+x$ **50.** $x^3(x-2)^2+2x^2(x-2)-(x+2)(x-2)$

51. One of possible factorizations of the polynomial $4x^2y^5-8xy^3$ is $2xy^3(2xy^2-4)$. Is this a complete factorization?

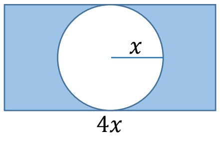
Use factoring the GCF strategy to ***solve*** *each formula* ***for the indicated variable****.*

52. $A=\boldsymbol{P}+\boldsymbol{P}r$, for $\boldsymbol{P}$ **53.** $M=\frac{1}{2}\boldsymbol{p}q+\frac{1}{2}\boldsymbol{p}r$, for $\boldsymbol{p}$

54. $2\boldsymbol{t}+c=k\boldsymbol{t}$, for $\boldsymbol{t}$ **55.** $w\boldsymbol{y}=3\boldsymbol{y}-x$, for $\boldsymbol{y}$

Write the area of each shaded region in factored form.

56.



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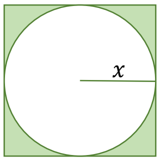

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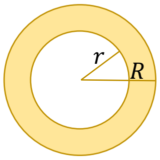

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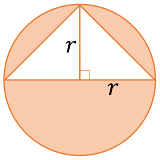
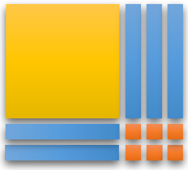


F2 Factoring Trinomials

In this section, we discuss factoring trinomials. We start with factoring quadratic trinomials of the form $x^2 + bx + c$, then quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, and finally trinomials reducible to quadratic by means of substitution.



Factorization of Quadratic Trinomials $x^2 + bx + c$

Factorization of a quadratic trinomial $x^2 + bx + c$ is the reverse process of the FOIL method of multiplying two linear binomials. Observe that

$$(x + p)(x + q) = x^2 + qx + px + pq = x^2 + (p + q)x + pq$$

GUESSING
METHOD

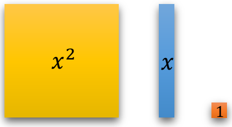
So, to reverse this multiplication, we look for two numbers p and q, such that the product pq equals to the free term c and the sum $p + q$ equals to the middle coefficient b of the trinomial.

$$x^2 + \underbrace{b}_{(p+q)} x + \underbrace{c}_{pq} = (x + p)(x + q)$$

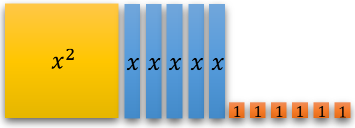
For example, to factor $x^2 + 5x + 6$, we think of two integers that multiply to 6 and add to 5. Such integers are 2 and 3, so $x^2 + 5x + 6 = (x + 2)(x + 3)$. Since multiplication is commutative, the order of these factors is not important.

VISUALIZATION
OF FACTORING

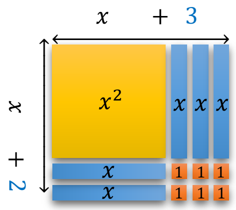
This could also be illustrated geometrically, using algebra tiles.



The area of a square with the side length x is equal to x^2. The area of a rectangle with the dimensions x by 1 is equal to x, and the area of a unit square is equal to 1. So, the trinomial $x^2 + 5x + 6$ can be represented as



To factor this trinomial, we would like to rearrange these tiles to fulfill a rectangle.



The area of such rectangle can be represented as the product of its length, $(x + 3)$, and width, $(x + 2)$ which becomes the factorization of the original trinomial.

In the trinomial examined above, the signs of the middle and the last terms are both positive. To analyse how different signs of these terms influence the signs used in the factors, observe the next three examples.

To factor x^2-5x+6, we look for two integers that multiply to 6 and add to -5. Such integers are -2 and -3, so $x^2-5x+6=(x-2)(x-3)$.

To factor x^2+x-6, we look for two integers that multiply to -6 and add to 1. Such integers are -2 and 3, so $x^2+x-6=(x-2)(x+3)$.

To factor x^2-x-6, we look for two integers that multiply to -6 and add to -1. Such integers are 2 and -3, so $x^2-x-6=(x+2)(x-3)$.

Observation: A **positive constant c** in a trinomial x^2+bx+c tells us that the integers p and q in the factorization $(x+p)(x+q)$ are both of the **same sign** and their **sum** is the middle coefficient b. In addition, if b is positive, both p and q are positive, and if b is negative, both p and q are negative.

A **negative constant c** in a trinomial x^2+bx+c tells us that the integers p and q in the factorization $(x+p)(x+q)$ are of **different signs** and the **difference** of their absolute values is the middle coefficient b. In addition, the integer whose absolute value is larger takes the sign of the middle coefficient b.

These observations are summarized in the following **Table of Signs**.

Assume that $|p| \geq |q|$.

sum b	product c	p	q	**comments**
+	+	+	+	b is the ***sum*** of p and q
−	+	−	−	b is the ***sum*** of p and q
+	−	+	−	b is the ***difference*** $\lvert p\rvert - \lvert q\rvert$
−	−	−	+	b is the ***difference*** $\lvert q\rvert - \lvert p\rvert$

Example 1 ► Factoring Trinomials with the Leading Coefficient Equal to 1

Factor each trinomial, if possible.

a. $x^2-10x+24$

b. $x^2+9x-36$

c. $x^2-39xy-40y^2$

d. x^2+7x+9

Solution ►

a. To factor the trinomial $x^2-10x+24$, we look for two integers with a product of 24 and a sum of -10. The two integers are fairly easy to guess, -4 and -6. However, if one wishes to follow a more methodical way of finding these numbers, one can list the possible two-number factorizations of 24 and observe the sums of these numbers.

For simplicity, the table doesn't include signs of the integers. The signs are determined according to the **Table of Signs**.

product = 24 (pairs of factors of 24)	**sum = −10** (sum of factors)
1 · 24	25
2 · 12	14
3 · 8	11
4 · 6	10

Bingo!

Since the product is positive and the sum is negative, both integers must be negative. So, we take -4 and -6.

Thus, $x^2 - 10x + 24 = (\boldsymbol{x} - \mathbf{4})(\boldsymbol{x} - \mathbf{6})$. The reader is encouraged to check this factorization by multiplying the obtained binomials.

b. To factor the trinomial $x^2 + 9x - 36$, we look for two integers with a product of -36 and a sum of 9. So, let us list the possible factorizations of 36 into two numbers and observe the differences of these numbers.

product = −36 (pairs of factors of 36)	**sum = 9** (difference of factors)
1 · 36	35
2 · 18	16
3 · 12	9
4 · 9	5
6 · 6	0

This row contains the solution, so there is no need to list any of the subsequent rows.

Since the product is negative and the sum is positive, the integers are of different signs and the one with the larger absolute value assumes the sign of the sum, which is positive. So, we take 12 and -3.

Thus, $x^2 + 9x - 36 = (\boldsymbol{x} + \mathbf{12})(\boldsymbol{x} - \mathbf{3})$. Again, the reader is encouraged to check this factorization by multiplying the obtained binomials.

c. To factor the trinomial $x^2 - 39xy - 40y^2$, we look for two binomials of the form $(x + ?y)(x + ?y)$ where the question marks are two integers with a product of -40 and a sum of 39. Since the two integers are of different signs and the absolute values of these integers differ by 39, the two integers must be -40 and 1.

Therefore, $x^2 - 39xy - 40y^2 = (\boldsymbol{x} - \mathbf{40}\boldsymbol{y})(\boldsymbol{x} + \boldsymbol{y})$.

Suggestion: Create a table of pairs of factors only if guessing the two integers with the given product and sum becomes too difficult.

d. When attempting to factor the trinomial $x^2 + 7x + 9$, we look for a pair of integers that would multiply to 9 and add to 7. There are only two possible factorizations of 9: $9 \cdot 1$ and $3 \cdot 3$. However, neither of the sums, $9 + 1$ or $3 + 3$, are equal to 7. So, there is no possible way of factoring $x^2 + 7x + 9$ into two linear binomials with integral coefficients. Therefore, if we admit only integral coefficients, this polynomial is **not factorable**.

Factorization of Quadratic Trinomials $ax^2 + bx + c$ with $a \neq 0$

Before discussing factoring quadratic trinomials with a leading coefficient different than 1, let us observe the multiplication process of two linear binomials with integral coefficients.

$$(mx + p)(nx + q) = mnx^2 + mqx + npx + pq = \underbrace{\boldsymbol{a}}_{mn} x^2 + \underbrace{\boldsymbol{b}}_{(mq+np)} x + \underbrace{\boldsymbol{c}}_{pq}$$

To reverse this process, notice that this time, we are looking for four integers m, n, p, and q that satisfy the conditions

$$mn = a,\ \ pq = c,\ \ mq + np = b,$$

where a, b, c are the coefficients of the quadratic trinomial that needs to be factored. This produces a lot more possibilities to consider than in the guessing method used in the case of the leading coefficient equal to 1. However, if at least one of the outside coefficients, a or c, are prime, the guessing method still works reasonably well.

For example, consider $2x^2 + x - 6$. Since the coefficient $a = 2 = mn$ is a prime number, there is only one factorization of a, which is $1 \cdot 2$. So, we can assume that $m = 2$ and $n = 1$. Therefore,

$$2x^2 + x - 6 = (2x \pm |p|)(x \mp |q|)$$

Since the constant term $c = -6 = pq$ is negative, the binomial factors have different signs in the middle. Also, since pq is negative, we search for such p and q that the inside and outside products **differ** by the middle term $b = x$, up to its sign. The only factorizations of 6 are $1 \cdot 6$ and $2 \cdot 3$. So we try

GUESSING METHOD

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{1)(x}_{x} \mp 6}_{12x})$$

differs by $11x$ → too much

Observe that these two trials can be disregarded at once as 2 is not a common factor of all the terms of the trinomial, while it is a common factor of the terms of one of the binomials.

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{6)(x}_{6x} \mp 1}_{2x})$$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{2)(x}_{2x} \mp 3}_{6x})$$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{3)(x}_{3x} \mp 2}_{4x})$$

differs by x → perfect!

Then, since the difference between the inner and outer products should be positive, the larger product must be positive and the smaller product must be negative. So, we distribute the signs as below.

$$2x^2 + x - 6 = (\underbrace{2x - \underbrace{3)(x}_{-3x} + 2}_{4x})$$

In the end, it is a good idea to multiply the product to check if it results in the original polynomial. We leave this task to the reader.

What if the outside coefficients of the quadratic trinomial are both composite? Checking all possible distributions of coefficients m, n, p, and q might be too cumbersome. Luckily, there is another method of factoring, called **decomposition**.

The decomposition method is based on the reverse FOIL process.
Suppose the polynomial $6x^2 + 19x + 15$ factors into $(mx + p)(nx + q)$. Observe that the FOIL multiplication of these two binomials results in the four term polynomial,

$$mnx^2 + mqx + npx + pq,$$

which after combining the two middle terms gives us the original trinomial. So, reversing these steps would lead us to the factored form of $6x^2 + 19x + 15$.

To reverse the FOIL process, we would like to:

This product is often referred to as the **master product** or the ***ac*-product**.

- Express the middle term, $19x$, as a sum of two terms, mqx and npx, such that the product of their coefficients, $mnpq$, is equal to the product of the outside coefficients $ac = 6 \cdot 15 = 90$.
- Then, factor the four-term polynomial by grouping.

Thus, we are looking for two integers with the product of 90 and the sum of 19. One can check that 9 and 10 satisfy these conditions. Therefore,

DECOMPOSITION METHOD

$$\begin{aligned} & 6x^2 + 19x + 15 \\ &= 6x^2 + 9x + 10x + 15 \\ &= 3x(2x + 3) + 5(2x + 3) \\ &= (2x + 3)(3x + 5) \end{aligned}$$

Example 2 ► Factoring Trinomials with the Leading Coefficient Different than 1

Factor completely each trinomial.

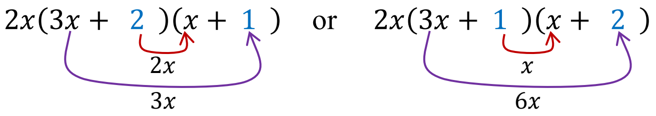
a. $6x^3 + 14x^2 + 4x$ **b.** $-6y^2 - 10 + 19y$

c. $18a^2 - 19ab - 12b^2$ **d.** $2(x + 3)^2 + 5(x + 3) - 12$

Solution ► **a.** First, we factor out the GCF, which is $2x$. This gives us

$$6x^3 + 14x^2 + 4x = 2x(3x^2 + 7x + 2)$$

The outside coefficients of the remaining trinomial are prime, so we can apply the guessing method to factor it further. The first terms of the possible binomial factors must be $3x$ and x while the last terms must be 2 and 1. Since both signs in the trinomial are positive, the signs used in the binomial factors must be both positive as well. So, we are ready to give it a try:



The first distribution of coefficients does not work as it would give us $2x + 3x = 5x$ for the middle term. However, the second distribution works as $x + 6x = 7x$, which matches the middle term of the trinomial. So,

$$6x^3 + 14x^2 + 4x = \mathbf{2x(3x + 1)(x + 2)}$$

b. Notice that the trinomial is not arranged in decreasing order of powers of y. So, first, we rearrange the last two terms to achieve the decreasing order. Also, we factor out the -1, so that the leading term of the remaining trinomial is positive.

$$-6y^2 - 10 + 19y = -6y^2 + 19y - 10 = -(6y^2 - 19y + 10)$$

Then, since the outside coefficients are composite, we will use the decomposition method of factoring. The ac-product equals to 60 and the middle coefficient equals to -19. So, we are looking for two integers that multiply to 60 and add to -19. The integers that satisfy these conditions are -15 and -4. Hence, we factor

$$\begin{aligned} &-(6y^2 - 19y + 10) \\ &= -(6y^2 - 15y - 4y + 10) \\ &= -[3y(2y - 5) - 2(2y - 5)] \\ &= \mathbf{-(2y - 5)(3y - 2)} \end{aligned}$$

the **square bracket** is essential because of the negative sign outside

remember to **reverse the sign!**

c. There is no common factor to take out of the polynomial $18a^2 - 19ab - 12b^2$. So, we will attempt to factor it into two binomials of the type $(ma \pm pb)(na \mp qb)$, using the decomposition method. The ac-product equals $-12 \cdot 18 = -2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ and the middle coefficient equals -19. To find the two integers that multiply to the ac-product and add to -19, it is convenient to group the factors of the product

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

in such a way that the products of each group differ by 19. It turns out that grouping all the 2's and all the 3's satisfy this condition, as 8 and 27 differ by 19. Thus, the desired integers are -27 and 8, as the sum of them must be -19. So, we factor

$$\begin{aligned} &18a^2 - 19ab - 12b^2 \\ &= 18a^2 - 27ab + 8ab - 12b^2 \\ &= 9a(2a - 3b) + 4b(2a - 3b) \\ &= \mathbf{(2a - 3b)(9a + 4b)} \end{aligned}$$

d. To factor $2(x+3)^2 + 5(x+3) - 12$, first, we notice that treating the group $(x+3)$ as another variable, say a, simplifies the problem to factoring the quadratic trinomial

$$2a^2 + 5a - 12$$

This can be done by the guessing method. Since

$$2a^2 + 5a - 12 = (2a - 3)(a + 4),$$

$$-3a$$

$$8a$$

then

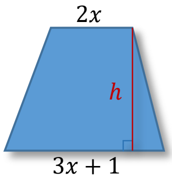
$$\begin{aligned} 2(x+3)^2 + 5(x+3) - 12 &= [2(x+3) - 3][(x+3) + 4] \\ &= (2x + 6 - 3)(x + 3 + 4) \\ &= \mathbf{(2x + 3)(x + 7)} \end{aligned}$$

Note 1: Polynomials that can be written in the form $\boldsymbol{a}(\quad)^2 + \boldsymbol{b}(\quad) + \boldsymbol{c}$, where $a \neq 0$ and ( ) represents any nonconstant polynomial expression, are referred to as **quadratic in form**. To factor such polynomials, it is convenient to **replace** the expression in the bracket by **a single variable**, different than the original one. This was illustrated in *Example 2d* by substituting a for $(x+3)$. However, when using this **substitution method**, we must **remember to leave the final answer in terms of the original variable**. So, after factoring, we replace a back with $(x+3)$, and then simplify each factor.

Note 2: Some students may feel comfortable factoring polynomials quadratic in form directly, without using substitution.

Example 3 ► **Application of Factoring in Geometry Problems**

Suppose that the area in square meters of a trapezoid is given by the polynomial $5x^2 - 9x - 2$. If the two bases are $2x$ and $(3x + 1)$ meters long, then what polynomial represents the height of the trapezoid?



Solution ► Using the formula for the area of a trapezoid, we write the equation

$$\frac{1}{2}h(a+b) = 5x^2 - 9x - 2$$

Since $a + b = 2x + (3x+1) = 5x + 1$, then we have

$$\frac{1}{2}h(5x+1) = 5x^2 - 9x - 2,$$

which after factoring the right-hand side gives us

$$\frac{1}{2}h(5x+1) = (5x+1)(x-2).$$

To find h, it is enough to divide the above equation by the common factor $(5x+1)$ and then multiply it by 2. So,

$$h = 2(x-2) = \mathbf{2x-4}.$$

F.2 Exercises

1. If $ax^2 + bx + c$ has no monomial factor, can either of the possible binomial factors have a monomial factor?
2. Is $(2x+5)(2x-4)$ a complete factorization of the polynomial $4x^2 + 2x - 20$?

3. When factoring the polynomial $-2x^2 - 7x + 15$, students obtained the following answers:

$$(-2x + 3)(x + 5),\ \ (2x - 3)(-x - 5),\ \text{ or }\ -(2x - 3)(x + 5)$$

Which of the above factorizations are correct?

4. Is the polynomial $x^2 - x + 2$ factorable or is it prime?

Fill in the missing factor.

5. $x^2 - 4x + 3 = (\qquad)(x - 1)$

6. $x^2 + 3x - 10 = (\qquad)(x - 2)$

7. $x^2 - xy - 20y^2 = (x + 4y)(\qquad)$

8. $x^2 + 12xy + 35y^2 = (x + 5y)(\qquad)$

Factor, if possible.

9. $x^2 + 7x + 12$

10. $x^2 - 12x + 35$

11. $y^2 + 2y - 48$

12. $a^2 - a - 42$

13. $x^2 + 2x + 3$

14. $p^2 - 12p - 27$

15. $m^2 - 15m + 56$

16. $y^2 + 3y - 28$

17. $18 - 7n - n^2$

18. $20 + 8p - p^2$

19. $x^2 - 5xy + 6y^2$

20. $p^2 + 9pq + 20q^2$

Factor completely.

21. $-x^2 + 4x + 21$

22. $-y^2 + 14y + 32$

23. $n^4 - 13n^3 - 30n^2$

24. $y^3 - 15y^2 + 54y$

25. $-2x^2 + 28x - 80$

26. $-3x^2 - 33x - 72$

27. $x^4y + 7x^2y - 60y$

28. $24ab^2 + 6a^2b^2 - 3a^3b^2$

29. $40 - 35t^{15} - 5t^{30}$

30. $x^4y^2 + 11x^2y + 30$

31. $64n - 12n^5 - n^9$

32. $24 - 5x^a - x^{2a}$

33. If a polynomial $x^2 + \square x + 36$ with an unknown coefficient *b* by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of *b*?

Fill in the missing factor.

34. $2x^2 + 7x + 3 = (\qquad)(x + 3)$

35. $3x^2 - 10x + 8 = (\qquad)(x - 2)$

36. $4x^2 + 8x - 5 = (2x - 1)(\qquad)$

37. $6x^2 - x - 15 = (2x + 3)(\qquad)$

Factor completely.

38. $2x^2 - 5x - 3$

39. $6y^2 - y - 2$

40. $4m^2 + 17m + 4$

41. $6t^2 - 13t + 6$

42. $10x^2 + 23x - 5$

43. $42n^2 + 5n - 25$

44. $3p^2 - 27p + 24$

45. $-12x^2 - 2x + 30$

46. $6x^2 + 41xy - 7y^2$

47. $18x^2 + 27xy + 10y^2$

48. $8 - 13a + 6a^2$

49. $15 - 14n - 8n^2$

50. $30x^4 + 3x^3 - 9x^2$

51. $10x^3 - 6x^2 + 4x^4$

52. $2y^6 + 7xy^3 + 6x^2$

53. $9x^2y^2 - 4 + 5xy$

54. $16x^2y^3 + 3y - 16xy^2$

55. $4p^4 - 28p^2q + 49q^2$

56. $4(x-1)^2 - 12(x-1) + 9$

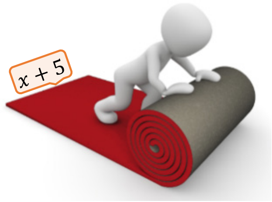
57. $2(a+2)^2 + 11(a+2) + 15$

58. $4x^{2a} - 4x^a - 3$

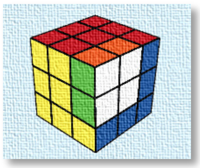
59. If a polynomial $3x^2 + \boxed{}x - 20$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of $\boxed{b}$?

60. The volume of a case of apples is $2x^3 - 3x^2 - 2x$ cubic feet, and the height of the case is $(x - 2)$ feet. Find a polynomial representing the area of the bottom of the case?

61. Suppose the width of a rectangular runner carpet is $(x + 5)$ feet. If the area of the carpet is $(3x^2 + 17x + 10)$ square feet, find the polynomial that represents the length of the carpet.



F3 Special Factoring and a General Strategy of Factoring



Recall that in *Section P2*, we considered formulas that provide a shortcut for finding special products, such as a product of two **conjugate** binomials,

$$(a+b)(a-b)=\boldsymbol{a^2-b^2},$$

or the **perfect square** of a binomial,

$$(a\pm b)^2=\boldsymbol{a^2\pm 2ab+b^2}.$$

Since factoring reverses the multiplication process, these formulas can be used as shortcuts in factoring binomials of the form $\boldsymbol{a^2-b^2}$ **(difference of squares)**, and trinomials of the form $\boldsymbol{a^2\pm 2ab+b^2}$ **(perfect square)**. In this section, we will also introduce a formula for factoring binomials of the form $\boldsymbol{a^3\pm b^3}$ **(sum or difference of cubes)**. These special product factoring techniques are very useful in simplifying expressions or solving equations, as they allow for more efficient algebraic manipulations.

At the end of this section, we give a summary of all the factoring strategies shown in this chapter.

Difference of Squares

Out of the special factoring formulas, the easiest one to use is the difference of squares,

$$\boxed{\boldsymbol{a^2-b^2=(a+b)(a-b)}}$$

Figure 3.1 shows a geometric interpretation of this formula. The area of the yellow square, a^2, diminished by the area of the blue square, b^2, can be rearranged to a rectangle with the length of $(a+b)$ and the width of $(a-b)$.

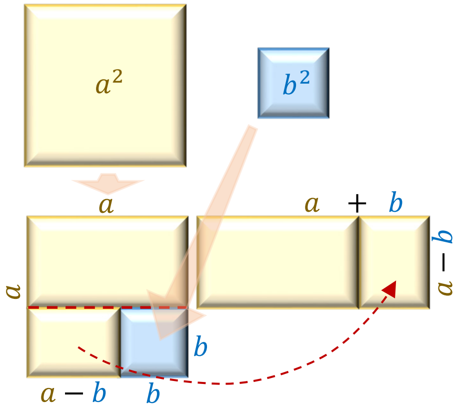


Figure 3.1

To factor a difference of squares $\boldsymbol{a^2-b^2}$, first, identify $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared, and then, form two factors, the sum $(\boldsymbol{a+b})$, and the difference $(\boldsymbol{a-b})$, as illustrated in the example below.

Example 1 ► **Factoring Differences of Squares**

Factor each polynomial completely.

a. $25x^2-1$ **b.** $3.6x^4-0.9y^6$

c. x^4-81 **d.** $16-(a-2)^2$

Solution ► **a.** First, we rewrite each term of $25x^2-1$ as a perfect square of an expression.

$$25x^2-1=(\overset{\overset{a}{\downarrow}}{5x})^2-\overset{\overset{b}{\downarrow}}{1}{}^2$$

Then, treating $5x$ as the a and 1 as the b in the difference of squares formula $a^2-b^2=(a+b)(a-b)$, we factor:

$$\begin{array}{c} a^2 - b^2 = (a + b)(a - b) \\ \downarrow \quad \downarrow \quad\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ 25x^2 - 1 = (5x)^2 - 1^2 = (\mathbf{5x + 1})(\mathbf{5x - 1}) \end{array}$$

b. First, we factor out 0.9 to leave the coefficients in a perfect square form. So,

$$3.6x^4 - 0.9y^6 = 0.9(4x^4 - y^6)$$

Then, after writing the terms of $4x^4 - y^6$ as perfect squares of expressions that correspond to a and b in the difference of squares formula $a^2 - b^2 = (a + b)(a - b)$, we factor

$$0.9(4x^4 - y^6) = 0.9[(\overset{\overset{a}{\downarrow}}{2x^2})^2 - (\overset{\overset{b}{\downarrow}}{y^3})^2] = \mathbf{0.9(2x^2 + y^3)(2x^2 - y^3)}$$

c. Similarly as in the previous two examples, $x^4 - 81$ can be factored by following the difference of squares pattern. So,

$$x^4 - 81 = (x^2)^2 - (9)^2 = (x^2 + 9)(x^2 - 9)$$

However, this factorization is not complete yet. Notice that $x^2 - 9$ is also a difference of squares, so the original polynomial can be factored further. Thus,

$$x^4 - 81 = (x^2 + 9)(x^2 - 9) = \mathbf{(x^2 + 9)(x + 3)(x - 3)}$$

Attention: The sum of squares, $x^2 + 9$, cannot be factored using real coefficients.

Generally, except for a common factor, a quadratic binomial of the form $\boldsymbol{a^2 + b^2}$ is **not factorable** over the real numbers.

Recall that
$\boldsymbol{a^2 + b^2 \neq (a + b)^2}$

d. Following the difference of squares formula, we have

$$\begin{aligned} 16 - (a - 2)^2 &= 4^2 - (a - 2)^2 \\ &= [4 + (a - 2)][4 - (a - 2)] \\ &= (4 + a - 2)(4 - a + 2) \\ &= \mathbf{(2 + a)(6 - a)} \end{aligned}$$

Remember to use brackets after the negative sign!

work out the inner brackets

combine like terms

Perfect Squares

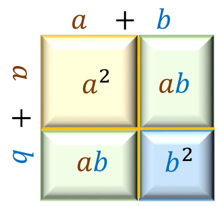


Figure 3.2

Another frequently used special factoring formula is the **perfect square** of a sum or a difference.

$$\boldsymbol{a^2 + 2ab + b^2 = (a + b)^2}$$

or

$$\boldsymbol{a^2 - 2ab + b^2 = (a - b)^2}$$

Figure 3.2 shows the geometric interpretation of the perfect square of a sum. We encourage the reader to come up with a similar interpretation of the perfect square of a difference.

To factor a perfect square trinomial $\boldsymbol{a^2 \pm 2ab + b^2}$, we find $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared. Then, depending on the middle sign, we use $\boldsymbol{a}$ and $\boldsymbol{b}$ to form the perfect square of the sum $\boldsymbol{(a+b)^2}$, or the perfect square of the difference $\boldsymbol{(a-b)^2}$.

Example 2 ► Identifying Perfect Square Trinomials

Decide whether the given polynomial is a perfect square.

a. $9x^2 + 6x + 4$

b. $9x^2 + 4y^2 - 12xy$

c. $25p^4 + 40p^2 - 16$

d. $49y^6 + 84xy^3 + 36x^2$

Solution ►

a. Observe that the outside terms of the trinomial $9x^2 + 6x + 4$ are perfect squares, as $9x^2 = (3x)^2$ and $4 = 2^2$. So, the trinomial would be a perfect square if the middle terms would equal $2 \cdot 3x \cdot 2 = 12x$. Since this is not the case, our trinomial is **not a perfect square**.

Attention: Except for a common factor, trinomials of the type $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable** over the real numbers!

b. First, we arrange the trinomial in decreasing order of the powers of x. So, we obtain $9x^2 - 12xy + 4y^2$. Then, since $9x^2 = (3x)^2$, $4y^2 = (2y)^2$, and the middle term (except for the sign) equals $2 \cdot 3x \cdot 2y = 12xy$, we claim that the trinomial **is a perfect square**. Since the middle term is negative, this is the perfect square of a difference. So, the trinomial $9x^2 - 12xy + 4y^2$ can be seen as

$$\begin{array}{ccccccccccc} a^2 & - & 2 & a & b & + & b^2 & = & (a & - & b)^2 \\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (3x)^2 & - & 2 & \cdot 3x & \cdot 2y & + & (2y)^2 & = & (3x & - & 2y)^2 \end{array}$$

c. Even though the coefficients of the trinomial $25p^4 + 40p^2 - 16$ and the distribution of powers seem to follow the pattern of a perfect square, the last term is negative, which makes it **not a perfect square**.

d. Since $49y^6 = (7y^3)^2$, $36x^2 = (6x)^2$, and the middle term equals $2 \cdot 7y^3 \cdot 6x = 84xy^3$, we claim that the trinomial **is a perfect square**. Since the middle term is positive, this is the perfect square of a sum. So, the trinomial $49y^6 + 84xy^3 + 36x^2$ can be seen as

$$\begin{array}{ccccccccccc} a^2 & + & 2 & a & b & + & b^2 & = & (a & + & b)^2 \\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (7y^3)^2 & + & 2 & \cdot 7y^3 & \cdot 6x & + & (6x)^2 & = & (7y^3 & + & 6x)^2 \end{array}$$

Example 3 ► Factoring Perfect Square Trinomials

Factor each polynomial completely.

a. $25x^2 + 10x + 1$

b. $a^2 - 12ab + 36b^2$

c. $m^2 - 8m + 16 - 49n^2$

d. $-4y^2 - 144y^8 + 48y^5$

Solution ►

a. The outside terms of the trinomial $25x^2 + 10x + 1$ are perfect squares of $5a$ and 1, and the middle term equals $2 \cdot 5x \cdot 1 = 10x$, so we can follow the perfect square formula. Therefore,

$$25x^2 + 10x + 1 = (\mathbf{5x} + \mathbf{1})^2$$

b. The outside terms of the trinomial $a^2 - 12ab + 36b^2$ are perfect squares of a and $6b$, and the middle term (disregarding the sign) equals $2 \cdot a \cdot 6b = 12ab$, so we can follow the perfect square formula. Therefore,

$$a^2 - 12ab + 36b^2 = (\boldsymbol{a} - \boldsymbol{6b})^2$$

c. Observe that the first three terms of the polynomial $m^2 - 8m + 16 - 49n^2$ form a perfect square of $m - 4$ and the last term is a perfect square of $7n$. So, we can write

$$m^2 - 8m + 16 - 49n^2 = (m - 4)^2 - (7n)^2$$

This is not in factored form yet!

Notice that this way we have formed a difference of squares. So we can factor it by following the difference of squares formula

$$(m - 4)^2 - (7n)^2 = (\boldsymbol{m - 4 - 7n})(\boldsymbol{m - 4 + 7n})$$

d. As in any factoring problem, first we check the polynomial $-4y^2 - 144y^8 + 48y^5$ for a common factor, which is $4y^2$. To leave the leading term of this polynomial positive, we factor out $-4y^2$. So, we obtain

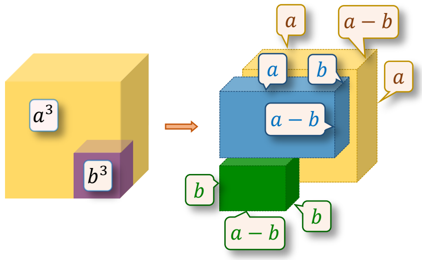
$$-4y^2 - 144y^8 + 48y^5$$
$$= -4y^2\,(1 + 36y^6 - 12y^3)$$
$$= -4y^2\,(36y^6 - 12y^3 + 1)$$

arrange the polynomial in decreasing powers

$$= \boldsymbol{-4y^2\,(6y^3 - 1)^2}$$

fold to the perfect square form

Sum or Difference of Cubes



$$a^3 - b^3 = a^2(a - b) + ab(a - b) + b^2(a - b)$$
$$= (a - b)(a^2 + ab + b^2)$$

Figure 3.3

The last special factoring formula to discuss in this section is the **sum or difference of cubes**.

$$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

or

$$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

The reader is encouraged to confirm these formulas by multiplying the factors in the right-hand side of each equation. In addition, we offer a geometric visualization of one of these formulas, the difference of cubes, as shown in *Figure 3.3*.

Hints for memorization of the sum or difference of cubes formulas:

- The binomial factor is a copy of the sum or difference of the terms that were originally cubed.
- The trinomial factor follows the pattern of a perfect square, except that the **middle term is single**, not doubled.
- The signs in the factored form follow the pattern ***S****ame*-***O****pposite*-***P****ositive* (**SOP**).

Example 4 ► Factoring Sums or Differences of Cubes

Factor each polynomial completely.

a. $8x^3 + 1$ **b.** $27x^7y - 125xy^4$

c. $2n^6 - 128$ **d.** $(p-2)^3 + q^3$

Solution ►

a. First, we rewrite each term of $8x^3 + 1$ as a perfect cube of an expression.

$$8x^3 + 1 = (\overset{\substack{a\\ \downarrow}}{2x})^3 + \overset{\substack{b\\ \downarrow}}{1}{}^3$$

Then, treating $2x$ as the a and 1 as the b in the sum of cubes formula $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, we factor:

$$8x^3 + 1 = \overset{\substack{a^3\\ \downarrow}}{(2x)^3} + \overset{\substack{b^3\\ \downarrow}}{1^3} = (\overset{\substack{a\\ \downarrow}}{2x} + \overset{\substack{b\\ \downarrow}}{1})(\overset{\substack{a^2\\ \downarrow}}{(2x)^2} - \overset{\substack{a\\ \downarrow}}{2x} \cdot \overset{\substack{b\\ \downarrow}}{1} + \overset{\substack{b^2\\ \downarrow}}{1^2})$$

$$= \mathbf{(2x+1)(4x^2 - 2x + 1)}$$

Quadratic trinomials of the form $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable**!

Notice that the trinomial $4x^2 - 2x + 1$ is not factorable anymore.

b. Since the two terms of the polynomial $27x^7y - 125xy^4$ contain the common factor xy, we factor it out and obtain

$$27x^7y - 125xy^4 = xy(27x^6 - 125y^3)$$

Observe that the remaining polynomial is a difference of cubes, $(3x^2)^3 - (5y)^3$. So, we factor,

$$27x^7y - 125xy^4 = xy[(3x^2)^3 - (5y)^3]$$

$$= xy(\overset{\substack{a\\ \downarrow}}{3x^2} - \overset{\substack{b\\ \downarrow}}{5y})[\overset{\substack{a^2\\ \downarrow}}{(3x^2)^2} + \overset{\substack{a\\ \downarrow}}{3x^2} \cdot \overset{\substack{b\\ \downarrow}}{5y} + \overset{\substack{b^2\\ \downarrow}}{(5y)^2}]$$

$$= \boldsymbol{xy(3x^2 - 5y)(9x^4 + 15x^2y + 25y^2)}$$

c. After factoring out the common factor 2, we obtain

$$2n^6 - 128 = 2(n^6 - 64)$$

Difference of squares or **difference of cubes?**

Notice that $n^6 - 64$ can be seen either as a difference of squares, $(n^3)^2 - 8^2$, or as a difference of cubes, $(n^2)^3 - 4^3$. It turns out that applying the **difference of squares** formula first **leads us to a complete factorization** while starting with the difference of cubes does not work so well here. See the two approaches below.

$$(n^3)^2 - 8^2$$
$$= (n^3 + 8)(n^3 - 8)$$
$$= (n + 2)(n^2 - 2n + 4)(n - 2)(n^2 + 2n + 4)$$

4 prime factors, so the factorization is complete

$$(n^2)^3 - 4^3$$
$$= (n^2 - 4)(n^4 + 4n^2 + 16)$$
$$= (n + 2)(n - 2)(n^4 + 4n^2 + 16)$$

There is no easy way of factoring this trinomial!

Therefore, the original polynomial should be factored as follows:

$$2n^6 - 128 = 2(n^6 - 64) = 2[(n^3)^2 - 8^2] = 2(n^3 + 8)(n^3 - 8)$$
$$= \mathbf{2(n + 2)(n^2 - 2n + 4)(n - 2)(n^2 + 2n + 4)}$$

d. To factor $(p - 2)^3 + q^3$, we follow the sum of cubes formula $(a + b)(a^2 - ab + b^2)$ by assuming $a = p - 2$ and $b = q$. So, we have

$$(p - 2)^3 + q^3 = (p - 2 + q)\,[(p - 2)^2 - (p - 2)q + q^2]$$
$$= (p - 2 + q)\,[p^2 - 4p + 4 - pq + 2q + q^2]$$
$$= \mathbf{(p + q - 2)\,[p^2 - pq + q^2 - 4p + 2q + 4]}$$

General Strategy of Factoring

Recall that a polynomial with integral coefficients is factored completely if all of its factors are prime over the integers.

How to Factorize Polynomials Completely?

1. Factor out all **common factors**. Leave the remaining polynomial with a positive leading term and integral coefficients, if possible.

2. Check the number of terms. If the polynomial has

- **more than three terms**, try to factor by **grouping**; a four term polynomial may require 2-2, 3-1, or 1-3 types of grouping.
- **three terms**, factor by **guessing**, **decomposition**, or follow the **perfect square** formula, if applicable.
- **two terms**, follow the **difference of squares**, or **sum or difference of cubes** formula, if applicable. Remember that sum of squares, $\boldsymbol{a^2 + b^2}$, is **not factorable** over the real numbers, except for possibly a common factor.

3. Keep in mind the **special factoring formulas**:

Difference of Squares	$\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$
Perfect Square of a Sum	$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$
Perfect Square of a Difference	$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$
Sum of Cubes	$\boldsymbol{a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)}$
Difference of Cubes	$\boldsymbol{a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)}$

4. **Keep factoring** each of the obtained factors until all of them are **prime** over the integers.

Example 5 ► Multiple-step Factorization

Factor each polynomial completely.

a. $80x^5 - 5x$ **b.** $4a^2 - 4a + 1 - b^2$

c. $(5r+8)^2 - 6(5r+8) + 9$ **d.** $(p-2q)^3 + (p+2q)^3$

Solution ► **a.** First, we factor out the GCF of $80x^5$ and $-5x$, which equals to $5x$. So, we obtain

$$80x^5 - 5x = 5x(16x^4 - 1)$$

repeated difference of squares

Then, we notice that $16x^4 - 1$ can be seen as the difference of squares $(4x^2)^2 - 1^2$. So, we factor further

$$80x^5 - 5x = 5x(4x^2+1)(4x^2-1)$$

The first binomial factor, $4x^2 + 1$, cannot be factored any further using integral coefficients as it is the sum of squares, $(2x)^2 + 1^2$. However, the second binomial factor, $4x^2 - 1$, is still factorable as a difference of squares, $(2x)^2 - 1^2$. Therefore,

$$80x^5 - 5x = \boldsymbol{5x\left(4x^2+1\right)(2x+1)(2x-1)}$$

This is a complete factorization as all the factors are prime over the integers.

3-1 type of grouping

b. The polynomial $4a^2 - 4a + 1 - b^2$ consists of four terms, so we might be able to factor it by grouping. Observe that the 2-2 type of grouping has no chance to succeed, as the first two terms involve only the variable a while the second two terms involve only the variable b. This means that after factoring out the common factor in each group, the remaining binomials would not be the same. So, the 2-2 grouping would not lead us to a factorization. However, the 3-1 type of grouping should help. This is because the first three terms form the perfect square, $(2a-1)^2$, and there is a subtraction before the last term b^2, which is also a perfect square. So, in the end, we can follow the difference of squares formula to complete the factoring process.

$$\begin{aligned} \underbrace{4a^2 - 4a + 1} - \underbrace{b^2} &= (2a-1)^2 - b^2 \\ &= \boldsymbol{(2a-1-b)(2a-1+b)} \end{aligned}$$

factoring by substitution

c. To factor $(5r+8)^2 - 6(5r+8) + 9$, it is convenient to substitute a new variable, say $\boldsymbol{a}$, for the expression $5r+8$. Then,

$$(5r+8)^2 - 6(5r+8) + 9 = \boldsymbol{a}^2 - 6\boldsymbol{a} + 9$$

perfect square!

$$= (\boldsymbol{a} - 3)^2$$

$$= (5r + 8 - 3)^2$$

go back to the original variable

$$= (5r+5)^2$$

Remember to represent the new variable by a **different letter** than the original variable!

Notice that $5r+5$ can still be factored by taking the 5 out. So, for a complete factorization, we factor further

$$(5r+5)^2 = \left(5(r+1)\right)^2 = \boldsymbol{25(r+1)^2}$$

multiple special formulas and simplifying

d. To factor $(p-2q)^3 + (p+2q)^3$, we follow the sum of cubes formula $(a+b)(a^2 - ab + b^2)$ by assuming $a = p - 2q$ and $b = p + 2q$. So, we have

$$(p-2q)^3 + (p+2q)^3$$

$$= (p - \cancel{2q} + p + \cancel{2q})\,[(p-2q)^2 - (p-2q)(p+2q) + (p+2q)^2]$$

$$= 2p\,[p^2 - \cancel{4pq} + 4q^2 - (p^2 - 4q^2) + p^2 + \cancel{4pq} + 4q^2]$$

$$= 2p\,(2p^2 + 8q^2 - p^2 + 4q^2) = \boldsymbol{2p\left(p^2 + 12q^2\right)}$$

F.3 Exercises

Determine whether each polynomial in problems 7-18 is a perfect square, a difference of squares, a sum or difference of cubes, or neither.

1. $0.25x^2 - 0.16y^2$

2. $x^2 - 14x + 49$

3. $9x^4 + 4x^2 + 1$

4. $4x^2 - (x+4)^2$

5. $125x^3 - 64$

6. $y^{12} + 0.008x^3$

7. $-y^4 + 16x^4$

8. $64 + 48x^3 + 9x^6$

9. $25x^6 - 10x^3y^2 + y^4$

10. $-4x^6 - y^6$

11. $-8x^3 + 27y^6$

12. $81x^2 - 16x$

13. Generally, the sum of squares is not factorable. For example, $x^2 + 9$ cannot be factored in integral coefficients. However, some sums of squares can be factored. For example, the binomial $25x^2 + 100$ can be factored. Factor the above example and discuss what makes a sum of two squares factorable.

14. Insert the correct signs into the blanks.

a. $8 + a^3 = (2\,\_\_\,a)(4\,\_\_\,2a\,\_\_\,a^2)$ **b.** $b^3 - 1 = (b\,\_\_\,1)(b^2\,\_\_\,b\,\_\_\,1)$

Factor each polynomial completely, if possible.

15. $x^2 - y^2$
16. $x^2 + 2xy + y^2$
17. $x^3 - y^3$
18. $16x^2 - 100$
19. $4z^2 - 4z + 1$
20. $x^3 + 27$
21. $4z^2 + 25$
22. $y^2 + 18y + 81$
23. $125 - y^3$
24. $144x^2 - 64y^2$
25. $n^2 + 20nm + 100m^2$
26. $27a^3b^6 + 1$
27. $9a^4 - 25b^6$
28. $25 - 40x + 16x^2$
29. $p^6 - 64q^3$
30. $16x^2z^2 - 100y^2$
31. $4 + 49p^2 + 28p$
32. $x^{12} + 0.008y^3$
33. $r^4 - 9r^2$
34. $9a^2 - 12ab - 4b^2$
35. $\frac{1}{8} - a^3$
36. $0.04x^2 - 0.09y^2$
37. $x^4 + 8x^2 + 1$
38. $-\frac{1}{27} + t^3$
39. $16x^6 - 121x^2y^4$
40. $9 + 60pq + 100p^2q^2$
41. $-a^3b^3 - 125c^6$
42. $36n^{2t} - 1$
43. $9a^8 - 48a^4b + 64b^2$
44. $9x^3 + 8$
45. $(x + 1)^2 - 49$
46. $\frac{1}{4}u^2 - uv + v^2$
47. $2t^4 - 128t$
48. $81 - (n + 3)^2$
49. $x^{2n} + 6x^n + 9$
50. $8 - (a + 2)^3$
51. $16z^4 - 1$
52. $5c^3 + 20c^2 + 20c$
53. $(x + 5)^3 - x^3$
54. $a^4 - 81b^4$
55. $0.25z^2 - 0.7z + 0.49$
56. $(x - 1)^3 + (x + 1)^3$
57. $(x - 2y)^2 - (x + y)^2$
58. $0.81p^8 + 9p^4 + 25$
59. $(x + 2)^3 - (x - 2)^3$

Factor each polynomial completely.

60. $3y^3 - 12x^2y$
61. $2x^2 + 50a^2 - 20ax$
62. $x^3 - xy^2 + x^2y - y^3$
63. $y^2 - 9a^2 + 12y + 36$
64. $64u^6 - 1$
65. $7m^3 + m^6 - 8$
66. $-7n^2 + 2n^3 + 4n - 14$
67. $a^8 - b^8$
68. $y^9 - y$
69. $(x^2 - 2)^2 - 4(x^2 - 2) - 21$
70. $8(p - 3)^2 - 64(p - 3) + 128$
71. $a^2 - b^2 - 6b - 9$
72. $25(2a - b)^2 - 9$
73. $3x^2y^2z + 25xyz^2 + 28z^3$
74. $x^{8a} - y^2$
75. $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$
76. $4x^2y^4 - 9y^4 - 4x^2z^4 + 9z^4$
77. $c^{2w+1} + 2c^{w+1} + c$

F4 Solving Polynomial Equations and Applications of Factoring

Many application problems involve solving polynomial equations. In Chapter L, we studied methods for solving linear, or first-degree, equations. Solving higher degree polynomial equations requires other methods, which often involve factoring. In this chapter, we study solving polynomial equations using the zero-product property, graphical connections between roots of an equation and zeros of the corresponding function, and some application problems involving polynomial equations or formulas that can be solved by factoring.

Zero-Product Property

Recall that to solve a linear equation, for example $2x + 1 = 0$, it is enough to isolate the variable on one side of the equation by applying reverse operations. Unfortunately, this method usually does not work when solving higher degree polynomial equations. For example, we would not be able to solve the equation $x^2 - x = 0$ through the reverse operation process, because the variable x appears in different powers.

So … how else can we solve it?

In this particular example, it is possible to guess the solutions. They are $x = 0$ and $x = 1$.

But how can we solve it algebraically?

It turns out that factoring the left-hand side of the equation $x^2 - x = 0$ helps. Indeed, $x(x - 1) = 0$ tells us that the product of x and $x - 1$ is 0. Since the product of two quantities is 0, at least one of them must be 0. So, either $x = 0$ or $x - 1 = 0$, which solves to $x = 1$.

The equation discussed above is an example of a second degree polynomial equation, more commonly known as a quadratic equation.

Definition 4.1 ► A **quadratic equation** is a second degree polynomial equation in one variable that can be written in the form,

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $a \neq 0$. This form is called **standard form**.

One of the methods of solving such equations involves factoring and the zero-product property that is stated below.

Zero-Product Property ► For any real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{ab = 0} \text{ if and only if } \boldsymbol{a = 0} \text{ or } \boldsymbol{b = 0}$$

This means that any product containing a factor of 0 is equal to 0, and conversely, if a product is equal to 0, then at least one of its factors is equal to 0.

Proof ► The implication "if $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$, then $\boldsymbol{ab = 0}$" is true by the multiplicative property of zero.

To prove the implication "if $\boldsymbol{ab = 0}$, then $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$", let us assume first that $a \neq 0$. (As, if $a = 0$, then the implication is already proven.)

Since $a \neq 0$, then $\frac{1}{a}$ exists. Therefore, both sides of $ab = 0$ can be multiplied by $\frac{1}{a}$ and we obtain

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$

$$b = 0,$$

which concludes the proof.

Attention: The zero-product property works only for a product equal to **0**. For example, the fact that $\boldsymbol{ab = 1}$ does not mean that either a or b equals to 1.

Example 1 ► **Using the Zero-Product Property to Solve Polynomial Equations**

Solve each equation.

a. $(x-3)(2x+5) = 0$ **b.** $2x(x-5)^2 = 0$

Solution ► **a.** Since the product of $x-3$ and $2x+5$ is equal to zero, then by the zero-product property at least one of these expressions must equal to zero. So,

$$x - 3 = 0 \quad \text{or} \quad 2x + 5 = 0$$

which results in $x = 3$ or $2x = -5$

$$x = -\frac{5}{2}$$

Thus, $\left\{-\frac{5}{2}, 3\right\}$ is the solution set of the given equation.

b. Since the product $2x(x-5)^2$ is zero, then either $x = 0$ or $x - 5 = 0$, which solves to $x = 5$. Thus, the solution set is equal to $\{\mathbf{0}, \mathbf{5}\}$.

Note 1: The factor of 2 does not produce any solution, as 2 is never equal to 0.

Note 2: The perfect square $(x-5)^2$ equals to 0 if and only if the base $x - 5$ equals to 0.

Solving Polynomial Equations by Factoring

To solve polynomial equations of second or higher degree by factoring, we

- **arrange** the polynomial **in decreasing order** of powers **on one side** of the equation,
- keep the **other side** of the equation **equal to 0**,
- **factor** the polynomial **completely**,
- use the zero-product property to **form linear equations for each factor**,
- **solve** the linear equations to find the roots (solutions) to the original equation.

Example 2 ► **Solving Quadratic Equations by Factoring**

Solve each equation by factoring.

a. $x^2 + 9 = 6x$

b. $15x^2 - 12x = 0$

c. $(x + 2)(x - 1) = 4(3 - x) - 8$

d. $(x - 3)^2 = 36x^2$

Solution ►

a. To solve $x^2 + 9 = 6x$ by factoring we need one side of this equation equal to 0. So, first, we move the $6x$ term to the left side of the equation,

$$x^2 + 9 - 6x = 0,$$

and arrange the terms in decreasing order of powers of x,

$$x^2 - 6x + 9 = 0.$$

Then, by observing that the resulting trinomial forms a perfect square of $x - 3$, we factor

$$(x - 3)^2 = 0,$$

which is equivalent to

$$x - 3 = 0,$$

and finally

$$x = 3.$$

So, the solution is $x = \mathbf{3}$.

b. After factoring the left side of the equation $15x^2 - 12x = 0$,

$$3x(5x - 4) = 0,$$

we use the zero-product property. Since 3 is never zero, the solutions come from the equations

$$x = \mathbf{0} \quad \text{or} \quad 5x - 4 = 0.$$

Solving the second equation for x, we obtain

$$5x = 4,$$

and finally

$$x = \frac{\mathbf{4}}{\mathbf{5}}.$$

So, the solution set consists of **0** and $\frac{\mathbf{4}}{\mathbf{5}}$.

c. To solve $(x + 2)(x - 1) = 4(3 - x) - 8$ by factoring, first, we work out the brackets and arrange the polynomial in decreasing order of exponents on the left side of the equation. So, we obtain

$$x^2 + x - 2 = 12 - 4x - 8$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

Now, we can read the solutions from each bracket, that is, $x = \mathbf{-6}$ and $x = \mathbf{1}$.

Observation: In the process of solving a linear equation of the form $ax + b = 0$, first we subtract b and then we divide by a. So the solution, sometimes referred to as the root, is $x = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$. This allows us to read the solution directly from the equation. For example, the solution to $x - 1 = 0$ is $x = 1$ and the solution to $2x - 1 = 0$ is $x = \frac{1}{2}$.

d. To solve $(x-3)^2 = 36x^2$, we bring all the terms to one side and factor the obtained difference of squares, following the formula $a^2 - b^2 = (a+b)(a-b)$. So, we have

$$(x-3)^2 - 36x^2 = 0$$
$$(x - 3 + 6x)(x - 3 - 6x) = 0$$
$$(7x-3)(-5x-3) = 0$$

Then, by the zero-product property,

$$7x - 3 = 0 \quad \text{or} \quad -5x - 3 = 0,$$

which results in

$$x = \mathbf{\frac{3}{7}} \quad \text{or} \quad x = -\mathbf{\frac{3}{5}}.$$

Example 3 ► **Solving Polynomial Equations by Factoring**

Solve each equation by factoring.

a. $2x^3 - 2x^2 = 12x$

b. $x^4 + 36 = 13x^2$

Solution ► **a.** First, we bring all the terms to one side of the equation and then factor the resulting polynomial.

$$2x^3 - 2x^2 = 12x$$
$$2x^3 - 2x^2 - 12x = 0$$
$$2x(x^2 - x - 6) = 0$$
$$2x(x-3)(x+2) = 0$$

By the zero-product property, the factors $x, (x-3)$ and $(x+2)$, give us the corresponding solutions, $0, 3,$ and -2. So, the solution set of the given equation is $\{\mathbf{0}, \mathbf{3}, \mathbf{-2}\}$.

b. Similarly as in the previous examples, we solve $x^4 + 36 = 13x^2$ by factoring and using the zero-product property. Since

$$x^4 - 13x^2 + 36 = 0$$

$$(x^2-4)(x^2-9)=0$$
$$(x+2)(x-2)(x+3)(x-3)=0,$$

then, the solution set of the original equation is $\{\boldsymbol{-2,2,-3,3}\}$

Observation: n-th degree polynomial equations may have up to n roots (solutions).

Factoring in Applied Problems

Factoring is a useful strategy when solving applied problems. For example, factoring is often used in **solving formulas** for a variable, in **finding roots** of a polynomial function, and generally, in any problem involving **polynomial equations** that can be solved by factoring.

Example 4 Solving Formulas with the Use of Factoring

Solve each formula for the specified variable.

a. $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$, for $\boldsymbol{h}$ **b.** $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solution

a. To solve $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$ for $\boldsymbol{h}$, we want to keep both terms containing $\boldsymbol{h}$ on the same side of the equation and bring the remaining terms to the other side. Here is an equivalent equation,

$$A-2wl=2\boldsymbol{h}w+2l\boldsymbol{h},$$

which, for convenience, could be written starting with h-terms:

$$2\boldsymbol{h}w+2l\boldsymbol{h}=A-2wl$$

Now, factoring $\boldsymbol{h}$ out causes $\boldsymbol{h}$ to appear in only one place, which is what we need to isolate it. So,

$$(2w+2l)\boldsymbol{h}=A-2wl$$
$$\boldsymbol{h}=\boldsymbol{\frac{A-2wl}{2w+2l}}$$

Notice: In the above formula, there is nothing that can be simplified. Trying to reduce 2 or $2w$ or l would be an error, as there is no essential common factor that can be carried out of the numerator.

b. When solving $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$ for $\boldsymbol{t}$, our goal is to, firstly, keep the variable $\boldsymbol{t}$ in the numerator and secondly, to keep it in a single place. So, we have

$$s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$$
$$s\boldsymbol{t}=2\boldsymbol{t}+3$$

factor t

$$\begin{aligned} s\boldsymbol{t} - 2\boldsymbol{t} &= 3 \\ \boldsymbol{t}(s-2) &= 3 \\ \boldsymbol{t} &= \boldsymbol{\frac{3}{s-2}}. \end{aligned}$$

Example 5 ► Finding Roots of a Polynomial Function

A toy-rocket is launched vertically with an initial velocity of 40 meters per second. If its height in meters after t seconds is given by the function

$$h(t) = -5t^2 + 40t,$$

in how many seconds will the rocket hit the ground?

Solution ► The rocket hits the ground when its height is 0. So, we need to find the time t for which $h(t) = 0$. Therefore, we solve the equation

$$-5t^2 + 40t = 0$$

for t. From the factored form

$$-5t(t-8) = 0$$

we conclude that the rocket is on the ground at times 0 and 8 seconds. So, the rocket hits the ground **8 seconds** after it was launched.

Example 6 ► Solving an Application Problem with the Use of Factoring

The height of a triangle is 1 meter less than twice the length of the base. If the area of the triangle is 14 m², how long are the base and the height?

Solution ► Let b and h represent the base and the height of the triangle, correspondingly. The first sentence states that h is 1 less than 2 times b. So, we record

$$h = 2b - 1.$$

Using the formula for area of a triangle, $A = \frac{1}{2}bh$, and the fact that $A = 14$, we obtain

$$14 = \frac{1}{2}b(2b-1).$$

Since this is a one-variable quadratic equation, we will attempt to solve it by factoring, after bringing all the terms to one side of the equation. So, we have

to clear the fraction, multiply each term by 2 before working out the bracket

$$\begin{aligned} 0 &= \frac{1}{2}b(2b-1) - 14 \\ 0 &= b(2b-1) - 28 \\ 0 &= 2b^2 - b - 28 \end{aligned}$$

$$0 = (2b + 7)(b - 4),$$

which by the zero-product property gives us $b = -\frac{7}{2}$ or $b = 4$. Since b represents the length of the base, it must be positive. So, the base is **4** meters long and the height is $h = 2b - 1 = 2 \cdot 4 - 1 = \mathbf{7}$ meters long.

F.4 Exercises

True or false.

1. If $xy = 0$ then $x = 0$ or $y = 0$.

2. If $ab = 1$ then $a = 1$ or $b = 1$.

3. If $x + y = 0$ then $x = 0$ or $y = 0$.

4. If $a^2 = 0$ then $a = 0$.

5. If $x^2 = 1$ then $x = 1$.

6. Which of the following equations is **not** in proper form for using the zero-product property.

a. $x(x-1) + 3(x-1) = 0$

b. $(x+3)(x-1) = 0$

c. $x(x-1) = 3(x-1)$

d. $(x+3)(x-1) = -3$

Solve each equation.

7. $3(x-1)(x+4) = 0$

8. $2(x+5)(x-7) = 0$

9. $(3x+1)(5x+4) = 0$

10. $(2x-3)(4x-1) = 0$

11. $x^2 + 9x + 18 = 0$

12. $x^2 - 18x + 80 = 0$

13. $2x^2 = 7 - 5x$

14. $3k^2 = 14k - 8$

15. $x^2 + 6x = 0$

16. $6y^2 - 3y = 0$

17. $(4-a)^2 = 0$

18. $(2b+5)^2 = 0$

19. $0 = 4n^2 - 20n + 25$

20. $0 = 16x^2 + 8x + 1$

21. $p^2 - 32 = -4p$

22. $19a + 36 = 6a^2$

23. $x^2 + 3 = 10x - 2x^2$

24. $3x^2 + 9x + 30 = 2x^2 - 2x$

25. $(3x+4)(3x-4) = -10x$

26. $(5x+1)(x+3) = -2(5x+1)$

27. $4(y-3)^2 - 36 = 0$

28. $3(a+5)^2 - 27 = 0$

29. $(x-3)(x+5)=-7$

30. $(x+8)(x-2)=-21$

31. $(2x-1)(x-3)=x^2-x-2$

32. $4x^2+x-10=(x-2)(x+1)$

33. $4(2x+3)^2-(2x+3)-3=0$

34. $5(3x-1)^2+3=-16(3x-1)$

35. $x^3+2x^2-15x=0$

36. $6x^3-13x^2-5x=0$

37. $25x^3=64x$

38. $9x^3=49x$

39. $y^4-26y^2+25=0$

40. $n^4-50n^2+49=0$

41. $x^3-6x^2=-8x$

42. $x^3-2x^2=3x$

43. $a^3+a^2-9a-9=0$

44. $2x^3-x^2-2x+1=0$

45. $5x^3+2x^2-20x-8=0$

46. $2x^3+3x^2-18x-27=0$

47. Discuss the validity of the following solution:

$$x^3=9x$$
$$x^2=9$$
$$x=3$$

How many solutions should we expect? What is the solution set of the original equation? What went wrong in the above procedure?

48. Given that $f(x)=x^2+14x+50$, find all values of x such that $f(x)=5$.

49. Given that $g(x)=2x^2-15x$, find all values of x such that $g(x)=-7$.

50. Given that $f(x)=2x^2+3x$ and $g(x)=-6x+5$, find all values of x such that $f(x)=g(x)$.

51. Given that $g(x)=2x^2+11x-16$ and $h(x)=5+9x-x^2$, find all values of x such that $g(x)=h(x)$.

Solve each equation for the specified variable.

52. $\boldsymbol{P}rt=A-\boldsymbol{P}$, for $\boldsymbol{P}$

53. $3\boldsymbol{s}+2p=5-r\boldsymbol{s}$, for $\boldsymbol{s}$

54. $5a+b\boldsymbol{r}=\boldsymbol{r}-2c$, for $\boldsymbol{r}$

55. $E=\frac{R+\boldsymbol{r}}{\boldsymbol{r}}$, for $\boldsymbol{r}$

56. $z=\frac{x+2\boldsymbol{y}}{\boldsymbol{y}}$, for $\boldsymbol{y}$

57. $c=\frac{-2\boldsymbol{t}+4}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solve each problem.

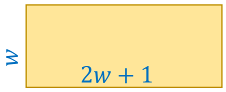
58. Bartek threw down a small rock from the top of a 120 m high observation tower. Suppose the distance travelled by the rock, in meters, is modelled by the function $d(t)=vt+4t^2$, where v is the initial velocity in m/s, and t is the time in seconds. In how many seconds will the rock hit the ground if it was thrown with the initial velocity of 4 m/s?

59. A camera is dropped from a hot-air balloon 320 meters above the ground. Suppose the height of the camera above the ground, in meters, is given by the function $h(t)=320-5t^2$, where t is the time in seconds. How long will it take for the camera to hit the ground?

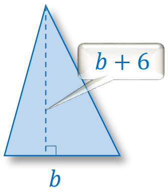
60. The sum of squares of two consecutive numbers is 85. Find the smaller number.

61. The difference between a number and its square is -156. Find the number.

62. The length of a rectangle is 1 centimeter more than twice the width. If the area of this rectangle is 105 cm^2, find its width and length.

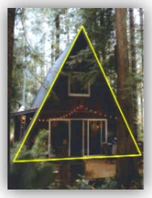


63. A postcard is 7 cm longer than it is wide. The area of this postcard is 144 cm^2. Find its length and width.



64. A triangle with the area of 80 cm^2 is 6 cm taller than the length of its base. Find the dimensions of the triangle.

65. A triangular house is 3 m taller than it is wide. If the cross-sectional area (see the accompanying picture) of the house is 35 m^2, what are the width and the height of this house?



66. Amira designs a rectangular flower bed with a pathway of uniform width around it. She has 42 square meters of ground available for the whole project (including the path). If the flower bed is planned to be 3 meters by 4 meters, how wide would be the pathway around it?

67. Suppose a rectangular flower bed is 5 m longer than it is wide. What are the dimensions of the flower bed if its area is 84 m^2 ?

68. Suppose a picture frame measures 10 cm by 18 cm, and it frames a picture with 48 cm^2 of area. How wide is the frame?

69. When 187 cm^2 picture is framed, its outside dimensions become 15 cm by 21 cm. How wide is the frame?

70. After lengthening each side of a square by 4 cm, the area of the enlarged square turns out to be 225 cm^2. How long is the side of the original square?

71. A square piece of drywall was used to fix a hole in a wall. The sides of the piece of drywall had to be shortened by 2 inches in order to cover the required area of 49 in^2. What were the dimensions of the original piece of drywall?

Attributions

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p.**229** Epic Cluster Rocket Launch by Steve Jurvetson / CC BY 2.0
p.**232** Triangular House by Kristin Ellis / Unsplash Licence
p.**232** Tulips by andreas160578 / Pixabay Licence

Review - ANSWERS

R1 Exercises

1. true

3. true

5. false

7. $\{1,2,3,4,5,6,7,8\}$

9. $\{2,4,6,\dots\}$

11. $\{4,5,6,7,8\}$

13. Answers may vary. Examples of correct answers: $\{n \in \mathbb{W} \mid n < 6\}$, $\{n \in \mathbb{Z} \mid 0 \le n \le 5\}$

15. $\{x \in \mathbb{R} | x > -3\}$, or $\{x | x > -3\}$

17. Answers may vary. An example of a correct answer: $\{n \in \mathbb{Z} | n = 3k, k \in \mathbb{Z}\}$

19. $\in$

21. $\not\subset$

23. $\notin$

25. $\in$

27. $=$

29. **a.** $\sqrt{16}$; **b.** $0, \sqrt{16}$; **c.** $0, \sqrt{16}$ **d.** $0.999\dots,\ -5.001,\ 0,\ 5\frac{3}{4},\ 1.4\overline{05},\ \frac{7}{8}, \sqrt{16},$

e. $\sqrt{2},\ 9.010010001\dots$ **f.** $0.999\dots,\ -5.001,\ 0,\ 5\frac{3}{4},\ 1.4\overline{05},\ \frac{7}{8},\ \sqrt{2},\ \sqrt{16},\ 9.010010001\dots$

31. $1.\overline{02} = \frac{101}{99} \in \mathbb{Q}$

33. $2.0\overline{125} = \frac{4021}{1998} \in \mathbb{Q}$

35. $5.22\overline{54} = \frac{1437}{275} \in \mathbb{Q}$

R2 Exercises

1. $-6 < -3$

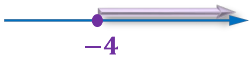
3. $17 \ge x$

5. $2x + 3 \ne 0$

7. $2 < x < 5$

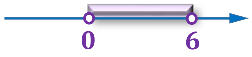
9. $-2 \le 2x < 6$

11. $[-4, \infty)$



13. $\left(-\infty, \frac{5}{2}\right)$

15. $(0, 6)$



17. $[-5, 16)$

19. -7

21. 8

23. -12

25. $<$

27. $=$

29. $\le$

31. 25

33. $\frac{1}{6}$

35. $|y - 5|$

37. $\{-5, 5\}$

39. $\{a - 5, a + 5\}$

R3 Exercises

1. true

3. true

5. false

7. true

9. false

11. $7 \cdot (5 \cdot 2)$

13. $-a$

15. 1

17. 0

19. $\frac{7}{20}$

21. $-\frac{1}{2}$

23. $-6x$

25. -8

27. -10

29. 18

31. $-\frac{41}{24}$ or $-1\frac{17}{24}$

33. $-\frac{101}{24}$ or $-4\frac{5}{24}$

35. -61

37. $-x + y$

39. $16x + 8y - 10$

41. $-x$

43. $-13x^2 + 12x$

45. $2\sqrt{x} - 4$

47. -2

49. $-25x^2$

51. $a + 7$

53. $-3x + 5$

55. $33a - 10$

57. $66x + 14$

59. $-12x + 615$

61. -8

63. 24

65. $\frac{8}{9}$

67. no

69. no

71. $29 \cdot 100 = 2900$

73. $\frac{8}{5} \cdot 30 = 48$

75. $9 \cdot 5 + 2 - (8 \cdot 3 + 1) = 22$

Linear Equations - ANSWERS

L1 Exercises

1. true

3. false

5. true

7. expression

9. equation

11. linear

13. not linear

15. linear

17. yes

19. No

21. $\frac{5}{6}$

23. -2

25. -1

27. $\mathbb{R}$; identity

29. $\emptyset$; contradiction

31. $-\frac{2}{3}$

33. -6

35. $\frac{13}{66}$

37. -1

39. -12

41. 3

43. $\frac{5}{32}$

45. $\frac{145}{23}$

47. 2500

L2 Exercise

1. A and C

3. $r = \frac{I}{Pt}$

5. $m = \frac{E}{c^2}$

7. $b = 2A - a$

9. $l = \frac{P-2w}{2}$ or $l = \frac{P}{2} - w$

11. $\pi = \frac{S}{rs+r^2}$

13. $C = \frac{5}{9}(F - 32)$

15. $p = 2Q + q$

17. $q = \frac{T-B}{Bt}$

19. $R = \frac{d}{1-st}$

21. **a.** $C(n) = 1.9n + 3.2$
b. $\$22.20$
c. 15 km

23. **a.** 5 ml
b. $d = \frac{c(a+12)}{a}$
c. 75 mg

25. **a.** $C = 1060d$
b. 7420

27. $L = \frac{A}{W}$

29. **a.** $t = \frac{I}{Pr}$
b. 3 year

31. **a.** $k = 120;\ N = 120P$
b. 75,778,800 bottles

33. 67 g

35. ~5 cm

37. \$3000

39. The area would decrease by 25%

41. ~105 barrels

L3 Exercises

1. $x - 7$

3. $\frac{1}{2}(x + y)$

5. $x^2 - y^2$

7. $n + (n + 1) + (n + 2) = 30$

9. $\frac{3x}{10}$

11. $0.03x - 100$

13. $x^2 - x$

15. 8

17. 238 and 239

19. 86.9%

21. 11, 13, 15

23. 33, 35, 37

25. $1850

27. ~158700

29. $9000 at 3%
$42000 at 6.5%

31. $6000 at 4.5%
$8000 at 5.25%

33. $4800

35. 39°, 63°, 78°

37. 8 ft by 16 ft

39. 7 nickels; 9 dimes

41. 12 kg of pecans
18 kg of cashews

43. 126 tickets for adults; 52 tickets for children

45. 8.22 $/kg

47. 20 grams

49. 375 ml

51. 40 ml

53. **a.** 1 hr 51 min; **b.** 1 hr 23 min

55. 6 km

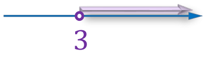
57. 1 min 12 sec

L4 Exercises

1. [2,3]

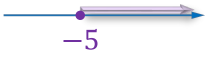
3. $(-\infty, 4)$

5. $x > 3$

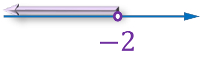


7. $-7 \le x \le 5$

9. $[-5, \infty)$



11. $(-\infty, -2)$



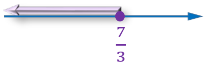
13. $(-4,1)$

15. $(-5,-2]$

17. yes

19. yes

21. $\left(-\infty,\frac{7}{3}\right]$

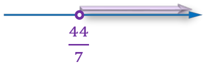


23. $(15,\infty)$

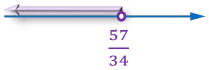
25. $[-9,\infty)$

27. $(-\infty,\infty)=\mathbb{R}$

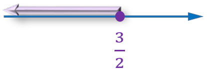
29. $\left(\frac{44}{7},\infty\right)$



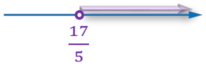
31. $\left(-\infty,\frac{57}{34}\right)$



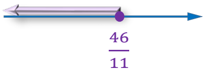
33. $\left(-\infty,\frac{3}{2}\right]$



35. $\left(\frac{17}{5},\infty\right)$

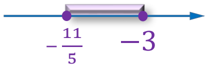


37. $\left(-\infty,\frac{46}{11}\right]$



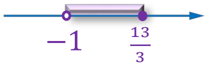
39. $\left[\frac{1}{5},\frac{11}{5}\right]$

41. $\left[-\frac{11}{3},-3\right]$



43. $(10,14)$

45. $\left(-1,\frac{13}{3}\right]$



47. $x+5>12$
$x\in(7,\infty)$

49. $3x+2\geq 8$
$x\in[2,\infty)$

51. $\frac{1}{2}(x+3)\leq 12$
$x\in(-\infty,21]$

53. $-6<2x<8$
$x\in(-3,4)$

55. at least 87%

57. up to 112 days

59. between −5°C and 20°C

61. more than \$30,000

63. 65 cheques

L5 Exercises

1. $\{1,\ 3\}$

3. $\{1,\ 3,\ 5\}$

5. $\emptyset$

7. $\{5\}$

9. $[1,3]$

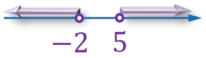
11. $(0,7]$

13. $(-\infty,\infty)$

15. $\{1\}$

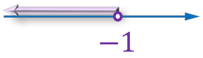
17. $(-2,\infty)$

19. $(-\infty,-2)\cup(5,\infty)$

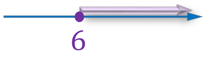


21. $(-2,1)$

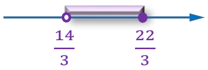
23. $(-\infty,-1)$



25. $[6,\infty)$

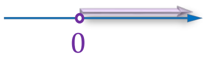


27. $\left(\frac{14}{3},\frac{22}{3}\right]$

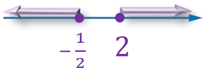


29. $(-\infty,\infty)=\mathbb{R}$

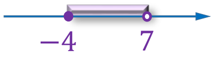
31. $(0,\infty)$



33. $\left(-\infty,\frac{1}{2}\right]\cup[2,\infty)$



35. $[-4,7)$



37. 8.5 to 11.5 hr/day

39. at least 14 and at most 24

41. **a.** $\{Education, Humanities, Nursing, Veterinary\ Medicine\}$
b. $\{Nursing\}$
c. $\{Education, Humanities, Business, Mathematics, Dentistry, Veterinary\ Medicine\}$
d. $\{Business, Mathematics, Dentistry\}$

L6 Exercises

1. $2x^2$

3. $\frac{5}{|y|}$

5. $7x^4|y|^3$

7. $\frac{x^2}{|y|}$

9. $\frac{x^2}{2}$

11. $(x-1)^2$

13. **a.** $\emptyset$ **b.** 1 **c.** 2

15. $\{-4,4\}$

17. $\{-5,11\}$

19. $\left\{0,\frac{10}{3}\right\}$

21. $\{-28,16\}$

23. no solution

25. $\{-2,2\}$

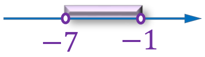
27. $\{-7,8\}$

29. $\left\{-\frac{3}{5},5\right\}$

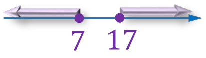
31. $\left\{-\frac{40}{3},-\frac{20}{7}\right\}$

33. $\left\{\frac{20}{17},\frac{40}{13}\right\}$

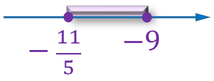
35. $(-7,-1)$



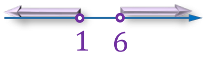
37. $(-\infty,7]\cup[17,\infty)$



39. $\left[-\frac{11}{5},1\right]$

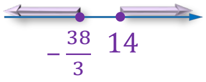


41. $(-\infty,1)\cup(6,\infty)$

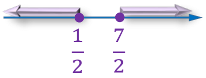


43. $[-72,120]$

45. $\left(-\infty,-\frac{38}{3}\right]\cup[14,\infty)$



47. $\left(-\infty, \frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$



49. $[-5, -3]$

51. $\{-6\}$

53. $\emptyset$

55. $\mathbb{R}$

57. **a.** $|M - 370| \leq 50$
b. $M \in [320, 420]$

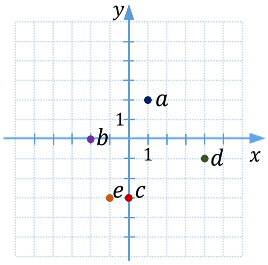
59. $|C - 1200| \leq 100$
$C \in [1100, 1300]$

61. $|r - 85| \leq 25$

Graphs and Linear Functions - ANSWERS

G1 Exercises

1.

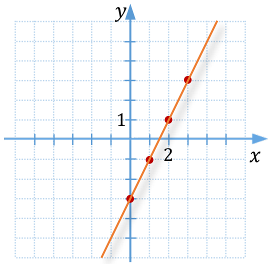


3. yes

5. no

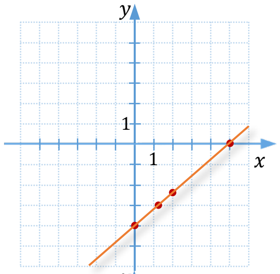
7.

x	y
−3	3
0	2
3	1
6	0



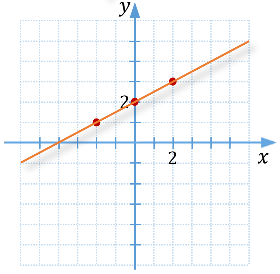
9.

x	y
0	−4
5	0
2	$-\frac{12}{5}$
$\frac{5}{4}$	−3



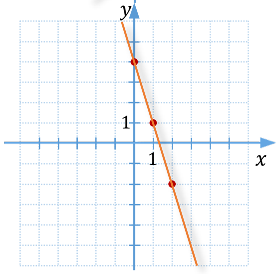
11.

x	y
0	2
2	3
−2	1



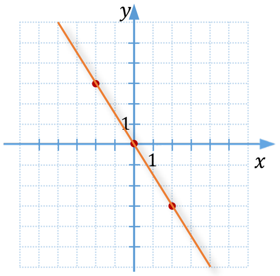
13.

x	y
0	4
1	1
2	−2



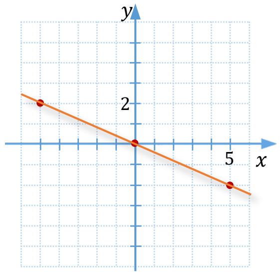
15.

x	y
−2	3
0	0
2	−3



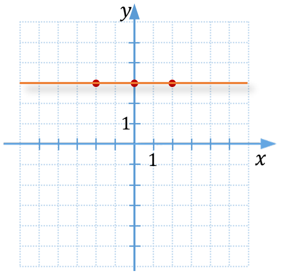
17.

x	y
0	0
5	−2
−5	2



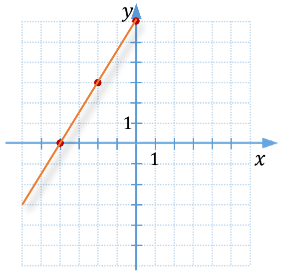
19.

x	y
−2	3
0	3
2	3



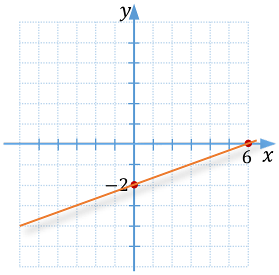
21.

x	y
0	6
−2	3
−4	0



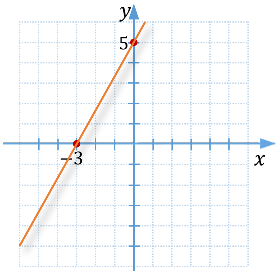
23.

x	y
6	0
0	-2



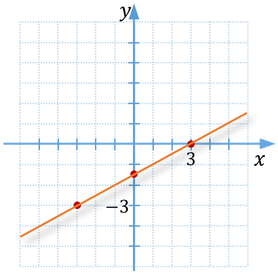
25.

x	y
−3	0
0	5



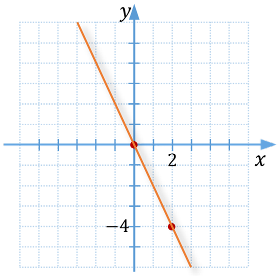
27.

x	y
3	0
0	$-\frac{3}{2}$
−3	-3



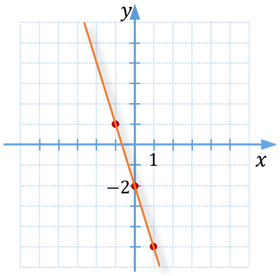
29.

x	y
0	0
2	-4



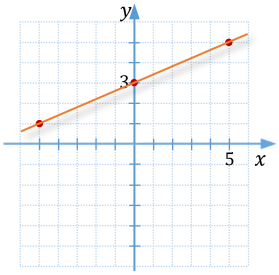
31. y-int. $= 2$

slope $= -3$



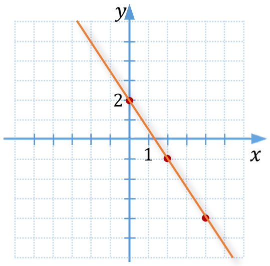
33. y-int. $= 3$

slope $= \frac{2}{5}$



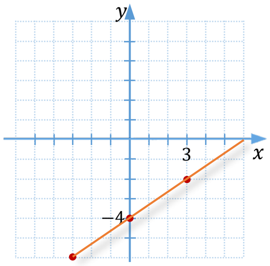
35. y-int. $= 2$

slope $= -\frac{3}{2}$



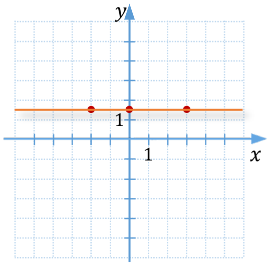
37. y-int. $= -4$

slope $= \frac{2}{3}$



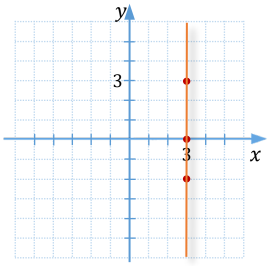
39. y-int. $= \frac{3}{2}$

slope $= 0$



41. y-int. $= none$

slope $= undefined$



43. $\left(\frac{3}{2},0\right)$

45. $\left(-\frac{9}{2},8\right)$

47. $\left(\frac{11}{20},-\frac{17}{12}\right)$

49. $(3,-4)$

51. $(3,10)$

G2 Exercises

1. $-\frac{1}{3}$

3. 4

5. $\frac{1}{2}$

7. $\frac{4}{5}$

9. undefined

11. -1

13. $\frac{4}{9}$

15. $y = -3x - 5$

17. $y = -\frac{2}{5}x + \frac{14}{5}$

19. $y = -1$

21. $\frac{1}{2}$

23. $\frac{2}{3}$

25. $-\frac{5}{3}$

27. 0

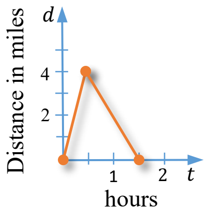
29. 3

31. $a - C, b - A, c - D, d - B$

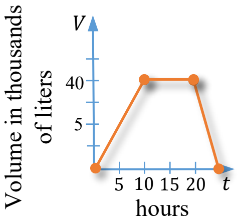
33. For the first 4 years, the pay raise was 0 $\%/year$.

35. On average, between 6 and 16 years old boys grow 6.7 $cm/year$.

37.



39.



41. 375 km/hr

43. perpendicular

45. parallel

47. neither

49. perpendicular

51. not collinear

G3 Exercises

1. $x + 2y = -14$

3. $4x - 5y = 20$

5. $4x + 6y = -9$

7. $y = \frac{1}{6}x - \frac{5}{6}$

9. $y = \frac{4}{5}x - 2$

11. $y = \frac{4}{5}x - 2$

13. $y = \frac{1}{4}x + 2$

15. $y = -x + 3$

17. $y = \frac{1}{2}x + \frac{7}{2}$
$x - 2y = -7$

19. $y = \frac{3}{2}x - 1$
$3x - 2y = 2$

21. $y = -x + 3$
$x + y = 3$

23. $y = -\frac{7}{6}x + \frac{4}{3}$
$7x + 6y = 8$

25. $y = \frac{5}{4}x - \frac{1}{3}$
$15x - 12y = 4$

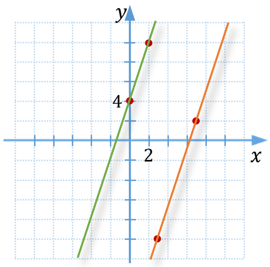
27. $y = 7$

29. $x = -1$

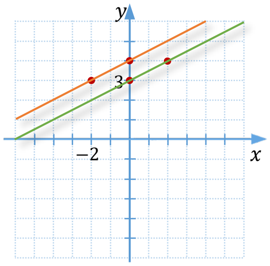
31. $y = 6$

33. $x = -\frac{3}{4}$

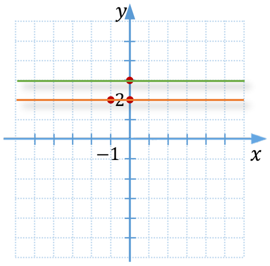
35. $3x - y = 19$



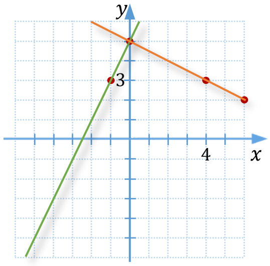
37. $x - 2y = -8$



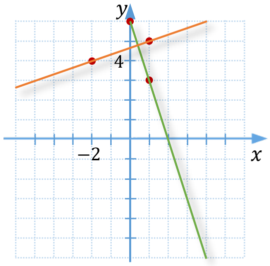
39. $y = 2$



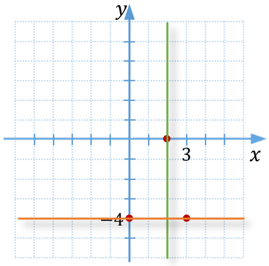
41. $x + 2y = 10$



43. $x - 3y = -14$



45. $y = -4$



47. $C = 49.95n + 80$; $679.40

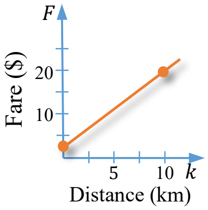
49. **a.** $C = 23d + 60$;
b. 6 days

51. $N = \frac{17}{3}t + 8$

53. **a.** $C = 800y - 1581200$;
b. The slope of 800 indicates that the annual tuition and fees for out-of-state students at Oxford University was increasing by 800$/year between 2007 and 2016.
c. $36400

55. $A = 180t + 2000$

57. **a.** $F = 1.75k + 2.5$
b.



c. the charge per kilometer
d. 12 km

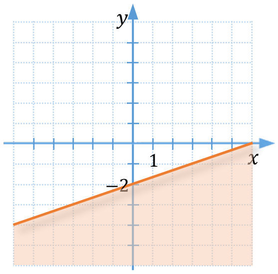
G4 Exercises

1. yes; yes

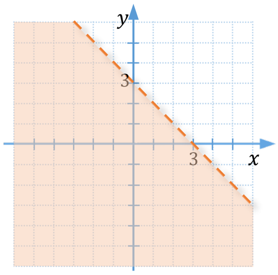
3. no; yes

5. a. – II; b. – IV; c. – I; d. – III;

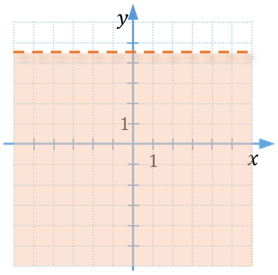
7.



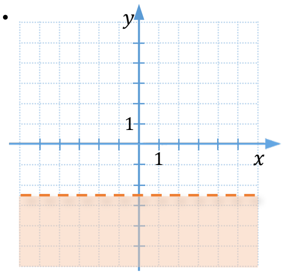
9.



11.



13.



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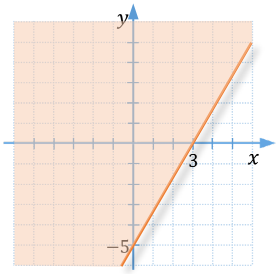
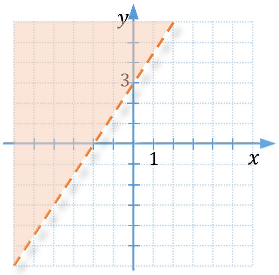
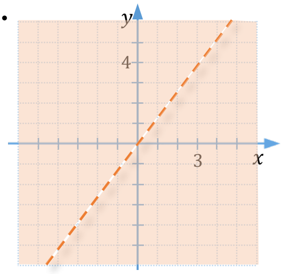


17.



19.



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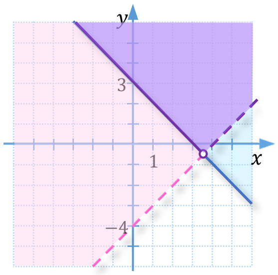
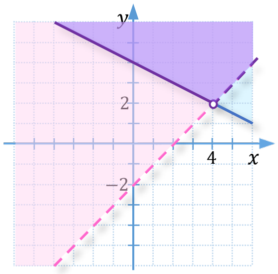
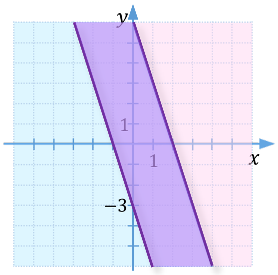


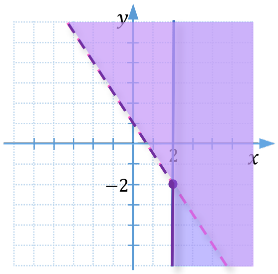
23.



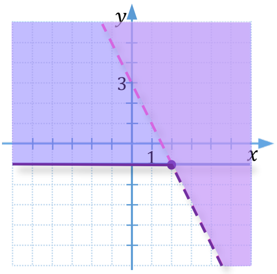
25.



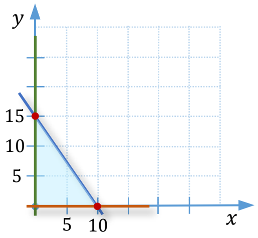
27.



29.



31.



G5 Exercises

1. not a function
domain = $\{0,2\}$
range = $\{2,3,4\}$

3. function
domain = $\{2,3,4,5\}$
range = $\{2,3,4,5\}$

5. not a function
domain = $\{a,b\}$
range = $\{2,4,5\}$

7. function
domain = $\{a,b,c\}$
range = $\{2,4\}$

9. not a function
domain = $\{0,1\}$
range = $\{-2,-1,1,2\}$

11. function
domain = $\{3,6,9,12\}$
range = $\{1,2\}$

13. function
domain = $\mathbb{R}$
range = $[0,\infty)$

15. function
domain = $\mathbb{R}$
range = $\mathbb{R}$

17. not a function
domain = $\mathbb{R}$
range = $[-4,4]$

19. not a function
domain = $\mathbb{R}$
range = $\mathbb{R}$

21. function
domain = $\mathbb{R}$

23. function
domain = $\mathbb{R}$

25. not a function
domain = $\mathbb{R}$

27. not a function
domain = $[0,\infty)$

29. function
domain = $[0,\infty)$

31. function
domain = $\mathbb{R}\setminus\{-5\}$

33. function
domain = $\mathbb{R}\setminus\{2\}$

35. not a function
domain = $\mathbb{R}$

37. not a function
domain = $\mathbb{R}$

39. function
domain = $\mathbb{R}$

41. not a function
domain = $[-2,2]$

G6 Exercises

1. **a.** 2 **b.** 3

3. **a.** 1 **b.** $\{-1,0\}$

5. **a.** 4 **b.** 2

7. **a.** -1 **b.** $\{-5,1\}$

9. $f(1) = 2$

11. $g(-1) = -4$

13. $f(p) = -3p + 5$

15. $g(-x) = -x^2 - 2x - 1$

17. $f(a + 1) = -3a + 2$

19. $g(x - 1) = -x^2 + 4x - 4$

21. $f(2 + h) = -3h - 1$

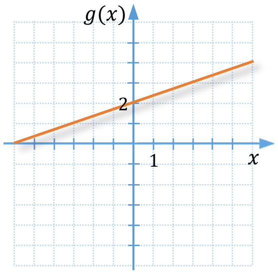
23. $g(a + h) = -a^2 - 2ah - h^2 + 2a + 2h - 1$

25. $f(3) - g(3) = 0$

27. $3g(x) + f(x) = -3x^2 + 3x + 2$

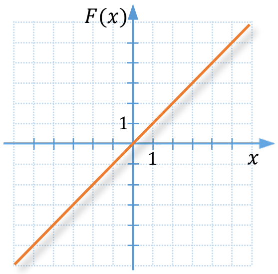
29. line; 4; $-2x + 6$; 4; $(1,4)$

31.



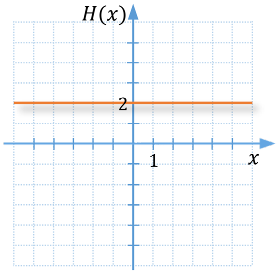
domain $= \mathbb{R}$
range $= \mathbb{R}$

33.



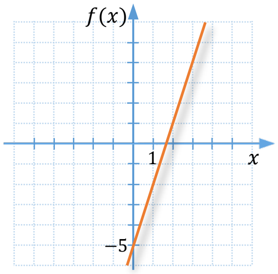
domain $= \mathbb{R}$
range $= \mathbb{R}$

35.



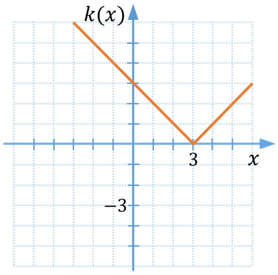
domain $= \mathbb{R}$
range $= \{2\}$

37.



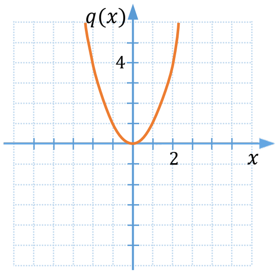
domain $= \mathbb{R}$
range $= \mathbb{R}$

39.



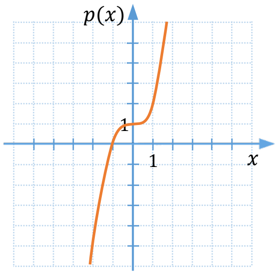
domain $= \mathbb{R}$
range $= [0, \infty)$

41.



domain $= \mathbb{R}$
range $= [0, \infty)$

43.



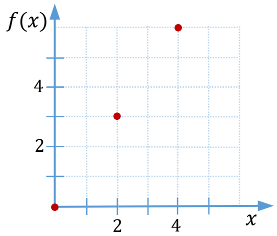
domain $= \mathbb{R}$
range $= \mathbb{R}$

45. **a.**

x	***f(x)***
0	0
2	3.00
4	6.00

b. $f(x) = 1.5x$

c.



47. **a.** $C(d) = 24.6d + 18.8$ **b.** $C(4) = 117.20$; The cost of renting the car for 4 days is \$117.20.
c. $d = 7$

49. **a.** $t \in [0,20]$; $f(t) \in [0,600]$ **b.** 5 minutes; 10 minutes **c.** 600 meters
d. $f(15) = 300$; In 15 minutes, the person is 300 meters from home.

51. The height of water in the bathtub decreases quickly, then remains constant, and finally increases slowly until it reaches half of the original height.

Systems of Linear Equations - ANSWERS

E1 Exercises

1. system

3. Consistent

5. inconsistent

7. empty

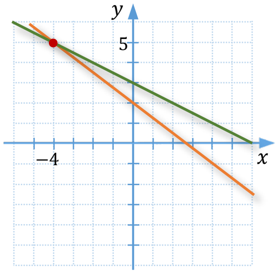
9. one

11. opposite

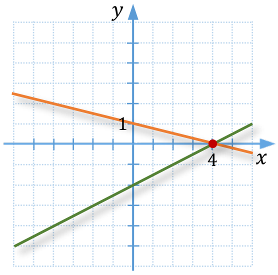
13. yes

15. no

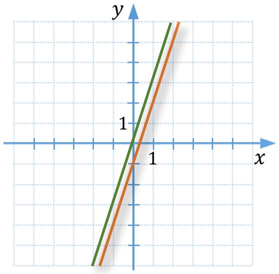
17. $(-4, 5)$
consistent;
independent



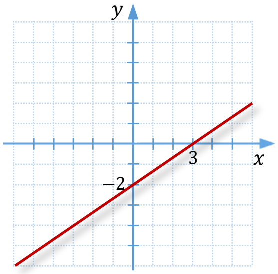
19. $(4, 0)$
consistent;
independent



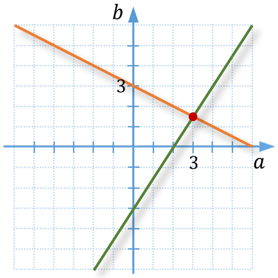
21. no solution;
inconsistent;
independent



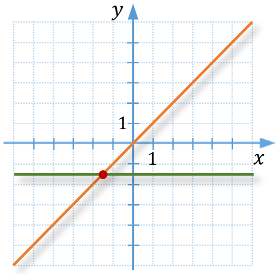
23. $\{(x, y) | 2x - 3y = 6\}$
consistent;
dependent



25. $\left(3, \frac{3}{2}\right)$
consistent;
independent



27. $\left(-\frac{3}{2}, -\frac{3}{2}\right)$
consistent;
independent



29. $(-1, -1)$

31. $\left(\frac{7}{3}, \frac{1}{3}\right)$

33. no solution;
parallel lines

35. $(5, 1)$

37. $(4, 6)$

39. $(4.2, -4.4)$

41. $(12, 8)$

43. $(4, -1)$

45. $(-1, 1)$

47. infinitely many
solutions; same line

49. $\left(\frac{140}{13}, -\frac{50}{13}\right)$

51. $\left(\frac{10}{21}, \frac{11}{14}\right)$

53. There are infinitely many other solutions. Any point satisfying the equation $5x + 3y = 14$ is a solution. For example, $(-2, 8)$ is another solution.

55. $\begin{cases} y = -\frac{2}{3}x + \frac{1}{3} \\ y = -\frac{2}{3}x + \frac{1}{3} \end{cases}$, infinitely many solutions

57. $(-3, 2)$

59. $(9, 4)$

61. infinitely many solutions: $\{(x, y) | x + 2y = 48\}$

63. $\left(\frac{2}{3}, \frac{1}{3}\right)$

65. $\left(-5, -\frac{5}{3}\right)$

67. $(0,\ 4)$

69. $\left(\frac{1}{a}, \frac{1}{b}\right)$

71. $\left(-\frac{3}{5a}, \frac{7}{5}\right)$

73. $[0, 30)$

75. 2017

77. since 2006; ~7%

E2 Exercises

1. 8 liters

3. $9.25n$

5. $r + c; r - c$

7. $69°, 21°$

9. base = 154 cm; heigth = 77 cm

11. 35 km

13. 13 gold; 11 silver; 9 bronze

15. 113 espressos; 339 cappuccinos

17. 322 adult tickets; 283 youth tickets

19. 2.49 $/egg salad sandwich; 3.99 $/meat sandwich

21. 416 $/week in New York; 340 $/week in Paris

23. $2300 at 3.25%; $2500 at 2.75%

25. $3200 at 3.7%; $2800 at 8.2%

27. 9 L of 4% brine; 3 L of 20% brine

29. $66\frac{2}{3}$ g of cottage cheese; 40 g of vanilla yogurt

31. 5 loonies; 9 quarters

33. houseboat: 12 km/h; current: 3 km/h

35. plane: 275 km/h; wind: 25 km/h

37. plane: 315 km/h; wind: 45 km/h

39. 480 km

41. 0.7 L

Polynomials and Polynomial Functions - ANSWERS

P1 Exercises

1. yes

3. no

5. 4; 1

7. 2; $\sqrt{2}$

9. $-\frac{2}{5}x^3+3x^2-x+5$; 3; $-\frac{2}{5}$

11. $x^5+8x^4+2x^3-3x$; 5; 1

13. $3q^4+q^2-2q+1$; 4; 3

15. first degree binomial

17. zero degree monomial

19. seventh degree monomial

21. -8

23. -12

25. -5

27. $2a-3$

29. -14

31. $6a-9$

33. $-x+13y$

35. $4xy+3x$

37. $6p^3-3p^2+p+2$

39. $3m+11$

41. $-x-4$

43. $-5x^2+4y^2-11z^2$

45. $-4x^2-3x-5$

47. $5r^6-r^5-7r^2+5$

49. $-5a^4-6a^3+9a^2-11$

51. $5x^2y^2-7y^3+17xy$

53. $-z^2+x+4m$

55. $10z^2-16z$

57. **a.** $(f+g)(x)=8x-8$ **b.** $(f-g)(x)=2x-4$

59. **a.** $(f+g)(x)=-2x^2-3x+1$ **b.** $(f-g)(x)=8x^2-7x-1$

61. **a.** $(f+g)(x)=-6x^{2n}-2x^n-1$ **b.** $(f-g)(x)=10x^{2n}-4x^n+7$

63. $(P-Q)(-2)=-1$

65. $(R-Q)(0)=-7$

67. $(P+Q)(a)=a^2+2a+1$

69. $(P+R)(2k)=4k^2+2k-6$

71. ~9.3 cm

73. **a.** $R(x)=56n$ **b.** $P(x)=24n-1500$ **c.** $P(100)=900$;
The profit from selling 100 dresses is $900.

P2 Exercises

1. **a.** no; $x^2\cdot x^4=x^6$ **b.** no; $-2x^2$ is in the simplest form **c.** yes **d.** yes **e.** no; $(a^2)^3=a^6$
f. no; $4^5\cdot 4^2=4^7$ **g.** no; $\frac{6^5}{3^2}=2^5\cdot 3^3$ **h.** no; $xy^0=x$ **i.** yes

3. $-8y^8$

5. $14x^3y^8$

7. $-27x^6y^3$

9. $\frac{-5x^3}{y^2}$

11. $\frac{64a^6}{b^2}$

13. $\frac{-125p^3}{q^9}$

15. $12a^5b^5$

17. $\frac{16y}{x^3}$

19. $64x^{18}y^6$

21. x^{2n-1}

23. 5^{2ab}

25. $-2x^2$

27. $x^{a^2-b^2}$

29. $-16x^7y^4$

31. $-6x^2+10x$

33. $-12x^5y+9x^4y^2$

35. $15k^4-10k^3+20k^2$

37. x^2+x-30

39. $6x^2+5x-6$

41. $6u^4-8u^3-30u^2$

43. $6x^3-7x^2-13x+15$

45. $6m^4-13m^2n^2+5n^4$

47. a^2-4b^2

49. $a^2-4ab+4b^2$

51. y^3+27

53. $2x^4-4x^3y-x^2y^2+3xy^3-2y^4$

55. true

57. true

59. false; $(2-1)^3 \neq 2^3-1^3$

61. $25x^2-16$

63. $\frac{1}{4}x^2-9y^2$

65. x^4-49y^6

67. $0.64a^2+0.16ab+0.04b^2$

69. x^2-6x+9

71. $25x^2-60xy+36y^2$

73. $4n^2-\frac{4}{3}n+\frac{1}{9}$

75. $x^8y^4+6x^4y^2+9$

77. $4x^4-12x^2y^3+9y^6$

79. $8a^5+40a^4b+50a^4b^2$

81. $x^4-x^2y^2$

83. x^4-1

85. $a^4-2a^2b^2+b^4$

87. $4x^2+12xy+9y^2-25$

89. $4k^2=12k+4hk-6h+h^2+9$

91. $x^{4a}-y^{4b}$

93. $101\cdot 99=(100+1)(100-1)=10000-1=9999$

95. $505\cdot 495=(500+5)(500-5)=250000-25=249975$

97. x^2-x-12

99. $(fg)(x)=15x^2-28x+12$

101. $(fg)(x)=-3x^4+8x^3+22x^2-45x$

103. $(PR)(x)=x^3-2x^2-4x+8$

105. $(PQ)(a)=2a^3-8a$

107. $(PQ)(3)=30$

109. $(QR)(x)=2x^2-4x$

111. $(QR)(a+1)=2a^2-2$

113. $P(2a+3)=4a^2+12a+5$

115. $4x^3-40x^2+96x$

P3 Exercises

1. False; When dividing powers with the same bases, we subtract exponents. So, the quotient will be a fourth-degree polynomial.

3. $4x^2-3x+1$

5. $2xy-6$

7. $-3a^3+5a^2-4a$

9. $8-\frac{9}{x}+\frac{3}{2x^2}$

11. $\frac{2b}{a}+\frac{5}{3}+\frac{3c}{a}$ **13.** $y+5$ **15.** $t-4 \quad \mathrm{R}\ -21$

17. $2a^2-a+2 \quad \mathrm{R}\ 6$ **19.** $2z^2-4z+1 \quad \mathrm{R}\ -10$ **21.** $3x+1 \quad \mathrm{R}\ -3x-7$

23. $3k^2+4k+1$ **25.** $\frac{5}{4}t+1 \quad \mathrm{R}\ -5$ **27.** p^2+p+1

29. $y^3-2y^2+4y-8 \quad \mathrm{R}\ 32$ **31.** $Q(x)=2x^2-x+6;\ R(x)=4$

33. $\left(\frac{f}{g}\right)(x)=3x-2;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\{0\}$ **35.** $\left(\frac{f}{g}\right)(x)=x-6;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\{-6\}$

37. $\left(\frac{f}{g}\right)(x)=x+1;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\left\{\frac{3}{2}\right\}$ **39.** $\left(\frac{f}{g}\right)(x)=4x^2-10x+25;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\left\{-\frac{5}{2}\right\}$

41. $\left(\frac{R}{Q}\right)(x)=\frac{x-2}{2x}$ **43.** $\left(\frac{R}{P}\right)(x)=\frac{1}{x+2},\ x\neq 2$ **45.** $\left(\frac{R}{Q}\right)(0)=DNE$

47. $\left(\frac{R}{P}\right)(-2)=DNE$ **49.** $\left(\frac{P}{R}\right)(a)=a+2$ **51.** $\frac{1}{2}\left(\frac{Q}{R}\right)(x)=\frac{x}{x-2}$

53. **a.** $L=3x-2$ **b.** 10 m

P4 Exercises

1. $(f\circ g)(1)=18$ **3.** $(f\circ g)(x)=x^2-10x+27$ **5.** $(f\circ h)(-1)=27$

7. $(f\circ h)(x)=4x^2-12x+11$ **9.** $(h\circ g)(-2)=11$

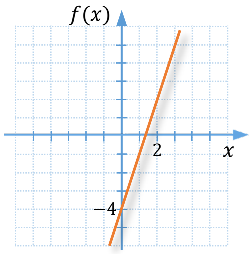
11. $(h\circ g)(x)=-2x+7$ **13.** $(f\circ f)(2)=38$ **15.** $(h\circ h)(x)=4x-9$

17. $(g\circ f)(x)=30.48x$ computes the number of centimeters in x feet

19. **a.** $r=\frac{C}{2\pi}$ **b.** $A=\frac{C^2}{4\pi}$ **c.** $A(6\pi)=9\pi$

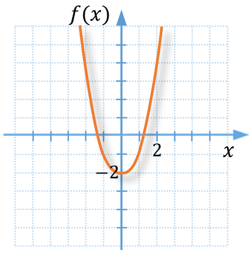
21. No. It is 40.5% off. To find the new price we use composition of functions $(f\circ g)(x)$ where $f(x)=.85x$ and $g(x)=.7x$. So, the discount is $x-f\big(g(x)\big)=x-.85\cdot .7x=(1-.595)x=.405x$. Thus, the dress was overall discounted by 40.5%.

23.



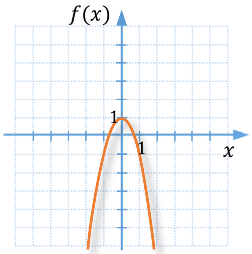
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

25.



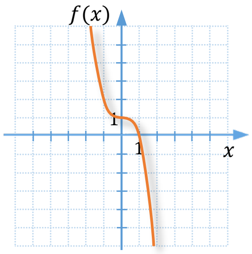
Domain: $\mathbb{R}$
Range: $[-2,\infty)$

27.



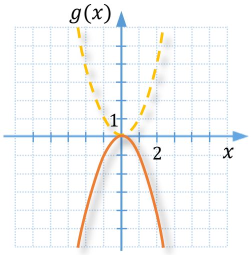
Domain: $\mathbb{R}$
Range: $(-\infty,1]$

29.



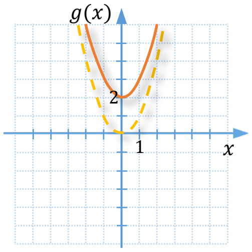
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

31.



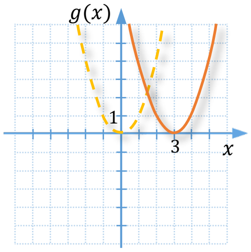
symmetry in x-axis

33.



translation: 2 up

35.



translation: 3 to the right

Factoring - ANSWERS

F1 Exercises

1. false

3. Both are correct but the second one is preferable as the binomial factor has integral coefficients.

5. $7a^3b^5$

7. $x(x-3)$

9. $(x-2y)$

11. $x^{-4}(x+2)^{-2}$

13. $8k(k^2+3)$

15. $-6a^2(6a^2-a-3)$

17. $5x^2y^2(y-2x)$

19. $(a-2)(y^2-3)$

21. $2n(n-2)$

23. $(4x-y)(4x+1)$

25. $-(p-3)(p^2-10p+19)$

27. $k^{-4}(k^2+2)$

29. $-p^{-5}(2p^3-p^2-3)$

31. $-x^{-2}y^{-3}(2xy-5)$

33. $(a^2-7)(2a+1)$

35. $-(xy+3)(x-2)$

37. $(x^2-y)(x-y)$

39. $-(y-3)(x^2+z^2)$

41. $(x-6)(y+3)$

43. $(x^2-a)(y^2-b)$

45. $(x^n+1)(y-3)$

47. $2(s+1)(3r-7)$

49. $x(x-1)(x^3+x^2-1)$

51. no, as $(2xy^2-4)$ can still be factored to $2(xy^2-2)$

53. $p=\frac{2M}{q+r}$

55. $y=\frac{x}{3-w}$

57. $A=(4-\pi)x^2$

59. $A=(\pi-1)r^2$

F2 Exercises

1. no

3. All of them; however, the preferable answer is $-(2x-3)(x+5)$.

5. $x-3$

7. $x-5y$

9. $(x+3)(x+4)$

11. $(y+8)(y-6)$

13. not factorable

15. $(m-7)(m-8)$

17. $-(n+9)(n-2)$

19. $(x-2y)(x-3y)$

21. $-(x+3)(x-7)$

23. $n^2(n+2)(n-15)$

25. $-2(x-10)(x-4)$

27. $y(x^2+12)(x^2-5)$

29. $-5(t^{13}+8)(t^{13}-1)$

31. $-n(n^4+16)(n^4-3)$

33. $\pm12, \pm13, \pm15, \pm20, \pm37$

35. $3x-4$

37. $3x-5$

39. $(2y+1)(3y-2)$

41. $(6t-1)(t-6)$

43. $(6n+5)(7n-5)$

45. $-2(2x-3)(3x+5)$

47. $(6x+5y)(3x+2y)$

49. $-(2n+5)(4n-3)$

51. $2x^2(2x-1)(x+3)$

53. $(9xy-4)(xy+1)$

55. $(2p^2-7q)^2$

57. $(2a+9)(a+5)$

59. $\pm3, \pm4, \pm11, \pm17, \pm28, \pm59$

61. $(3x+2)$ feet

F3 Exercises

1. difference of squares

3. neither

5. difference of cubes

7. difference of squares

9. perfect square

11. difference of cubes

13. $25x^2 + 100 = 25(x^2 + 4)$; The sum of squares is factorable in integral coefficients only if the two terms have a common factor.

15. $(x + y)(x - y)$

17. $(x - y)(x^2 + xy + y^2)$

19. $(2z - 1)^2$

21. not factorable

23. $(5 - y)(25 + 5y + y^2)$

25. $(n + 10m)^2$

27. $(3a^2 + 5b^3)(3a^2 - 5b^3)$

29. $(p^2 - 4q)(p^4 + 4p^2q + 16q^2)$

31. $(7p + 2)^2$

33. $r^2(r + 3)(r - 3)$

35. $\frac{1}{8}(1 - 2a)(1 + 2a + 4a^2)$ or $\left(\frac{1}{2} - a\right)\left(\frac{1}{4} + \frac{1}{2}a + a^2\right)$

37. not factorable

39. $x^2(4x^2 + 11y^2)(4x^2 - 11y^2)$

41. $-(ab + 5c^2)(a^2b^2 - 5abc^2 + 25c^4)$

43. $(3a^4 - 8b)^2$

45. $(x + 8)(x - 6)$

47. $2t(t - 4)(t^2 + 4t + 16)$

49. $(x^n + 3)^2$

51. $(4z^2 + 1)(2z + 1)(2z - 1)$

53. $5(3x^2 + 15x + 25)$

55. $0.01(5z - 7)^2$ or $(0.5z - 0.7)^2$

57. $-3y(2x - y)$

59. $4(3x^2 + 4)$

61. $2(x - 5a)^2$

63. $(y + 6 + 3a)(y + 6 - 3a)$

65. $(m + 2)(m^2 - 2m + 4)(m - 1)(m^2 + m + 1)$

67. $(a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$

69. $(x^2 + 1)(x + 3)(x - 3)$

71. $(a + b + 3)(a - b - 3)$

72. $z(3xy + 4z)(xy + 7z)$

75. $(x^2 + 1)(x + 1)(x - 1)^3$

77. $c(c^w + 1)^2$

F4 Exercises

1. true

3. false

5. false

7. $x \in \{-4, 1\}$

9. $x \in \left\{-\frac{4}{5}, -\frac{1}{3}\right\}$

11. $x \in \{-6, -3\}$

13. $x \in \left\{-\frac{7}{2}, 1\right\}$

15. $x \in \{-6, 0\}$

17. $x \in \{4\}$

19. $x \in \left\{\frac{5}{2}\right\}$

21. $x \in \{-8, 4\}$

23. $x \in \left\{\frac{1}{3}, 3\right\}$

25. $x \in \left\{-2, \frac{8}{9}\right\}$

27. $x \in \{0, 6\}$

29. $x \in \{-4, 2\}$

31. $x \in \{1, 5\}$

33. $x \in \left\{-\frac{15}{8}, -1\right\}$

35. $x \in \{-5, 0, 3\}$

37. $x \in \left\{-\frac{8}{5}, 0, \frac{8}{5}\right\}$

39. $x \in \{-5, -1, 1, 5\}$

41. $x \in \{0, 2, 4\}$

43. $x \in \{-3, -1, 3\}$

45. $x \in \left\{-2, -\frac{2}{5}, 2\right\}$

47. 3; $\{-3,0,3\}$; Do not divide by x as x can be equal to zero. Also, $\sqrt{x^2} = |x|$ so in the last step, we should obtain $x = \pm 3$. The safest way to solve polynomial equations is by factoring and using the zero-product property.

49. $x \in \left\{\frac{1}{2}, 7\right\}$

51. $x \in \left\{-3, \frac{7}{3}\right\}$

53. $s = \frac{5-2p}{r+3}$

55. $r = \frac{R}{E-1}$

57. $t = \frac{4}{c+2}$

59. 8 seconds

61. -12 or 13

63. width = 9 cm; length = 16 cm

65. width = 7 m; height = 10 m

67. 7 m by 12 m

69. 2 cm

71. 9 in