Advanced (Algebraic) Mathematics 2

For Math 073 – Camosun College Edition

Adapted from *Intermediate Algebra and Trigonometry – Camosun College Edition* by Crystal Lomas. For information about the adaptation, refer to the Copyright and Licensing statement.

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Advanced (Algebraic) Mathematics 1 and 2 are adapted from the Intermediate Algebra and Trigonometry – Camosun College Edition, which was adapted by **Crystal Lomas** from Anna Kuczynska's original open textbook.

These adapted editions were further reorganized by **Puja Gupta** to align with the learning outcomes for Math 072 and Math 073 at Camosun College.

- Minor changes and corrections were applied throughout the text.
- Section RD6 Complex Numbers was added.
- Content about secant, cosecant, and cotangent was added to Sections T2 and T3.
- Dependency on trigonometric identities was removed and replaced with alternate explanations where appropriate in Sections T2, T3, T5.
- Calculator instruction in Section T3 was changed to reference scientific calculators.

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Polynomials and Polynomial Functions

One of the simplest types of algebraic expressions is a polynomial. Polynomials are formed only by addition and multiplication of variables and constants. Since both addition and multiplication produce unique values for any given inputs, polynomials are in fact functions. Some of the simplest polynomial functions are linear functions, such as $P(x) = 2x + 1$, and quadratic functions, such as $Q(x) = x^2 + x - 6$. Due to their comparably simple form, polynomial functions appear in a wide variety of areas of mathematics and science, economics, business, and many other areas of life. Polynomial functions are often used to model various natural phenomena, such as the shape of a mountain, the distribution of temperature, the trajectory of projectiles, etc. The shape and properties of polynomial functions are helpful when constructing such structures as roller coasters or bridges, solving optimization problems, or even analysing stock market prices.

In this chapter, we will introduce polynomial terminology, perform operations on polynomials, and evaluate and compose polynomial functions.

P1 Addition and Subtraction of Polynomials

Terminology of Polynomials

Recall that products of constants, variables, or expressions are called **terms** (see *Section R3, Definition 3.1*). **Terms** that are **products** of only **numbers** and **variables** are called **monomials**. Examples of monomials are $-2x$, xy^2, $\frac{2}{3}x^3$, etc.

Definition 1.1 ► A **polynomial** is a sum of monomials.

A **polynomial** in a single variable is the sum of terms of the form ax^n, where a is a **numerical coefficient**, x is the variable, and n is a whole number.

An ***n*-th degree polynomial** in x-variable has the form

$$\boldsymbol{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0},$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$.

Note: A polynomial can always be considered as a sum of monomial terms even though there are negative signs when writing it.
For example, polynomial $x^2 - 3x - 1$ can be seen as the sum of signed terms

$$x^2 + -3x + -1$$

Definition 1.2 ► The **degree of a monomial** is the sum of exponents of all its variables.

For example, the degree of $5x^3y$ is 4, as the sum of the exponent of x^3, which is 3 and the exponent of y, which is 1. To record this fact, we write $\deg(5x^3y) = 4$.

The **degree of a polynomial** is the highest degree out of all its terms.

For example, the degree of $2x^2y^3 + 3x^4 - 5x^3y + 7$ is 5 because $\deg(2x^2y^3) = 5$ and the degrees of the remaining terms are not greater than 5.

Polynomials that are sums of two terms, such as $x^2 - 1$, are called **binomials**.
Polynomials that are sums of three terms, such as $x^2 + 5x - 6$ are called **trinomials**.

The **leading term** of a polynomial is the highest degree term.
The **leading coefficient** is the numerical coefficient of the leading term.
So, the leading term of the polynomial $1 - x - x^2$ is $-x^2$, even though it is not the first term. The leading coefficient of the above polynomial is -1, as $-x^2$ can be seen as $(-1)x^2$.

A first degree term is often referred to as a **linear term**. A second degree term can be referred to as a **quadratic term**. A zero degree term is often called a **constant** or a **free term**.

Below are the parts of an n-th degree polynomial in a single variable x:

$$\substack{\textit{leading}\\\textit{coefficient}} \longrightarrow \underbrace{\boxed{a_n}x^n}_{\substack{\textit{leading}\\\textit{term}}} + a_{n-1}x^{n-1} + \cdots + \underbrace{a_2x^2}_{\substack{\textit{quadratic}\\\textit{term}}} + \underbrace{a_1x}_{\substack{\textit{linear}\\\textit{term}}} + \underbrace{a_0}_{\substack{\textit{constant}\\\textit{(free)}\\\textit{term}}}$$

Note: Single variable polynomials are usually arranged in descending powers of the variable. Polynomials in more than one variable are arranged in decreasing degrees of terms. If two terms are of the same degree, they are arranged with respect to the descending powers of the variable that appers first in alphabetical order.

For example, polynomial $x^2 + x - 3x^4 - 1$ is customarily arranged as follows

$$-3x^4 + x^2 + x - 1,$$

while polynomial $3x^3y^2 + 2y^6 - y^2 + 4 - x^2y^3 + 2xy$ is usually arranged as below.

$$\underbrace{2y^6}_{\substack{\textit{6th}\\\textit{degree}\\\textit{term}}} \underbrace{+3x^3y^2 - x^2y^3}_{\substack{\textit{5th degree terms}\\\textit{arranged}\\\textit{with respect to } x}} \underbrace{+2xy - y^2}_{\substack{\textit{2nd degree}\\\textit{terms arranged}\\\textit{with respect to } x}} \underbrace{+4}_{\substack{\textit{zero}\\\textit{degree}\\\textit{term}}}$$

Example 1 ► Writing Polynomials in Descending Order and Identifying Parts of a Polynomial

Suppose $P = x - 6x^3 - x^6 + 4x^4 + 2$ and $Q = 2y - 3xyz - 5x^2 + xy^2$. For each polynomial:

a. Write the polynomial in descending order.
b. State the degree of the polynomial and the number of its terms.
c. Identify the leading term, the leading coefficient, the coefficient of the linear term, the coefficient of the quadratic term, and the free term of the polynomial.

Solution ► **a.** After arranging the terms in descending powers of x, polynomial P becomes

$$-x^6 + 4x^4 - 6x^3 + x + 2,$$

while polynomial Q becomes

$$xy^2 - 3xyz - 5x^2 + 2y.$$

Notice that the first two terms, xy^2 and $-3xyz$, are both of the same degree. So, to decide which one should be written first, we look at powers of x. Since these powers are again the same, we look at powers of y. This time, the power of y in xy^2 is higher than the power of y in $-3xyz$. So, the term xy^2 should be written first.

b. The polynomial P has **5 terms**. The highest power of x in P is 6, so the **degree** of the polynomial P is **6**.
The polynomial Q has **4 terms**. The highest degree terms in Q are xy^2 and $-3xyz$, both third degree. So the **degree** of the polynomial Q is **3**.

c. The leading term of the polynomial $P = -x^6 + 4x^4 - 6x^3 + x - 2$ is $-x^6$, so the **leading coefficient** equals $-\mathbf{1}$.
The linear term of P is x, so the **coefficient of the linear term** equals **1**.
P doesn't have any quadratic term so the coefficient of the quadratic term equals **0**.
The **free term** of P equals $-\mathbf{2}$.

$x = 1 \cdot x$
$-x = (-1)x$

The leading term of the polynomial $Q = xy^2 - 3xyz - 5x^2 + 2y$ is xy^2, so the **leading coefficient** is equal to **1**.
The linear term of Q is $2y$, so the **coefficient of the linear term** equals **2**.
The quadratic term of Q is $-5x^2$, so the **coefficient of the quadratic term** equals $-\mathbf{5}$.
The polynomial Q does not have a free term, so the **free term** equals **0**.

Example 2 ► **Classifying Polynomials**

Describe each polynomial as a *constant*, *linear*, *quadratic*, or *n-th degree* polynomial. Decide whether it is a *monomial*, *binomial*, or *trinomial*, if applicable.

a. $x^2 - 9$ **b.** $-3x^7y$
c. $x^2 + 2x - 15$ **d.** π
e. $4x^5 - x^3 + x - 7$ **f.** $x^4 + 1$

Solution ►

a. $x^2 - 9$ is a second degree polynomial with two terms, so it is a **quadratic binomial**.

b. $-3x^7y$ is an **8-th degree monomial**.

c. $x^2 + 2x - 15$ is a second degree polynomial with three terms, so it is a **quadratic trinomial**.

d. π is a 0-degree term, so it is a **constant monomial**.

e. $4x^5 - x^3 + x - 7$ is a **5-th degree polynomial**.

f. $x^4 + 1$ is a **4-th degree binomial**.

Polynomials as Functions and Evaluation of Polynomials

Each term of a polynomial in one variable is a product of a number and a power of the variable. The polynomial itself is either one term or a sum of several terms. Since taking a power of a given value, multiplying, and adding given values produce unique answers, polynomials are also functions. While f, g, or h are the most commonly used letters to represent functions, other letters can also be used. To represent polynomial functions, we customarily use capital letters, such as P, Q, R, etc.

Any polynomial function P of degree n, has the form

$$\boldsymbol{P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$, and $n \in \mathbb{W}$.

Since polynomials are functions, they can be evaluated for different x-values.

Example 3 ► **Evaluating Polynomials**

Given $P(x) = 3x^3 - x^2 + 4$, evaluate the following expressions:

a. $P(0)$ **b.** $P(-1)$
c. $2 \cdot P(1)$ **d.** $P(a)$

Solution ►

a. $P(0) = 3 \cdot 0^3 - 0^2 + 4 = \mathbf{4}$

b. $P(-1) = 3 \cdot (-1)^3 - (-1)^2 + 4 = 3 \cdot (-1) - 1 + 4 = -3 - 1 + 4 = \mathbf{0}$

When evaluating at negative x-values, it is essential to use brackets in place of the variable before substituting the desired value.

c. $2 \cdot P(1) = 2 \cdot \underbrace{(3 \cdot 1^3 - 1^2 + 4)}_{this\ is\ P(1)} = 2 \cdot (3 - 1 + 4) = 2 \cdot 6 = \mathbf{12}$

d. To find the value of $P(a)$, we replace the variable x in $P(x)$ with a. So, this time the final answer,

$$P(a) = \mathbf{3a^3 - a^2 + 4},$$

is an expression in terms of a rather than a specific number.

Since polynomials can be evaluated at any real x-value, then the **domain** (*see Section G3, Definition 5.1*) of any polynomial is the set $\mathbb{R}$ of all real numbers.

Addition and Subtraction of Polynomials

Recall that terms with the same variable part are referred to as **like terms** (see *Section R3, Definition 3.1*). Like terms can be **combined** by adding their coefficients. For example,

$$\underbrace{2x^2y - 5x^2y = (2-5)x^2y}_{\substack{by\ distributive\ property\\ (factoring)}} = -3x^2y$$

In practice, this step is not necessary to write.

Unlike terms, such as $2x^2$ and $3x$, **cannot be combined**.

Example 4 ► Simplifying Polynomial Expressions

Simplify each polynomial expression.

a. $5x - 4x^2 + 2x + 7x^2$ **b.** $8p - (2 - 3p) + (3p - 6)$

Solution ► **a.** To simplify $5x - 4x^2 + 2x + 7x^2$, we combine like terms, starting from the highest degree terms. It is suggested to underline the groups of like terms, using different type of underlining for each group, so that it is easier to see all the like terms and not to miss any of them. So,

$$\underline{\underline{5x}}\ \underline{-4x^2}\ \underline{\underline{+2x}}\ \underline{+7x^2} = \mathbf{3x^2 + 7x}$$

Remember that the sign in front of a term belongs to this term.

b. To simplify $8p - (2 - 3p) + (3p - 6)$, first we remove the brackets using the distributive property of multiplication and then we combine like terms. So, we have

$-(2-3p)$
$= (-1)(2-3p)$

$$8p - (2 - 3p) + (3p - 6)$$

$$= \underline{8p}\ - 2\ \underline{+3p}\ \underline{+3p} - 6$$

$$= \mathbf{14p - 8}$$

Example 5 ► Adding or Subtracting Polynomials

Perform the indicated operations.

a. $(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$
b. $(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$
c. $[9p - (3p - 2)] - [4p - (3 - 7p) + p]$

Solution ► **a.** To add polynomials, combine their like terms. So,

remove any bracket preceeded by a "+" sign

$$(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$$

$$= 6a^5 - 4a^3 \underline{\underline{+3a}}\ \uwave{-1} + 2a^4 + a^2 \underline{\underline{-5a}}\ \uwave{+9}$$

$$= \mathbf{6a^5 + 2a^4 - 4a^3 + a^2 - 2a + 8}$$

b. To subtract a polynomial, add its opposite. In practice, remove any bracket preceeded by a negative sign by reversing the signs of all the terms of the polynomial inside the bracket. So,

$$(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$$

$$= \underline{4y^3} - 3y^2 \underline{\underline{+y}}\ \uwave{+6}\ \underline{-y^3}\ \underline{\underline{-3y}}\ \uwave{+2}$$

$$= \mathbf{3y^3 - 3y^2 - 2y + 8}$$

To remove a bracket preceeded by a "−" sign, reverse each sign inside the bracket.

c. First, perform the operations within the square brackets and then subtract the resulting polynomials. So,

$$
\begin{aligned}
&[9p-(3p-2)]-[4p-(3-7p)+p] \\
&=[9p-3p+2]-[4p-3+7p+p] \\
&=[6p+2]-[12p-3] \\
&=6p+2-12p+3 \\
&=\boldsymbol{-6p+5}
\end{aligned}
$$

collect like terms before removing the next set of brackets

Addition and Subtraction of Polynomial Functions

Similarly as for polynomials, addition and subtraction can also be defined for general functions.

Definition 1.3 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **sum function $\boldsymbol{f+g}$** is defined as

$$(\boldsymbol{f}+\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})$$

and the **difference function $\boldsymbol{f-g}$** is defined as

$$(\boldsymbol{f}-\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the sum or difference function is the intersection $\boldsymbol{D_f \cap D_g}$ of the domains of the two functions.

A frequently used application of a sum or difference of polynomial functions comes from the business area. The fact that profit P equals revenue R minus cost C can be recorded using function notation as

$$P(x)=(R-C)(x)=R(x)-C(x),$$

where x is the number of items produced and sold. Then, if $R(x)=6.5x$ and $C(x)=3.5x+900$, the profit function becomes

$$\boldsymbol{P(x)}=R(x)-C(x)=6.5x-(3.5x+900)=6.5x-3.5x-900=\boldsymbol{3x-900}.$$

Example 6 ► **Adding or Subtracting Polynomial Functions**

Suppose $P(x)=x^2-6x+4$ and $Q(x)=2x^2-1$. Find the following:

a. $(P+Q)(x)$ and $(P+Q)(2)$
b. $(P-Q)(x)$ and $(P-Q)(-1)$
c. $(P+Q)(k)$
d. $(P-Q)(2a)$

Solution ► **a.** Using the definition of the sum of functions, we have

$$\boldsymbol{(P+Q)(x)} = P(x) + Q(x) = \underbrace{x^2 - 6x + 4}_{P(x)} + \underbrace{2x^2 - 1}_{Q(x)} = \boldsymbol{3x^2 - 6x + 3}$$

Therefore, $\boldsymbol{(P+Q)(2)} = 3 \cdot 2^2 - 6 \cdot 2 + 3 = 12 - 12 + 3 = \mathbf{3}$.

Alternatively, $(P+Q)(2)$ can be calculated without refering to the function $(P+Q)(x)$, as shown below.

$$\boldsymbol{(P+Q)(2)} = P(2) + Q(2) = \underbrace{2^2 - 6 \cdot 2 + 4}_{P(2)} + \underbrace{2 \cdot 2^2 - 1}_{Q(2)}$$

$$= 4 - 12 + 4 + 8 - 1 = \mathbf{3}.$$

b. Using the definition of the difference of functions, we have

$$\boldsymbol{(P-Q)(x)} = P(x) - Q(x) = \underbrace{x^2 - 6x + 4}_{P(x)} - \underbrace{(2x^2 - 1)}_{Q(x)}$$

$$= x^2 - 6x + 4 - 2x^2 + 1 = \boldsymbol{-x^2 - 6x + 5}$$

To evaluate $(P-Q)(-1)$, we will take advantage of the difference function calculated above. So, we have

$$\boldsymbol{(P-Q)(-1)} = -(-1)^2 - 6(-1) + 5 = -1 + 6 + 5 = \mathbf{10}.$$

c. By *Definition 1.3*,

$$\boldsymbol{(P+Q)(k)} = P(k) + Q(k) = k^2 - 6k + 4 + 2k^2 - 1 = \boldsymbol{3k^2 - 6k + 3}$$

Alternatively, we could use the sum function already calculated in the solution to *Example 6a*. Then, the result is instant: $\boldsymbol{(P+Q)(k) = 3k^2 - 6k + 3}$.

d. To find $(P-Q)(2a)$, we will use the difference function calculated in the solution to *Example 6b*. So, we have

$$\boldsymbol{(P-Q)(2a)} = -(2a)^2 - 6(2a) + 5 = \boldsymbol{-4a^2 - 12a + 5}.$$

P.1 Exercises

Determine whether the expression is a monomial.

1. $-\pi x^3 y^2$

2. $5x^{-4}$

3. $5\sqrt{x}$

4. $\sqrt{2}x^4$

Identify the degree and coefficient.

5. xy^3

6. $-x^2 y$

7. $\sqrt{2}xy$

8. $-3\pi x^2 y^5$

Arrange each polynomial in descending order of powers of the variable. Then, identify the degree and the leading coefficient of the polynomial.

9. $5 - x + 3x^2 - \frac{2}{5}x^3$
10. $7x + 4x^4 - \frac{4}{3}x^3$
11. $8x^4 + 2x^3 - 3x + x^5$

12. $4y^3 - 8y^5 + y^7$
13. $q^2 + 3q^4 - 2q + 1$
14. $3m^2 - m^4 + 2m^3$

State the degree of each polynomial and identify it as a monomial, binomial, trinomial, or n-th degree polynomial if $n > 2$.

15. $7n - 5$
16. $4z^2 - 11z + 2$
17. 25

18. $-6p^4q + 3p^3q^2 - 2pq^3 - p^4$
19. $-mn^6$
20. $16k^2 - 9p^2$

Let $P(x) = -2x^2 + x - 5$ *and* $Q(x) = 2x - 3$. *Evaluate each expression.*

21. $P(-1)$
22. $P(0)$
23. $2P(1)$
24. $P(a)$

25. $Q(-1)$
26. $Q(5)$
27. $Q(a)$
28. $Q(3a)$

29. $3Q(-2)$
30. $3P(a)$
31. $3Q(a)$
32. $Q(a+1)$

Simplify each polynomial expression.

33. $5x + 4y - 6x + 9y$
34. $4x^2 + 2x - 6x^2 - 6$

35. $6xy + 4x - 2xy - x$
36. $3x^2y + 5xy^2 - 3x^2y - xy^2$

37. $9p^3 + p^2 - 3p^3 + p - 4p^2 + 2$
38. $n^4 - 2n^3 + n^2 - 3n^4 + n^3$

39. $4 - (2 + 3m) + 6m + 9$
40. $2a - (5\mathrm{a} - 3) - (7a - 2)$

41. $6 + 3x - (2x + 1) - (2x + 9)$
42. $4y - 8 - (-3 + y) - (11y + 5)$

Perform the indicated operations.

43. $(x^2 - 5y^2 - 9z^2) + (-6x^2 + 9y^2 - 2z^2)$
44. $(7x^2y - 3xy^2 + 4xy) + (-2x^2y - xy^2 + xy)$

45. $(-3x^2 + 2x - 9) - (x^2 + 5x - 4)$
46. $(8y^2 - 4y^3 - 3y) - (3y^2 - 9y - 7y^3)$

47. $(3r^6 + 5) + (-7r^2 + 2r^6 - r^5)$
48. $(5x^{2a} - 3x^a + 2) + (-x^{2a} + 2x^a - 6)$

49. $(-5a^4 + 8a^2 - 9) - (6a^3 - a^2 + 2)$
50. $(3x^{3a} - x^a + 7) - (-2x^{3a} + 5x^{2a} - 1)$

51. $(10xy - 4x^2y^2 - 3y^3) - (-9x^2y^2 + 4y^3 - 7xy)$

52. Subtract $(-4x + 2z^2 + 3m)$ from the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$.

53. Subtract the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$ from $(-4x + 2z^2 + 3m)$.

54. $[2p - (3p - 6)] - \big[\big(5p - (8 - 9p)\big) + 4p\big]$

55. $-[3z^2 + 5z - (2z^2 - 6z)] + \big[\big(8z^2 - (5z - z^2)\big) + 2z^2\big]$

56. $5k - (5k - [2k - (4k - 8k)]) + 11k - (9k - 12k)$

For each pair of functions, find ***a)*** $(f+g)(x)$ *and* ***b)*** $(f-g)(x)$.

57. $f(x)=5x-6,\ \ g(x)=-2+3x$

58. $f(x)=x^2+7x-2,\ \ g(x)=6x+5$

59. $f(x)=3x^2-5x,\ \ g(x)=-5x^2+2x+1$

60. $f(x)=2x^n-3x-1,\ \ g(x)=5x^n+x-6$

61. $f(x)=2x^{2n}-3x^n+3,\ \ g(x)=-8x^{2n}+x^n-4$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x+5}$, *and* $\boldsymbol{R(x)=x-2}$. *Find each of the following.*

62. $(P+R)(-1)$

63. $(P-Q)(-2)$

64. $(Q-R)(3)$

65. $(R-Q)(0)$

66. $(R-Q)(k)$

67. $(P+Q)(a)$

68. $(Q-R)(a+1)$

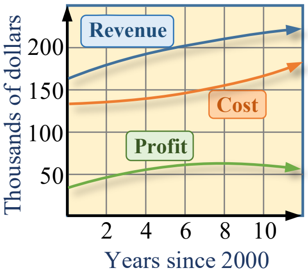
69. $(P+R)(2k)$

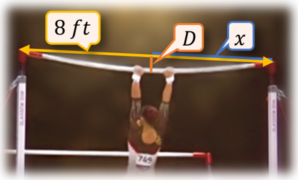
Solve each problem.

70. Suppose that during the years 2000-2012 the revenue R and the cost C of a particular business are modelled by the polynomials

$$R(t)=-0.296t^2+9.72t+164 \text{ and } C(t)=0.154t^2+2.15t+135,$$

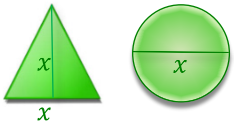
where t represents the number of years since 2000 and both $R(t)$ and $C(t)$ are in thousands of dollars. Write a polynomial that models the profit $P(t)$ of this business during the years 2000-2012.





71. Suppose that the deflection D of an 8 feet-long gymnastic bar can be approximated by the polynomial function $D(x)=0.037x^4-0.59x^3+2.35x^2$, where x is the distance in feet from one end of the bar and D is in centimeters. To the nearest tenths of a centimeter, determine the maximum deflection for this bar, assuming that it occurs at the middle of the bar.

72. Write a polynomial that can be used to calculate the sum of areas of a triangle with the base and height of length x and a circle with diameter x. Determine the total area of the two shapes for $x=5$ centimeters. *Round the answer to the nearest centimeter square.*



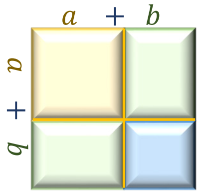
73. Suppose the cost in dollars of sewing n dresses is given by $C(n)=32n+1500$. If the dresses can be sold for \$56 each, complete the following.

a. Write a function $R(n)$ that gives the revenue for selling n dresses.

b. Write a formula $P(n)$ for the profit. *Recall that profit is defined as the difference between revenue and cost.*

c. Evaluate $P(100)$ and interpret the answer.

P2 Multiplication of Polynomials



As shown in the previous section, addition and subtraction of polynomials results in another polynomial. This means that the **set of polynomials** is **closed under** the operation of **addition** and **subtraction**. In this section, we will show that the set of polynomials is also closed under the operation of **multiplication**, meaning that a product of polynomials is also a polynomial.

Properties of Exponents

Since multiplication of polynomials involves multiplication of powers, let us review properties of exponents first.

Recall:

power

exponent

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}$$

base

For example, $x^4 = x \cdot x \cdot x \cdot x$ and we read it "x to the fourth power" or shorter "x to the fourth". If $n = 2$, the power x^2 is customarily read "x squared". If $n = 3$, the power x^3 is often read "x cubed".

Let $a \in \mathbb{R}$, and $m, n \in \mathbb{W}$. The table below shows basic exponential rules with some examples justifying each rule.

Power Rules for Exponents

General Rule	Description	Example
$a^m \cdot a^n = a^{m+n}$	To **multiply powers** of the same bases, keep the base and **add** the **exponents**.	$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x)$ $= x^{2+3} = x^5$
$\frac{a^m}{a^n} = a^{m-n}$	To **divide powers** of the same bases, keep the base and **subtract** the **exponents**.	$\frac{x^5}{x^2} = \frac{(x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x})}{(\cancel{x} \cdot \cancel{x})}$ $= x^{5-2} = x^3$
$(a^m)^n = a^{mn}$	To raise a **power to a power, multiply the exponents.**	$(x^2)^3 = (x \cdot x)(x \cdot x)(x \cdot x)$ $= x^{2 \cdot 3} = x^6$
$(ab)^n = a^n b^n$	To raise a **product to a power**, raise **each factor to that power**.	$(2x)^3 = 2^3 x^3$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	To raise a **quotient to a power**, raise the **numerator and** the **denominator to that power**.	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2}$
$a^0 = 1$ **for** $a \neq 0$ 0^0 is undefined	A **nonzero number** raised **to the power of zero** equals **one**.	$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1$

Example 1 ► **Simplifying Exponential Expressions**

Simplify.

a. $(-3xy^2)^4$

b. $(5p^3q)(-2pq^2)$

c. $\left(\frac{-2x^5}{x^2y}\right)^3$

d. $x^{2a}x^a$

Solution ►

a. To simplify $(-3xy^2)^4$, we apply the fourth power to each factor in the bracket. So,

$$(-3xy^2)^4 = \underbrace{(-3)^4}_{\substack{\text{even power}\\ \text{of a negative}\\ \text{is a positive}}} \cdot x^4 \cdot \underbrace{(y^2)^4}_{\substack{\text{multiply}\\ \text{exponents}}} = \mathbf{3^4x^4y^8}$$

$p = p^1$

b. To simplify $(5p^3q)(-2pq^2)$, we multiply numbers, powers of p, and powers of q. So,

$$(5p^3q)(-2pq^2) = (-2)\cdot 5 \cdot \underbrace{p^3\cdot p}_{\substack{\text{add}\\ \text{exponents}}} \cdot \underbrace{q\cdot q^2}_{\substack{\text{add}\\ \text{exponents}}} = \mathbf{-10p^4q^3}$$

c. To simplify $\left(\frac{-2x^5}{x^2y}\right)^3$, first we reduce the common factors and then we raise every factor of the numerator and denominator to the third power. So, we obtain

$$\left(\frac{-2x^{\overset{3}{\cancel{5}}}}{\cancel{x}^2y}\right)^3 = \left(\frac{-2x^3}{y}\right)^3 = \frac{(-2)^3(x^3)^3}{y^3} = \mathbf{\frac{-8x^9}{y^3}}$$

d. When multiplying powers with the same bases, we add exponents, so $x^{2a}x^a = \mathbf{x^{3a}}$

Multiplication of Polynomials

Multiplication of polynomials involves finding products of monomials. To multiply monomials, we use the commutative property of multiplication and the product rule of powers.

Example 2 ► **Multiplying Monomials**

Find each product.

a. $(3x^4)(5x^3)$

b. $(5b)(-2a^2b^3)$

c. $-4x^2(3xy)(-x^2y)$

Solution ►

a. $$(3x^4)(5x^3) = 3\cdot \underbrace{x^4\cdot 5}_{\substack{\text{commutative}\\ \text{property}}} \cdot x^3 = 3\cdot 5\cdot \underbrace{x^4\cdot x^3}_{\substack{\text{product}\\ \text{rule of powers}}} = \mathbf{15x^7}$$

b. $$(5b)(-2a^2b^3) = 5(-2)a^2bb^3 = \mathbf{-10a^2b^4}$$

c. $$-4x^2(3xy)(-x^2y) = \underbrace{(-4)\cdot 3\cdot(-1)}_{\substack{\text{multiply}\\ \text{coefficients}}}\,\underbrace{x^2xx^2}_{\substack{\text{apply product}\\ \text{rule of powers}}}\,\underbrace{yy} = \mathbf{12x^5y^2}$$

To find the product of monomials, find the following:

- the final **sign**,
- the **number**,
- the **power**.

The intermediate steps are not necessary to write. The final answer is immediate if we follow the order: **sign**, **number**, **power** of each variable.

To multiply polynomials by a monomial, we use the distributive property of multiplication.

Example 3 ► **Multiplying Polynomials by a Monomial**

Find each product.

a. $-2x(3x^2 - x + 7)$ **b.** $(5b - ab^3)(3ab^2)$

Solution ► **a.** To find the product $-2x(3x^2 - x + 7)$, we distribute the monomial $-2x$ to each term inside the bracket. So, we have

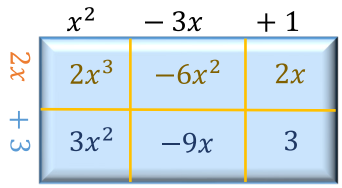
$$-2x(3x^2 - x + 7) = \underbrace{-2x(3x^2) - 2x(-x) - 2x(7)}_{\textit{this step can be done mentally}} = \boldsymbol{-6x^3 + 2x^2 - 14x}$$

b. $$(5b - ab^3)(3ab^2) = \underbrace{5b(3ab^2) - ab^3(3ab^2)}_{\textit{this step can be done mentally}}$$

$$= 15ab^3 - 3a^2b^5 = \boldsymbol{-3a^2b^5 + 15ab^3}$$

arranged in decreasing order of powers

When multiplying polynomials by polynomials we **multiply each term of the first polynomial by each term of the second polynomial**. This process can be illustrated with finding areas of a rectangle whose sides represent each polynomial. For example, we multiply $(2x + 3)(x^2 - 3x + 1)$ as shown below



	x^2	$-3x$	$+1$
$2x$	$2x^3$	$-6x^2$	$2x$
$+3$	$3x^2$	$-9x$	3

$$\begin{array}{rl} \text{So,}\quad (2x+3)(x^2-3x+1) = & 2x^3 - 6x^2 + 2x \\ & \quad\ \ + 3x^2 - 9x + 3 \\ \hline & = \boldsymbol{2x^3 - 3x^2 - 7x + 3} \end{array}$$

line up like terms to combine them

Example 4 ► **Multiplying Polynomials by Polynomials**

Find each product.

a. $(3y^2 - 4y - 2)(5y - 7)$ **b.** $4a^2(2a - 3)(3a^2 + a - 1)$

Solution ► **a.** To find the product $(3y^2 - 4y - 2)(5y - 7)$, we can distribute the terms of the second bracket over the first bracket and then collect the like terms. So, we have

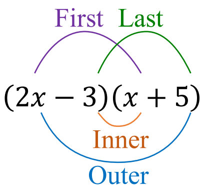
$$\begin{array}{rl} (3y^2 - 4y - 2)(5y - 7) = & 15y^3 - 20y^2 - 10y \\ & \qquad - 21y^2 + 28y + 14 \\ \hline & = \boldsymbol{15y^3 - 41y^2 + 18y + 14} \end{array}$$

b. To find the product $4a^2(2a - 3)(3a^2 + a - 1)$, we will multiply the two brackets first, and then multiply the resulting product by $4a^2$. So,

$$4a^2(2a-3)(3a^2+a-1) = 4a^2\underbrace{\left(6a^3 + 2a^2 - 2a - 9a^2 - 3a + 3\right)}_{\text{collect like terms before removing the bracket}}$$

$$= 4a^2(6a^3 - 7a^2 - 5a + 3) = \mathbf{24a^5 - 28a^4 - 20a^3 + 12a^2}$$

In multiplication of binomials, it might be convenient to keep track of the multiplying terms by following the **FOIL** mnemonic, which stands for multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials. Here is how it works:



$$(2x-3)(x+5) = 2x^2 \underbrace{+10x - 3x}_{\substack{\text{the } \textbf{sum} \text{ of the} \\ \textbf{Outer} \text{ and } \textbf{Inner} \\ \text{terms becomes} \\ \text{the } \textbf{middle} \text{ term}}} - 15 = \mathbf{2x^2 + 7x - 15}$$

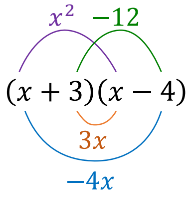
Example 5 ► Using the FOIL Method in Binomial Multiplication

Find each product.

a. $(x+3)(x-4)$ **b.** $(5x-6)(2x+3)$

Solution ►

a. To find the product $(x+3)(x-4)$, we may follow the **FOIL** method



$$(x+3)(x-4) = \mathbf{x^2 - x - 12}$$

To find the **linear (middle)** term try to add the inner and outer products mentally.

b. Observe that the linear term of the product $(5x-6)(2x+3)$ is equal to the sum of $-12x$ and $15x$, which is $3x$. So, we have

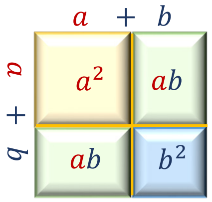
$$(5x-6)(2x+3) = \mathbf{10x^2 + 3x - 18}$$

Special Products

Suppose we want to find the product $(a+b)(a+b)$. This can be done via the FOIL method

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b}) = a^2 + ab + ab + b^2 = \boldsymbol{a^2 + 2ab + b^2},$$

or via the geometric visualization:



Observe that since the products of the inner and outer terms are both equal to ab, we can use a shortcut and write the middle term of the final product as $2ab$. We encorage the reader to come up with similar observations regarding the product $(a-b)(a-b)$. This regularity in multiplication of a binomial by itself leads us to the **perfect square formula**:

$$\boxed{(\boldsymbol{a} \pm \boldsymbol{b})^2 = \boldsymbol{a}^2 \pm 2\boldsymbol{ab} + \boldsymbol{b}^2}$$

In the above notation, the " $\pm$ " sign is used to record two formulas at once, the perfect square of a sum and the perfect square of a difference. It tells us to either use a " $+$ " in both places, or a " $-$ " in both places. The $\boldsymbol{a}$ and $\boldsymbol{b}$ can actually represent any expression. For example, to expand $(2x-y^2)^2$, we can apply the perfect square formula by treating $2x$ as a and y^2 as b. Here is the calculation.

$$(2x-y^2)^2 = (2x)^2 - 2(2x)y^2 + (y^2)^2 = \boldsymbol{4x^2 - 4xy^2 + y^4}$$

Conjugate binomials have the **same first terms** and **opposite second terms**.

Another interesting pattern can be observed when multiplying two **conjugate** brackets, such as $(a+b)$ and $(a-b)$. Using the FOIL method,

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}-\boldsymbol{b}) = a^2 + \cancel{ab} - \cancel{ab} - b^2 = \boldsymbol{a^2 - b^2},$$

we observe that the products of the inner and outer terms are opposite. So, they add to zero and the product of conjugate brackets becomes the difference of squares of the two terms. This regularity in multiplication of conjugate brackets leads us to the **difference of squares formula**.

$$\boxed{(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}^2 - \boldsymbol{b}^2}$$

Again, $\boldsymbol{a}$ and $\boldsymbol{b}$ can represent any expression. For example, to find the product $(3x+0.1y^2)(3x-0.1y^2)$, we can apply the difference of squares formula by treating $3x$ as a and $0.1y^2$ as b. Here is the calculation.

$$(\boldsymbol{3x+0.1y^3})(\boldsymbol{3x-0.1y^3}) = (3x)^2 - (0.1y^3)^2 = \boldsymbol{9x^2 - 0.01y^6}$$

We encourage the use of the above formulas whenever applicable, as it allows for more efficient calculations and helps to observe patterns useful in future factoring.

Example 6 ►

Using Special Product Formulas in Polynomial Multiplication

Find each product. Apply special products formulas, if applicable.

a. $(5x+3y)^2$ **b.** $(x+y-5)(x+y+5)$

Solution ► **a.** Applying the perfect square formula, we have

$$(5x+3y)^2 = (5x)^2 + 2(5x)3y + (3y)^2 = \boldsymbol{25x^2 + 30xy + 9y^2}$$

b. The product $(x+y-5)(x+y+5)$ can be found by multiplying each term of the first polynomial by each term of the second polynomial, using the distributive property. However, we can find the product $(x+y-5)(x+y+5)$ in a more efficient way by applying the difference of squares formula. Treating the expression $x+y$ as the first term a and the 5 as the second term b in the formula $(a+b)(a-b)=a^2-b^2$, we obtain

$$(x+y-5)(x+y+5)=(x+y)^2-5^2$$

$$=\underbrace{\boldsymbol{x^2+2xy+y^2}}_{\substack{\text{here we apply}\\ \text{the perfect square}\\ \text{formula}}}\ \boldsymbol{-25}$$

Caution: *The perfect square formula shows that* $(\boldsymbol{a}+\boldsymbol{b})^2 \neq \boldsymbol{a}^2+\boldsymbol{b}^2$.
The difference of squares formula shows that $(\boldsymbol{a}-\boldsymbol{b})^2 \neq \boldsymbol{a}^2-\boldsymbol{b}^2$.
More generally, $(\boldsymbol{a}\pm\boldsymbol{b})^{\boldsymbol{n}} \neq \boldsymbol{a}^{\boldsymbol{n}}\pm\boldsymbol{b}^{\boldsymbol{n}}$ *for any natural* $n \neq 1$.

Product Functions

The operation of multiplication can be defined not only for polynomials but also for general functions.

Definition 2.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **product function**, denoted $\boldsymbol{f\cdot g}$ or $\boldsymbol{fg}$, is defined as

$$(\boldsymbol{f}\cdot\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the product function is the intersection $\boldsymbol{D_f}\cap\boldsymbol{D_g}$of the domains of the two functions.

Example 7 ► **Multiplying Polynomial Functions**

Suppose $P(x)=x^2-4x$ and $Q(x)=3x+2$. Find the following:

a. $(PQ)(x)$, $(PQ)(-2)$, and $P(-2)Q(-2)$
b. $(QQ)(x)$ and $(QQ)(1)$
c. $2(PQ)(k)$

Solution ► **a.** Using the definition of the product function, we have

$$(PQ)(x)=P(x)\cdot Q(x)=(x^2-4x)(3x+2)=3x^3+2x^2-12x^2-8x$$

$$=\boldsymbol{3x^3-10x^2-8x}$$

To find $(PQ)(-2)$, we substitute $x=-2$ to the above polynomial function. So,

$$(PQ)(-2)=3(-2)^3-10(-2)^2-8(-2)=3\cdot(-8)-10\cdot 4+16$$

$$=-24-40+16=\boldsymbol{-48}$$

To find $P(-2)Q(-2)$, we calculate

$$P(-2)Q(-2)=\left((-2)^2-4(-2)\right)(3(-2)+2)=(4+8)(-6+2)=12\cdot(-4)$$

$$=\boldsymbol{-48}$$

Observe that both expressions result in the same value. This was expected, as by the definition, $(PQ)(-2) = P(-2) \cdot Q(-2)$.

b. Using the definition of the product function as well as the perfect square formula, we have

$$(QQ)(x) = Q(x) \cdot Q(x) = [Q(x)]^2 = (3x+2)^2 = \mathbf{9x^2 + 12x + 4}$$

Therefore, $(QQ)(1) = 9 \cdot 1^2 + 12 \cdot 1 + 4 = 9 + 12 + 4 = \mathbf{25}$.

c. Since $(PQ)(x) = 3x^3 - 10x^2 - 8x$, as shown in the solution to *Example 7a*, then $(PQ)(k) = 3k^3 - 10k^2 - 8k$. Therefore,

$$2(PQ)(k) = 2[3k^3 - 10k^2 - 8k] = \mathbf{6k^3 - 20k^2 - 16k}$$

P.2 Exercises

1. Decide whether each expression has been simplified correctly. If not, correct it.

a. $x^2 \cdot x^4 = x^8$
b. $-2x^2 = 4x^2$
c. $(5x)^3 = 5^3x^3$
d. $-\left(\frac{x}{5}\right)^2 = -\frac{x^2}{25}$
e. $(a^2)^3 = a^5$
f. $4^5 \cdot 4^2 = 16^7$
g. $\frac{6^5}{3^2} = 2^3$
h. $xy^0 = 1$
i. $(-x^2y)^3 = -x^6y^3$

Simplify each expression.

2. $3x^2 \cdot 5x^3$
3. $-2y^3 \cdot 4y^5$
4. $3x^3(-5x^4)$
5. $2x^2y^5(7xy^3)$
6. $(6t^4s)(-3t^3s^5)$
7. $(-3x^2y)^3$
8. $\frac{12x^3y}{4xy^2}$
9. $\frac{15x^5y^2}{-3x^2y^4}$
10. $(-2x^5y^3)^2$
11. $\left(\frac{4a^2}{b}\right)^3$
12. $\left(\frac{-3m^4}{n^3}\right)^2$
13. $\left(\frac{-5p^2q}{pq^4}\right)^3$
14. $3a^2(-5a^5)(-2a)^0$
15. $-3a^3b(-4a^2b^4)(ab)^0$
16. $\frac{(-2p)^2pq^3}{6p^2q^4}$
17. $\frac{(-8xy)^2y^3}{4x^5y^4}$
18. $\left(\frac{-3x^4y^6}{18x^6y^7}\right)^3$
19. $((-2x^3y)^2)^3$
20. $((-a^2b^4)^3)^5$
21. x^nx^{n-1}
22. $3a^{2n}a^{1-n}$
23. $(5^a)^{2b}$
24. $(-7^{3x})^{4y}$
25. $\frac{-12x^{a+1}}{6x^{a-1}}$
26. $\frac{25x^{a+b}}{-5x^{a-b}}$
27. $\left(x^{a+b}\right)^{a-b}$
28. $(x^2y)^n$

Find each product.

29. $8x^2y^3(-2x^5y)$

30. $5a^3b^5(-3a^2b^4)$

31. $2x(-3x+5)$

32. $4y(1-6y)$

33. $-3x^4y(4x-3y)$

34. $-6a^3b(2b+5a)$

35. $5k^2(3k^2-2k+4)$

36. $6p^3(2p^2+5p-3)$

37. $(x+6)(x-5)$

38. $(x-7)(x+3)$

39. $(2x+3)(3x-2)$

40. $3p(5p+1)(3p+2)$

41. $2u^2(u-3)(3u+5)$

42. $(2t+3)(t^2-4t-2)$

43. $(2x-3)(3x^2+x-5)$

44. $(a^2-2b^2)(a^2-3b^2)$

45. $(2m^2-n^2)(3m^2-5n^2)$

46. $(x+5)(x-5)$

47. $(a+2b)(a-2b)$

48. $(x+4)(x+4)$

49. $(a-2b)(a-2b)$

50. $(x-4)(x^2+4x+16)$

51. $(y+3)(y^2-3y+9)$

52. $(x^2+x-2)(x^2-2x+3)$

53. $(2x^2+y^2-2xy)(x^2-2y^2-xy)$

True or False? If it is false, show a counterexample by choosing values for a and b that would not satisfy the equation.

54. $(a+b)^2=a^2+b^2$

55. $a^2-b^2=(a-b)(a+b)$

56. $(a-b)^2=a^2+b^2$

57. $(a+b)^2=a^2+2ab+b^2$

58. $(a-b)^2=a^2+ab+b^2$

59. $(a-b)^3=a^3-b^3$

Find each product. Use the ***difference of squares*** *or the* ***perfect square*** *formula, if applicable.*

60. $(2p+3)(2p-3)$

61. $(5x-4)(5x+4)$

62. $\left(b-\frac{1}{3}\right)\left(b+\frac{1}{3}\right)$

63. $\left(\frac{1}{2}x-3y\right)\left(\frac{1}{2}x+3y\right)$

64. $(2xy+5y^3)(2xy-5y^3)$

65. $(x^2+7y^3)(x^2-7y^3)$

66. $(1.1x+0.5y)(1.1x-0.5y)$

67. $(0.8a+0.2b)(0.8a+0.2b)$

68. $(x+6)^2$

69. $(x-3)^2$

70. $(4x+3y)^2$

71. $(5x-6y)^2$

72. $\left(3a+\frac{1}{2}\right)^2$

73. $\left(2n-\frac{1}{3}\right)^2$

74. $(a^3b^2-1)^2$

75. $(x^4y^2+3)^2$

76. $(3a^2+4b^3)^2$

77. $(2x^2-3y^3)^2$

78. $3y(5xy^3+2)(5xy^3-2)$

79. $2a(2a^2+5ab)(2a^2+5ab)$

80. $3x(x^2y-xy^3)^2$

81. $(-xy+x^2)(xy+x^2)$

82. $(4p^2+3pq)(-3pq+4p^2)$

83. $(x+1)(x-1)(x^2+1)$

84. $(2x-y)(2x+y)(4x^2+y^2)$

85. $(a-b)(a+b)(a^2-b^2)$

86. $(a+b+1)(a+b-1)$

87. $(2x+3y-5)(2x+3y+5)$

88. $(3m+2n)(3m-2n)(9m^2-4n^2)$

89. $\left((2k-3)+h\right)^2$

90. $\left((4x+y)-5\right)^2$

91. $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}+y^{2b}\right)$

92. $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}-y^{2b}\right)$

Use the difference of squares formula, $(a+b)(a-b)=a^2-b^2$, *to find each product.*

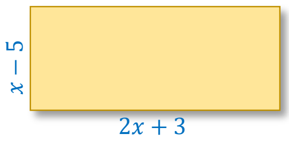
93. $101 \cdot 99$

94. $198 \cdot 202$

95. $505 \cdot 495$

Find the area of each figure. Express it as a polynomial in descending powers of the variable x.

96.



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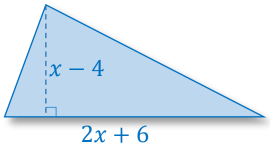

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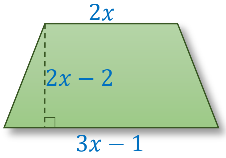


For each pair of functions, $\boldsymbol{f}$ *and* $\boldsymbol{g}$*, find the* ***product*** *function* $(\boldsymbol{fg})(\boldsymbol{x})$.

99. $f(x)=5x-6, \quad g(x)=-2+3x$

100. $f(x)=x^2+7x-2, \quad g(x)=6x+5$

101. $f(x)=3x^2-5x, \quad g(x)=9+x-x^2$

102. $f(x)=x^n-4, \quad g(x)=x^n+1$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x}$, *and* $\boldsymbol{R(x)=x-2}$. *Find each of the following.*

103. $(PR)(x)$

104. $(PQ)(x)$

105. $(PQ)(a)$

106. $(PR)(-1)$

107. $(PQ)(3)$

108. $(PR)(0)$

109. $(QR)(x)$

110. $(QR)\left(\frac{1}{2}\right)$

111. $(QR)(a+1)$

112. $P(a-1)$

113. $P(2a+3)$

114. $P(1+h)-P(1)$

Solve each problem.

115. Squares with x centimeters long sides are cut out from each corner of a rectangular piece of cardboard measuring 50 cm by 70 cm. Then the flaps of the remaining cardboard are folded up to construct a box. Find the volume $V(x)$ of the box in terms of the length x.

116. A rectangular flower-bed has a perimeter of 60 meters. If the rectangle is w meters wide, write a polynomial that can be used to determine the area $A(w)$ of the flower-bed in terms of w.

P3 Division of Polynomials

In this section we will discuss dividing polynomials. The result of division of polynomials is not always a polynomial. For example, $x+1$ divided by x becomes

$$\frac{x+1}{x}=\frac{x}{x}+\frac{1}{x}=1+\frac{1}{x},$$

which is not a polynomial. Thus, the set of polynomials is not closed under the operation of division. However, we can perform division with remainders, mirroring the algorithm of division of natural numbers. We begin with dividing a polynomial by a monomial and then by another polynomial.

Division of Polynomials by Monomials

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial, and then simplify each quotient. In other words, we use the reverse process of addition of fractions, as illustrated below.

$$\boldsymbol{\frac{a+b}{d}=\frac{a}{d}+\frac{b}{d}}$$

Example 1 ► **Dividing Polynomials by Monomials**

Divide and simplify.

a. $(6x^3+15x^2-2x)\div(3x)$ **b.** $\frac{xy^2-8x^2y+6x^3y^2}{-2xy^2}$

Solution ►

a. $(6x^3+15x^2-2x)\div(3x)=\frac{6x^3+15x^2-2x}{3x}=\frac{\overset{2}{\cancel{6}}x^{\overset{2}{\cancel{3}}}}{\cancel{3}\cancel{x}}+\frac{\overset{5}{\cancel{15}}x^{\cancel{2}}}{\cancel{3}\cancel{x}}-\frac{2\cancel{x}}{3\cancel{x}}=\boldsymbol{2x^2+5x-\frac{2}{3}}$

b. $\frac{xy^2-8x^2y+6x^3y^2}{-2xy^2}=-\frac{\cancel{x}\cancel{y^2}}{2\cancel{x}\cancel{y^2}}+\frac{\overset{4}{\cancel{8}}x^{\cancel{2}}\cancel{y}}{\cancel{2}\cancel{x}y^{\cancel{2}}}-\frac{\overset{3}{\cancel{6}}x^{\overset{2}{\cancel{3}}}\cancel{y^2}}{\cancel{2}\cancel{x}\cancel{y^2}}=\boldsymbol{-\frac{1}{2}+\frac{4x}{y}-3x^2}$

Division of Polynomials by Polynomials

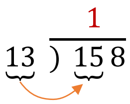
To divide a polynomial by another polynomial, we follow an algorithm similar to the long division algorithm used in arithmetic. For example, observe the steps taken in the long division algorithm when dividing 158 by 13 and the corresponding steps when dividing x^2+5x+8 by $x+3$.

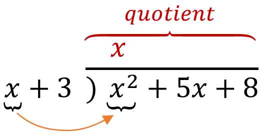
Step 1: Place the dividend under the long division symbol and the divisor in front of this symbol.

$$13\overline{)\,158}\qquad\qquad \underbrace{x+3}_{divisor}\,\overline{)\,\underbrace{x^2+5x+8}_{dividend}}$$

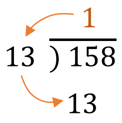
Remember: Both polynomials should be written in **decreasing order of powers**. Also, any **missing terms** after the leading term should be displayed with a **zero coefficient**. This will ensure that the terms in each column are of the same degree.

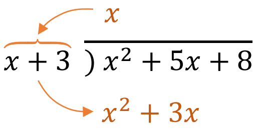
Step 2: Divide the first term of the dividend by the first term of the divisor and record the quotient above the division symbol.



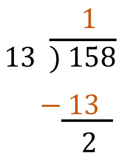


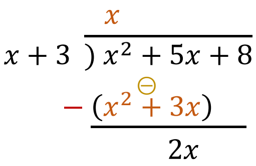
Step 3: Multiply the quotient from *Step 2* by the divisor and write the product under the dividend, lining up the columns with the same degree terms.



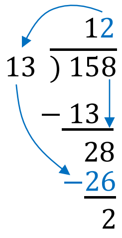


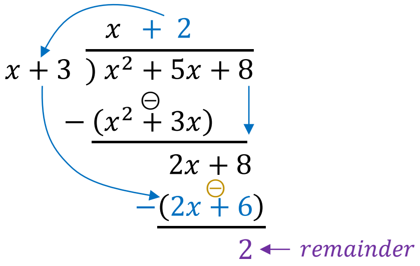
Step 4: Underline and subtract by adding opposite terms in each column. We suggest recording the new sign in a circle, so that it is clear what is being added.





Step 5: Drop the next term (or digit) and repeat the algorithm until the degree of the remainder is lower than the degree of the divisor.





In the example of long division of numbers, we have $158 = 13 \cdot 12 + 2$.

So, the quotient can be written as $\frac{158}{13} = \mathbf{12} + \frac{\mathbf{2}}{\mathbf{13}}$.

In the example of long division of polynomials, we have

$$x^2 + 5x + 8 = (x + 3) \cdot (x + 2) + 2.$$

So, the quotient can be written as $\frac{x^2+5x+8}{x+3} = \boldsymbol{x + 2} + \frac{\mathbf{2}}{\boldsymbol{x+3}}$.

Generally, if P, D, Q, and R are polynomials, such that $P(x) = D(x) \cdot Q(x) + R(x)$, then the ratio of polynomials P and D can be written as

$$\boxed{\frac{\boldsymbol{P(x)}}{\boldsymbol{D(x)}} = \boldsymbol{Q(x)} + \frac{\boldsymbol{R(x)}}{\boldsymbol{D(x)}},}$$

where $Q(x)$ is the quotient polynomial, and $R(x)$ is the remainder from the division of $P(x)$ by the divisor $D(x)$.

Observe: The degree of the remainder must be lower than the degree of the divisor, as otherwise, we could apply the division algorithm one more time.

Example 2 Dividing Polynomials by Polynomials

Divide.

a. $(3x^3 - 2x^2 + 5) \div (x^2 - 3)$ **b.** $\frac{2p^3 + 2p + 3p^2}{5 + 2p}$

Solution

a. When writing the polynomials in the long division format, we use a zero placeholder term in place of the missing linear terms in both the dividend and the divisor. So, we have

$$
\begin{array}{r l}
 & \mathbf{3x \;\; - \;\; 2} \\
\cline{2-2}
x^2 + 0x - 3 & \big)\, 3x^3 \; - \; 2x^2 \; + \; 0x + 5 \\
 & -\,(\, 3x^3 \; + \; 0x^2 \; \overset{\oplus}{-} \; 9x) \\
\cline{2-2}
 & \qquad\qquad -2x^2 \; + \; 9x + 5 \\
 & \qquad\quad -\,(-2x^2 \; - \; 0x \overset{\ominus}{+} 6) \\
\cline{2-2}
 & \qquad\qquad\qquad\quad \mathbf{9x - 1}
\end{array}
$$

Thus, $(3x^3 - 2x^2 + 5) \div (x^2 - 3) = 3x - 2 + \frac{9x - 1}{x^2 - 3}$.

b. To perform this division, we arrange both polynomials in decreasing order of powers, and replace the constant term in the dividend with a zero. So, we have

$$
\begin{array}{r l}
 & \boldsymbol{p^2 - \; p \; + \; \tfrac{7}{2}} \\
\cline{2-2}
2p + 5 & \big)\, 2p^3 + \; 3p^2 + 2p \; + \; 0 \\
 & -\,(\, 2p^3 \overset{\ominus}{+} 5p^2) \\
\cline{2-2}
 & \qquad\quad -2p^2 + 2p \\
 & \qquad -\,(-2p^2 \overset{\oplus}{-} 5p) \\
\cline{2-2}
 & \qquad\qquad\qquad 7p + 0 \\
 & \qquad\qquad -\left(7p \overset{\ominus}{+} \frac{35}{2}\right) \\
\cline{2-2}
 & \qquad\qquad\qquad\quad \boldsymbol{-\frac{35}{2}}
\end{array}
$$

Thus, $\frac{2p^3 + 2p + 3p^2}{5 + 2p} = p^2 - p + \frac{7}{2} + \frac{-\frac{35}{2}}{2p + 5} = \boldsymbol{p^2 - p + \frac{7}{2} - \frac{35}{4p + 10}}$.

Observe in the above answer that $\frac{-\frac{35}{2}}{2p + 5}$ is written in a simpler form, $-\frac{35}{4p + 10}$. This is because $\frac{-\frac{35}{2}}{2p + 5} = -\frac{35}{2} \cdot \frac{1}{2p + 5} = -\frac{35}{4p + 10}$.

Quotient Functions

Similarly as in the case of polynomials, we can define quotients of functions.

Definition 3.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **quotient function**, denoted $\boldsymbol{\frac{f}{g}}$, is defined as

$$\boldsymbol{\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}}.$$

The **domain** of the quotient function is the intersection of the domains of the two functions, D_f and D_g, excluding the x-values for which $g(x) = 0$. So,

$$\boldsymbol{D_{\frac{f}{g}} = D_f \cap D_g \backslash \{x | g(x) = 0\}}$$

Example 3 ► **Dividing Polynomial Functions**

Suppose $P(x) = 2x^2 - x - 6$ and $Q(x) = x - 2$. Find the following:

a. $\left(\frac{P}{Q}\right)(x)$ and $\left(\frac{P}{Q}\right)(-3)$,

b. $\left(\frac{P}{Q}\right)(2)$ and $\left(\frac{P}{Q}\right)(2a)$,

c. domain of $\frac{P}{Q}$.

Notice that this equation holds only for $x \neq 2$.

Solution ► **a.** By *Definition 3.1*, $\left(\frac{P}{Q}\right)(x) = \frac{P(x)}{Q(x)} = \frac{2x^2-x-6}{x-2} = \boldsymbol{2x+3}$

So, $\left(\frac{P}{Q}\right)(-3) = 2(-3) + 3 = \boldsymbol{-3}$. One can verify that the same value is found by evaluating $\frac{P(-3)}{Q(-3)}$.

b. Since the equation $\frac{(2x+3)(x-2)}{x-2} = 2x + 3$ is true only for $x \neq 2$, the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ cannot be used to evaluate $\left(\frac{P}{Q}\right)(2)$. However, by *Definition 3.1*, we have

$$\left(\frac{P}{Q}\right)(2) = \frac{P(2)}{Q(2)} = \frac{2(2)^2 - (2) - 6}{(2) - 2} = \frac{8-2-6}{0} = \frac{0}{0} = \boldsymbol{undefined}$$

$\left(\frac{P}{Q}\right)(2)$ is undefined, so 2 is not in the domain of $\frac{P}{Q}$

To evaluate $\left(\frac{P}{Q}\right)(2a)$, we first notice that if $a \neq 1$, then $2a \neq 2$. So, we can use the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ and evaluate $\left(\frac{P}{Q}\right)(2a) = 2(2a) + 3 = \boldsymbol{4a+3}$ **for all $\boldsymbol{a \neq 1}$**.

c. The domain of any polynomial is the set of all real numbers. So, the domain of $\frac{P}{Q}$ is the set of all real numbers except for the x-values for which the denominator $Q(x) =$

$x-2$ is equal to zero. Since the solution to the equation $x-2=0$ is $x=2$, then the value 2 must be excluded from the set of all real numbers. Therefore, $\boldsymbol{D_{\frac{P}{Q}}} = \mathbb{R} \setminus \{\mathbf{2}\}$.

P.3 Exercises

1. *True or False?* The quotient in a division of a six-degree polynomial by a second-degree polynomial is a third-degree polynomial. Justify your answer.

2. *True or False?* The remainder in a division of a polynomial by a second-degree polynomial is a first-degree polynomial. Justify your answer.

Divide.

3. $\frac{20x^3-15x^2+5x}{5x}$

4. $\frac{27y^4+18y^2-9y}{9y}$

5. $\frac{8x^2y^2-24xy}{4xy}$

6. $\frac{5c^3d+10c^2d^2-15cd^3}{5cd}$

7. $\frac{9a^5-15a^4+12a^3}{-3a^2}$

8. $\frac{20x^3y^2+44x^2y^3-24x^2y}{-4x^2y}$

9. $\frac{64x^3-72x^2+12x}{8x^3}$

10. $\frac{4m^2n^2-21mn^3+18mn^2}{14m^2n^3}$

11. $\frac{12ab^2c+10a^2bc+18abc^2}{6a^2bc}$

Divide.

12. $(x^2+3x-18) \div (x+6)$

13. $(3y^2+17y+10) \div (3y+2)$

14. $(x^2-11x+16) \div (x+8)$

15. $(t^2-7t-9) \div (t-3)$

16. $\frac{6y^3-y^2-10}{3y+4}$

17. $\frac{4a^3+6a^2+14}{2a+4}$

18. $\frac{4x^3+8x^2-11x+3}{4x+1}$

19. $\frac{10z^3-26z^2+17z-13}{5z-3}$

20. $\frac{2x^3+4x^2-x+2}{x^2+2x-1}$

21. $\frac{3x^3-2x^2+5x-4}{x^2-x+3}$

22. $\frac{4k^4+6k^3+3k-1}{2k^2+1}$

23. $\frac{9k^4+12k^3-4k-1}{3k^2-1}$

24. $\frac{2p^3+7p^2+9p+3}{2p+2}$

25. $\frac{5t^2+19t+7}{4t+12}$

26. $\frac{x^4-4x^3+5x^2-3x+2}{x^2+3}$

27. $\frac{p^3-1}{p-1}$

28. $\frac{x^3+1}{x+1}$

29. $\frac{y^4+16}{y+2}$

30. $\frac{x^5-32}{x-2}$

For each pair of polynomials, $P(x)$ and $D(x)$, find such polynomials $Q(x)$ and $R(x)$ that $\boldsymbol{P(x) = Q(x) \cdot D(x) + R(x)}$.

31. $P(x) = 4x^3-4x^2+13x-2$ and $D(x) = 2x-1$

32. $P(x) = 3x^3-2x^2+3x-5$ and $D(x) = 3x-2$

For each pair of functions, $\boldsymbol{f}$ *and* $\boldsymbol{g}$*, find the quotient function* $\left(\frac{f}{g}\right)(x)$ *and state its* ***domain****.*

33. $f(x) = 6x^2 - 4x, \ \ g(x) = 2x$

34. $f(x) = 6x^2 + 9x, \ \ g(x) = -3x$

35. $f(x) = x^2 - 36, \ \ g(x) = x + 6$

36. $f(x) = x^2 - 25, \ \ g(x) = x - 5$

37. $f(x) = 2x^2 - x - 3, \ \ g(x) = 2x - 3$

38. $f(x) = 3x^2 + x - 4, \ \ g(x) = 3x + 4$

39. $f(x) = 8x^3 + 125, \ \ g(x) = 2x + 5$

40. $f(x) = 64x^3 - 27, \ \ g(x) = 4x - 3$

Let $\boldsymbol{P(x) = x^2 - 4}$*,* $\boldsymbol{Q(x) = 2x}$*, and* $\boldsymbol{R(x) = x - 2}$*. Find each of the following. If the value can't be evaluated, say DNE (does not exist).*

41. $\left(\frac{R}{Q}\right)(x)$

42. $\left(\frac{P}{R}\right)(x)$

43. $\left(\frac{R}{P}\right)(x)$

44. $\left(\frac{R}{Q}\right)(2)$

45. $\left(\frac{R}{Q}\right)(0)$

46. $\left(\frac{P}{R}\right)(3)$

47. $\left(\frac{R}{P}\right)(-2)$

48. $\left(\frac{R}{P}\right)(2)$

49. $\left(\frac{P}{R}\right)(a)$, for $a \neq 2$

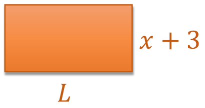
50. $\left(\frac{R}{Q}\right)\left(\frac{3}{2}\right)$

51. $\frac{1}{2}\left(\frac{Q}{R}\right)(x)$

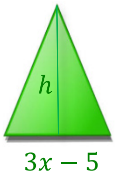
52. $\left(\frac{Q}{R}\right)(a - 1)$

Solve each problem.

53. The area A of a rectangle is $3x^2 + 7x - 6$ and its width W is $x + 3$.
a. Find a polynomial that represents the length L of the rectangle.
b. Find the length of the rectangle if the width is 7 meters.



54. The area A of a triangle is $6x^2 - 13x + 5$. Find the height h of the triangle whose base is $3x - 5$. What is the height of such a triangle if its base is 7 centimeters?



P4 Graphs of Basic Polynomial Functions

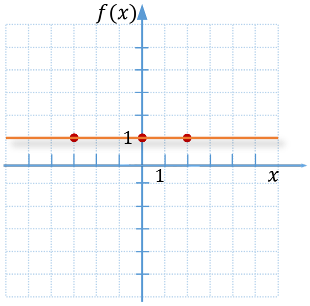
In this section, we will examine graphs of basic polynomial functions, such as constant, linear, quadratic, and cubic functions.

Graphs of Basic Polynomial Functions

Since polynomials are functions, they can be evaluated for different x-values and graphed in a system of coordinates. How do polynomial functions look like? Below, we graph several basic polynomial functions up to the third degree, and observe their shape, domain, and range.

CONSTANT

Let us start with a **constant function**, which is defined by a zero degree polynomial, such as $f(x) = 1$. In this example, for any real x-value, the corresponding y-value is constantly equal to 1. So, the graph of this function is a **horizontal line** with the y-intercept at 1.

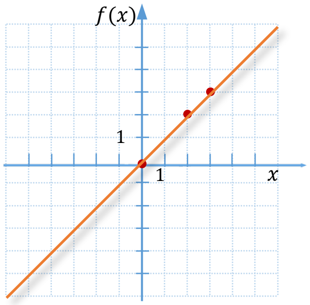


Domain: $\mathbb{R}$
Range: $\{1\}$

Generally, the graph of a **constant function**, $\boldsymbol{f(x) = c}$, is a horizontal line with the y-intercept at c. The domain of this function is $\mathbb{R}$ and the range is $\{c\}$.

LINEAR

The basic first degree polynomial function is the **identity function** given by the formula $\boldsymbol{f(x) = x}$. Since both coordinates of any point satisfying this equation are the same, the graph of the identity function is the diagonal line, as shown below.



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of any first degree polynomial function, $\boldsymbol{f(x) = mx + b}$ with $m \neq 0$, is a slanted line. So, the domain and range of such function is $\mathbb{R}$.

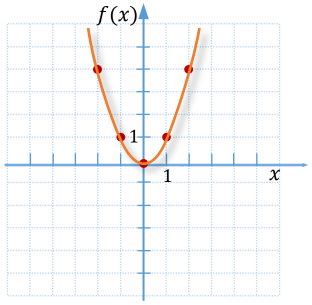
QUADRATIC

The basic second degree polynomial function is the **squaring function** given by the formula $\boldsymbol{f(x)=x^2}$. The shape of the graph of this function is refered to as the **basic parabola**. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x)=x^2$
-2	4
-1	1
0	0
1	1
2	4

vertex

symmetry about the y-axis



Domain: $\mathbb{R}$
Range: $[0,\infty)$

Generally, the graph of any second degree polynomial function, $\boldsymbol{f(x)=ax^2+bx+c}$ with $a\neq 0$, is a **parabola**. The domain of such function is $\mathbb{R}$ and the range depends on how the parabola is directed, with arms up or down.

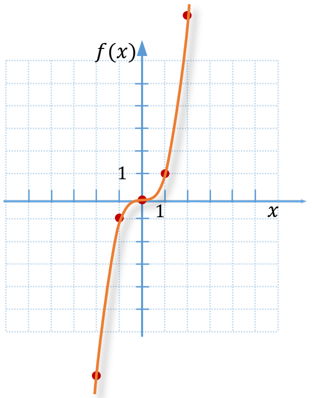
CUBIC

The basic third degree polynomial function is the **cubic function**, given by the formula $\boldsymbol{f(x)=x^3}$. The graph of this function has a shape of a 'snake'. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x)=x^3$
-2	-8
-1	-1
0	0
1	1
2	8

center

symmetry about the origin



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of a third degree polynomial function, $\boldsymbol{f(x)=ax^3+bx^2+cx+d}$ with $a\neq 0$, has a shape of a 'snake' with different size waves in the middle. The domain and range of such function is $\mathbb{R}$.

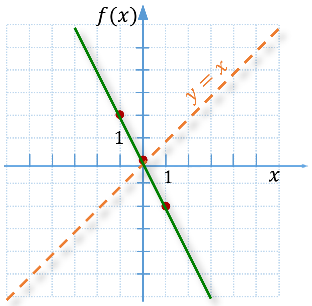
Example 1 ► Graphing Polynomial Functions

Graph each function using a table of values. Give the domain and range of each function by observing its graph. Then, on the same grid, graph the corresponding basic polynomial function. Observe and name the transformation(s) that can be applied to the basic shape in order to obtain the desired function.

a. $f(x) = -2x$ **b.** $f(x) = (x+2)^2$ **c.** $f(x) = x^3 - 2$

Solution ► **a.** The graph of $f(x) = -2x$ is a line passing through the origin and falling from left to right, as shown below in solid green.

x	$f(x) = -2x$
-1	2
0	0
1	-2



Domain of f: ℝ

Range of f: ℝ

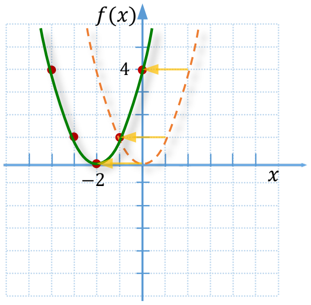
Observe that to obtain the green line, we multiply y-coordinates of the orange line by a factor of -2. Such a transformation is called a **dilation** in the **y-axis** by a factor of **-2**. This dilation can also be achieved by applying a **symmetry in the x-axis** first, and then **stretching** the resulting graph **in the y-axis** by a factor of **2**.

b. The graph of $f(x) = (x+2)^2$ is a parabola with a vertex at $(-2, 0)$, and its arms are directed upwards as shown below in solid green.

x	$f(x) = (x+2)^2$
-4	4
-3	1
-2	0
-1	1
0	4

vertex

symmetry about the line $x = -2$

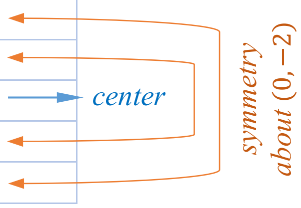


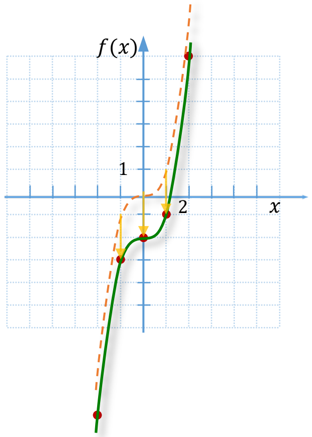
Domain: ℝ
Range: $[0, \infty)$

Observe that to obtain the solid green shape, it is enough to move the graph of the **basic parabola** by two units to the left. This transformation is called a **horizontal translation** by two units to the left. The translation to the left reflects the fact that the vertex of the parabola $f(x) = (x+2)^2$ is located at $x + 2 = 0$, which is equivalent to $x = -2$.

c. The graph of $f(x) = x^3 - 2$ has the shape of a basic cubic function with a center at $(0, -2)$.

x	$f(x) = x^3 - 2$
-2	-10
-1	-3
0	-2
1	-1
2	6





Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Observe that the solid green graph can be obtained by shifting the graph of the **basic cubic function** by two units down. This transformation is called a **vertical translation** by two units down.

P.4 Exercises

1. *True or False?* The graph of $x^2 + 3$ is the same shape as a basic parabola with a vertex at $(3,0)$.

***Graph** each function and state its **domain** and **range**.*

2. $f(x) = -2x + 3$

3. $f(x) = 3x - 4$

4. $f(x) = -x^2 + 4$

5. $f(x) = x^2 - 2$

6. $f(x) = \frac{1}{2}x^2$

7. $f(x) = -2x^2 + 1$

8. $f(x) = (x + 1)^2 - 2$

9. $f(x) = -x^3 + 1$

10. $f(x) = (x - 3)^3$

*Guess the **transformations** needed to apply to the graph of a basic parabola $f(x) = x^2$ to obtain the graph of the given function $g(x)$. Then **graph** both $f(x)$ and $g(x)$ on the same grid and confirm the the original guess.*

11. $g(x) = -x^2$

12. $g(x) = x^2 - 3$

13. $g(x) = x^2 + 2$

14. $g(x) = (x + 2)^2$

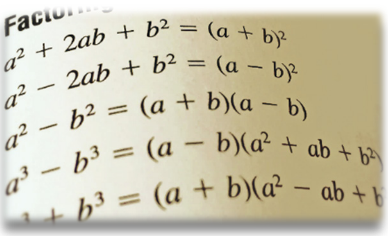
15. $g(x) = (x - 3)^2$

16. $g(x) = (x + 2)^2 - 1$

Attributions

p.**169** Roller Coaster in a Park by Priscilla Du Preez / Unsplash Licence

Factoring



Factoring is the reverse process of multiplication. Factoring polynomials in algebra has similar role as factoring numbers in arithmetic. Any number can be expressed as a product of prime numbers. For example, $6=2\cdot 3$. Similarly, any polynomial can be expressed as a product of **prime** polynomials, which are polynomials that cannot be factored any further. For example, $x^2+5x+6=(x+2)(x+3)$. Just as factoring numbers helps in simplifying or adding fractions, factoring polynomials is very useful in simplifying or adding algebraic fractions. In addition, it helps identify zeros of polynomials, which in turn allows for solving higher degree polynomial equations.

In this chapter, we will examine the most commonly used factoring strategies with particular attention to special factoring. Then, we will apply these strategies in solving polynomial equations.

F1 Greatest Common Factor and Factoring by Grouping

Prime Factors

When working with integers, we are often interested in their factors, particularly prime factors. Likewise, we might be interested in factors of polynomials.

Definition 1.1 ► To **factor** a polynomial means to write the polynomial as a **product** of 'simpler' polynomials. For example,

$$5x+10=5(x+2), \quad \text{or} \quad x^2-9=(x+3)(x-3).$$

In the above definition, 'simpler' means polynomials of **lower degrees** or polynomials with coefficients that **do not contain common factors** other than 1 or -1. If possible, we would like to see the polynomial factors, other than monomials, having **integral coefficients** and a **positive leading term**.

When is a polynomial factorization complete?

In the case of natural numbers, the complete factorization means a factorization into prime numbers, which are numbers divisible only by their own selves and 1. We would expect that similar situation is possible for polynomials. So, which polynomials should we consider as prime?

Observe that a polynomial such as $-4x+12$ can be written as a product in many different ways, for instance

$$-(4x+12), \quad 2(-2x+6), \quad 4(-x+3), \quad -4(x-3), \quad -12\left(\frac{1}{3}x+1\right), \text{ etc.}$$

Since the terms of $4x+12$ and $-2x+6$ still contain common factors different than 1 or -1, these polynomials are not considered to be factored completely, which means that they should not be called prime. The next two factorizations, $4(-x+3)$ and $-4(x-3)$ are both complete, so both polynomials $-x+3$ and $x-3$ should be considered as prime. But what about the last factorization, $-12\left(\frac{1}{3}x+1\right)$? Since the remaining binomial $\frac{1}{3}x+1$ does not have integral coefficients, such a factorization is not always desirable.

Here are some examples of **prime polynomials**:

- any monomials such as $-2x^2$, πr^2, or $\frac{1}{3}xy$;
- any linear polynomials with integral coefficients that have no common factors other than 1 or -1, such as $x-1$ or $2x+5$;
- some quadratic polynomials with integral coefficients that cannot be factored into any lower degree polynomials with integral coefficients, such as x^2+1 or x^2+x+1.

For the purposes of this course, we will assume the following definition of a prime polynomial.

Definition 1.2 ► A polynomial with integral coefficients is called **prime** if one of the following conditions is true

- it is a **monomial**, or
- the only **common factors** of its terms are **1 or -1** and it **cannot be factored into any lower degree polynomials** with integral coefficients.

Definition 1.3 ► A **factorization** of a polynomial with integral coefficients is **complete** if all of its **factors** are **prime**.

Here is an example of a polynomial factored completely:

$$-6x^3-10x^2+4x=-2x(3x-1)(x+2)$$

In the next few sections, we will study several factoring strategies that will be helpful in finding complete factorizations of various polynomials.

Greatest Common Factor

The first strategy of factoring is to factor out the **greatest common factor** (**GCF**).

Definition 1.4 ► The **greatest common factor** (**GCF**) of two or more terms is the largest expression that is a factor of all these terms.

In the above definition, the "largest expression" refers to the expression with the most factors, disregarding their signs.

To find the greatest common factor, we take the product of the least powers of each type of common factor out of all the terms. For example, suppose we wish to find the GCF of the terms

$$6x^2y^3,\ -18x^5y, \text{ and } 24x^4y^2.$$

First, we look for the GCF of 6, 18, and 24, which is 6. Then, we take the lowest power out of x^2, x^5, and x^4, which is x^2. Finally, we take the lowest power out of y^3, y, and y^2, which is y. Therefore,

$$\text{GCF}(6x^2y^3,\ -18x^5y,\ 24x^4y^2)=6x^2y$$

This GCF can be used to factor the polynomial $6x^2y^3-18x^5y+24x^4y^2$ by first seeing it as

$$6x^2y\cdot y^2-6x^2y\cdot 3x^3+6x^2y\cdot 4x^2y,$$

and then, using the **reverse distributing property**, 'pulling' the $6x^2y$ out of the bracket to obtain

$$6x^2y(y^2 - 3x^3 + 4x^2y).$$

Note 1: Notice that since 1 and -1 are factors of any expression, the GCF is defined up to the sign. Usually, we choose the positive GCF, but sometimes it may be convenient to choose the negative GCF. For example, we can claim that

$$\mathrm{GCF}(-2x, -4y) = 2 \quad \text{or} \quad \mathrm{GCF}(-2x, -4y) = -2,$$

depending on what expression we wish to leave after factoring the GCF out:

$$-2x - 4y = \underbrace{2}_{\substack{positive\\ GCF}} \underbrace{(-x - 2y)}_{\substack{negative\\ leading\\ term}} \text{ or } -2x - 4y = \underbrace{-2}_{\substack{negative\\ GCF}} \underbrace{(x + 2y)}_{\substack{positive\\ leading\\ term}}$$

Note 2: If the GCF of the terms of a polynomial is equal to 1, we often say that these terms do not have any common factors. What we actually mean is that the terms do not have a common factor other than 1, as factoring 1 out does not help in breaking the original polynomial into a product of simpler polynomials. See *Definition 1.1*.

Example 1 ► **Finding the Greatest Common Factor**

Find the greatest common factor for the given expressions.

a. $6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2$ **b.** $4\pi(y-x),\ 8\pi(x-y)$

c. $ab^2,\ a^2b,\ b,\ a$ **d.** $3x^{-1}y^{-3},\ x^{-2}y^{-2}z$

Solution ►

a. Since $\mathrm{GCF}(6,3,9) = 3$, the lowest power out of x^4, x^3, and x is x, and the lowest power out of $(x+1)^3$, $(x+1)$, and $(x+1)^2$ is $(x+1)$, then

$$\mathrm{GCF}(6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2) = \mathbf{3x(x+1)}$$

b. Since $y - x$ is opposite to $x - y$, then $y - x$ can be written as $-(x - y)$. So 4, π, and $(x - y)$ is common for both expressions. Thus,

$$\mathrm{GCF}\big(4\pi(y-x),\ 8\pi(x-y)\big) = \boldsymbol{4\pi(x-y)}$$

Note: The greatest common factor is unique up to its sign. Notice that in the above example, we could write $x - y$ as $-(y - x)$ and choose the GCF to be $4\pi(y - x)$.

c. The terms $ab^2,\ a^2b,\ b,$ and a have no common factor other than 1, so

$$\mathrm{GCF}(ab^2,\ a^2b,\ b,\ a) = \mathbf{1}$$

d. The lowest power out of x^{-1} and x^{-2} is x^{-2}, and the lowest power out of y^{-3} and y^{-2} is y^{-3}. Therefore,

$$\mathrm{GCF}(3x^{-1}y^{-3},\ x^{-2}y^{-2}z) = \boldsymbol{x^{-2}y^{-3}}$$

Example 2 ► **Factoring out the Greatest Common Factor**

Factor each expression by taking the greatest common factor out. Simplify the factors, if possible.

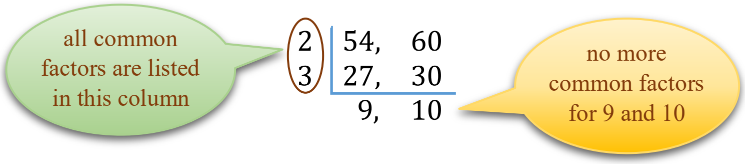
a. $54x^2y^2 + 60xy^3$

b. $ab - a^2b(a-1)$

c. $-x(x-5) + x^2(5-x) - (x-5)^2$

d. $x^{-1} + 2x^{-2} - x^{-3}$

Solution ► **a.** To find the greatest common factor of 54 and 60, we can use the method of dividing by any common factor, as presented below.



So, $\mathrm{GCF}(54, 60) = 2 \cdot 3 = 6$.

Since $\mathrm{GCF}(54x^2y^2,\ 60xy^3) = 6xy^2$, we factor the $6xy^2$ out by dividing each term of the polynomial $54x^2y^2 + 60xy^3$ by $6xy^2$, as below.

$$54x^2y^2 + 60xy^3$$
$$= \boldsymbol{6xy^2(9x + 10y)}$$

$\frac{54x^2y^2}{6xy^2} = 9x$

$\frac{60xy^3}{6xy^2} = 10y$

Note: Since factoring is the reverse process of multiplication, it can be checked by finding the product of the factors. If the product gives us the original polynomial, the factorization is correct.

b. First, notice that the polynomial has two terms, ab and $-a^2b(a-1)$. The greatest common factor for these two terms is ab, so we have

$$ab - a^2b(a-1) = ab(\mathbf{1} - a(a-1))$$

remember to leave **1** for the first term

$$= ab(1 - a^2 + a)$$
$$= ab(-a^2 + a + 1)$$

simplify and arrange in decreasing powers

$$= \boldsymbol{-ab(a^2 - a - 1)}$$

take the "–" out

Note: Both factorizations, $ab(-a^2+a+1)$ and $-ab(a^2-a-1)$ are correct. However, we customarily leave the polynomial in the bracket with a positive leading coefficient.

c. Observe that if we write the middle term $x^2(5-x)$ as $-x^2(x-5)$ by factoring the negative out of the $(5-x)$, then $(5-x)$ is the common factor of all the terms of the equivalent polynomial

$$-x(x-5)-x^2(x-5)-(x-5)^2.$$

Then notice that if we take $-(x-5)$ as the GCF, then the leading term of the remaining polynomial will be positive. So, we factor

$$-x(x-5)+x^2(5-x)-(x-5)^2$$
$$=-x(x-5)-x^2(x-5)-(x-5)^2$$
$$=-(x-5)\big(x+x^2+(x-5)\big)$$

simplify and arrange in decreasing powers

$$=-(x-5)\boldsymbol{(x^2+2x-5)}$$

d. The $\mathrm{GCF}(x^{-1},\ 2x^{-2},\ -x^{-3})=x^{-3}$, as -3 is the lowest exponent of the common factor x. So, we factor out x^{-3} as below.

$$x^{-1}+2x^{-2}-x^{-3}$$
$$=\boldsymbol{x^{-3}}\ \boldsymbol{(x^2+2x-1)}$$

the exponent 2 is found by subtracting -3 from -1

the exponent 1 is found by subtracting -3 from -2

To check if the factorization is correct, we multiply

$$x^{-3}\ (x^2+2x-1)$$

add exponents

$$=x^{-3}x^2+2x^{-3}x-1x^{-3}$$
$$=x^{-1}+2x^{-2}-x^{-3}$$

Since the product gives us the original polynomial, the factorization is correct.

Factoring by Grouping

When referring to a common factor, we have in mind a common factor other than 1.

Consider the polynomial $x^2+x+xy+y$. It consists of four terms that do not have any common factors. Yet, it can still be factored if we group the first two and the last two terms. The first group of two terms contains the common factor of x and the second group of two terms contains the common factor of y. Observe what happens when we factor each group.

$$\underbrace{x^2+x}+\underbrace{xy+y}$$
$$=x(x+1)+y(x+1)$$
$$=(x+1)(x+y)$$

now $(x+1)$ is the common factor of the entire polynomial

This method is called **factoring by grouping**, in particular, two-by-two grouping.

Warning: After factoring each group, make sure to write the "+" or "−" between the terms. Failing to write these signs leads to the false impression that the polynomial is already factored. For example, if in the second line of the above calculations we would fail to write the middle "+", the expression would look like a product $x(x+1)\ y(x+1)$, which is not the case. Also, since the expression $x(x+1)+y(x+1)$ is a sum, not a product, we should not stop at this step. We need to factor out the common bracket $(x+1)$ to leave it as a product.

A two-by-two grouping leads to a factorization only if **the binomials**, after factoring out the common factors in each group, **are the same**. Sometimes a rearrangement of terms is necessary to achieve this goal.

For example, the attempt to factor $x^3-15+5x^2-3x$ by grouping the first and the last two terms,

$$\underbrace{x^3-15}+\underbrace{5x^2-3x}$$

$$=(x^3-15)+x(5x-3)$$

does not lead us to a common binomial that could be factored out.

However, rearranging terms allows us to factor the original polynomial in the following ways:

$$\begin{aligned}&x^3-15+5x^2-3x\\&=\underbrace{x^3+5x^2}+\underbrace{-3x-15}\\&=x^2(x+5)-3(x+5)\\&=(x+5)(x^2-3)\end{aligned}$$

or

$$\begin{aligned}&x^3-15+5x^2-3x\\&=\underbrace{x^3-3x}+\underbrace{5x^2-15}\\&=x(x^2-3)+5(x^2-3)\\&=(x^2-3)(x+5)\end{aligned}$$

Factoring by grouping applies to polynomials with more than three terms. However, not all such polynomials can be factored by grouping. For example, if we attempt to factor x^3+x^2+2x-2 by grouping, we obtain

$$\underbrace{x^3+x^2}+\underbrace{2x-2}$$

$$=x^2(x+1)+2(x-1).$$

Unfortunately, the expressions $x+1$ and $x-1$ are not the same, so there is no common factor to factor out. One can also check that no other rearrangments of terms allows us for factoring out a common binomial. So, this polynomial cannot be factored by grouping.

Example 3 ► Factoring by Grouping

Factor each polynomial by grouping, if possible. Remember to check for the GCF first.

a. $2x^3 - 6x^2 + x - 3$ **b.** $5x - 5y - ax + ay$

c. $2x^2y - 8 - 2x^2 + 8y$ **d.** $x^2 - x + y + 1$

Solution ► **a.** Since there is no common factor for all four terms, we will attempt the two-by-two grouping method.

$$\underbrace{2x^3 - 6x^2} + \underbrace{x - 3}$$

$$= 2x^2(x-3) + 1(x-3)$$

$$= \boldsymbol{(x-3)(2x^2+1)}$$

write the **1** for the second term

b. As before, there is no common factor for all four terms. The two-by-two grouping method works only if the remaining binomials after factoring each group are exactly the same. We can achieve this goal by factoring $-a$, rather than a, out of the last two terms. So,

$$\underbrace{5x - 5y}\ \underbrace{-ax + ay}$$

$$= 5(x-y) - a(x-y)$$

$$= \boldsymbol{(x-y)(5 - a)}$$

reverse signs when ‘pulling’ a “–“ out

c. Notice that 2 is the GCF of all terms, so we factor it out first.

$$2x^2y - 8 - 2x^2 + 8y$$

$$= 2(x^2y - 4 - x^2 + 4y)$$

Then, observe that grouping the first and last two terms of the remaining polynomial does not help, as the two groups do not have any common factors. However, exchanging for example the second with the fourth term will help, as shown below.

$$= 2(\underbrace{x^2y + 4y}\ \underbrace{-x^2 - 4})$$

$$= 2[y(x^2+4) - (x^2+4)]$$

$$= \boldsymbol{2(x^2+4)(y-1)}$$

the square bracket is essential here because of the factor of 2

reverse signs when ‘pulling’ a “–” out

now, there is no need for the square bracket as multiplication is associative

d. The polynomial $x^2 - x + y + 1$ does not have any common factors for all four terms. Also, only the first two terms have a common factor. Unfortunately, when attempting to factor using the two-by-two grouping method, we obtain

$$x^2 - x + y + 1$$

$$= x(x-1) + (y+1),$$

which cannot be factored, as the expressions $x - 1$ and $y + 1$ are different.

One can also check that no other arrangement of terms allows for factoring of this polynomial by grouping. So, this polynomial cannot be factored by grouping.

Example 4 ► **Factoring in Solving Formulas**

Solve $ab = 3a + 5$ for a.

Solution ► First, we move the terms containing the variable a to one side of the equation,

$$ab = 3a + 5$$
$$ab - 3a = 5,$$

and then factor a out

$$a(b - 3) = 5.$$

So, after dividing by $b - 3$, we obtain $\boldsymbol{a = \frac{5}{b-3}}$.

F.1 Exercises

*In problems **1-2**, state whether the given sentence is **true** or **false**.*

1. The polynomial $6x + 8y$ is ***prime***.

2. The **GCF** of the terms of the polynomial $3(x - 2) + x(2 - x)$ is $(x - 2)(2 - x)$.

3. Observe the two factorizations of the polynomial $\frac{1}{2}x - \frac{3}{4}y$ performed by different students:

Student A: $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{2}(x - \frac{3}{2}y)$ *Student B:* $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{4}(2x - 3y)$

Are the two factorizations correct? Which one is preferable, and why?

Find the GCF with a positive coefficient for the given expressions.

4. $8xy,\ 10xz,\ -14xy$

5. $21a^3b^6,\ -35a^7b^5,\ 28a^5b^8$

6. $4x(x - 1),\ 3x^2(x - 1)$

7. $-x(x - 3)^2,\ x^2(x - 3)(x + 2)$

8. $9(a - 5),\ 12(5 - a)$

9. $(x - 2y)(x - 1),\ (2y - x)(x + 1)$

10. $-3x^{-2}y^{-3},\ 6x^{-3}y^{-5}$

11. $x^{-2}(x + 2)^{-2},\ -x^{-4}(x + 2)^{-1}$

Factor out the greatest common factor. Leave the remaining polynomial with a positive leading coeficient. Simplify the factors, if possible.

12. $9x^2 - 81x$

13. $8k^3 + 24k$

14. $6p^3 - 3p^2 - 9p^4$

15. $6a^3 - 36a^4 + 18a^2$

16. $-10r^2s^2 + 15r^4s^2$

17. $5x^2y^3 - 10x^3y^2$

18. $a(x - 2) + b(x - 2)$

19. $a(y^2 - 3) - 2(y^2 - 3)$

20. $(x - 2)(x + 3) + (x - 2)(x + 5)$

21. $(n - 2)(n + 3) + (n - 2)(n - 3)$

22. $y(x-1)+5(1-x)$

23. $(4x-y)-4x(y-4x)$

24. $4(3-x)^2-(3-x)^3+3(3-x)$

25. $2(p-3)+4(p-3)^2-(p-3)^3$

Factor out the least power of each variable.

26. $3x^{-3}+x^{-2}$

27. $k^{-2}+2k^{-4}$

28. $x^{-4}-2x^{-3}+7x^{-2}$

29. $3p^{-5}+p^{-3}-2p^{-2}$

30. $3x^{-3}y-x^{-2}y^2$

31. $-5x^{-2}y^{-3}+2x^{-1}y^{-2}$

Factor by grouping, if possible.

32. $20+5x+12y+3xy$

33. $2a^3+a^2-14a-7$

34. $ac-ad+bc-bd$

35. $2xy-x^2y+6-3x$

36. $3x^2+4xy-6xy-8y^2$

37. $x^3-xy+y^2-x^2y$

38. $3p^2+9pq-pq-3q^2$

39. $3x^2-x^2y-yz^2+3z^2$

40. $2x^3-x^2+4x-2$

41. $x^2y^2+ab-ay^2-bx^2$

42. $xy+ab+by+ax$

43. $x^2y-xy+x+y$

44. $xy-6y+3x-18$

45. x^ny-3x^n+y-5

46. $a^nx^n+2a^n+x^n+2$

Factor completely. Remember to check for the GCF first.

47. $5x-5ax+5abc-5bc$

48. $6rs-14s+6r-14$

49. $x^4(x-1)+x^3(x-1)-x^2+x$

50. $x^3(x-2)^2+2x^2(x-2)-(x+2)(x-2)$

51. One of possible factorizations of the polynomial $4x^2y^5-8xy^3$ is $2xy^3(2xy^2-4)$. Is this a complete factorization?

Use factoring the GCF strategy to ***solve*** *each formula* ***for the indicated variable****.*

52. $A=\boldsymbol{P}+\boldsymbol{P}r$, for $\boldsymbol{P}$

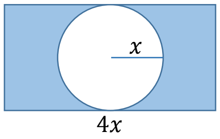
53. $M=\frac{1}{2}\boldsymbol{p}q+\frac{1}{2}\boldsymbol{p}r$, for $\boldsymbol{p}$

54. $2\boldsymbol{t}+c=k\boldsymbol{t}$, for $\boldsymbol{t}$

55. $w\boldsymbol{y}=3\boldsymbol{y}-x$, for $\boldsymbol{y}$

Write the area of each shaded region in factored form.

56.



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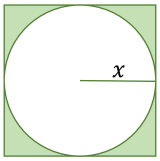

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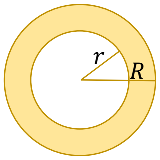

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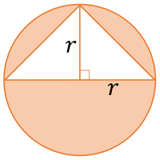
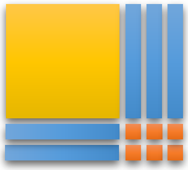


F2 Factoring Trinomials

In this section, we discuss factoring trinomials. We start with factoring quadratic trinomials of the form $x^2 + bx + c$, then quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, and finally trinomials reducible to quadratic by means of substitution.



Factorization of Quadratic Trinomials $x^2 + bx + c$

Factorization of a quadratic trinomial $x^2 + bx + c$ is the reverse process of the FOIL method of multiplying two linear binomials. Observe that

$$(x+p)(x+q) = x^2 + qx + px + pq = x^2 + (p+q)x + pq$$

GUESSING METHOD

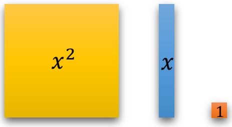
So, to reverse this multiplication, we look for two numbers p and q, such that the product pq equals to the free term c and the sum $p + q$ equals to the middle coefficient b of the trinomial.

$$x^2 + \underbrace{b}_{(p+q)} x + \underbrace{c}_{pq} = (x+p)(x+q)$$

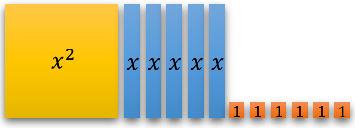
For example, to factor $x^2 + 5x + 6$, we think of two integers that multiply to 6 and add to 5. Such integers are 2 and 3, so $x^2 + 5x + 6 = (x+2)(x+3)$. Since multiplication is commutative, the order of these factors is not important.

VISUALIZATION OF FACTORING

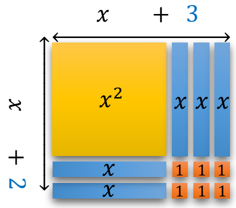
This could also be illustrated geometrically, using algebra tiles.



The area of a square with the side length x is equal to x^2. The area of a rectangle with the dimensions x by 1 is equal to x, and the area of a unit square is equal to 1. So, the trinomial $x^2 + 5x + 6$ can be represented as



To factor this trinomial, we would like to rearrange these tiles to fulfill a rectangle.



The area of such rectangle can be represented as the product of its length, $(x+3)$, and width, $(x+2)$ which becomes the factorization of the original trinomial.

In the trinomial examined above, the signs of the middle and the last terms are both positive. To analyse how different signs of these terms influence the signs used in the factors, observe the next three examples.

To factor $x^2 - 5x + 6$, we look for two integers that multiply to 6 and add to -5. Such integers are -2 and -3, so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

To factor $x^2 + x - 6$, we look for two integers that multiply to -6 and add to 1. Such integers are -2 and 3, so $x^2 + x - 6 = (x - 2)(x + 3)$.

To factor $x^2 - x - 6$, we look for two integers that multiply to -6 and add to -1. Such integers are 2 and -3, so $x^2 - x - 6 = (x + 2)(x - 3)$.

Observation: A **positive constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x + p)(x + q)$ are both of the **same sign** and their **sum** is the middle coefficient b. In addition, if b is positive, both p and q are positive, and if b is negative, both p and q are negative.

A **negative constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x + p)(x + q)$ are of **different signs** and the **difference** of their absolute values is the middle coefficient b. In addition, the integer whose absolute value is larger takes the sign of the middle coefficient b.

These observations are summarized in the following **Table of Signs**.

Assume that $|p| \geq |q|$.

sum $\boldsymbol{b}$	product $\boldsymbol{c}$	$\boldsymbol{p}$	$\boldsymbol{q}$	**comments**
$+$	$+$	$+$	$+$	b is the ***sum*** of p and q
$-$	$+$	$-$	$-$	b is the ***sum*** of p and q
$+$	$-$	$+$	$-$	b is the ***difference*** $\lvert p\rvert - \lvert q\rvert$
$-$	$-$	$-$	$+$	b is the ***difference*** $\lvert q\rvert - \lvert p\rvert$

Example 1 ► Factoring Trinomials with the Leading Coefficient Equal to 1

Factor each trinomial, if possible.

a. $x^2 - 10x + 24$ **b.** $x^2 + 9x - 36$

c. $x^2 - 39xy - 40y^2$ **d.** $x^2 + 7x + 9$

Solution ►

a. To factor the trinomial $x^2 - 10x + 24$, we look for two integers with a product of 24 and a sum of -10. The two integers are fairly easy to guess, -4 and -6. However, if one wishes to follow a more methodical way of finding these numbers, one can list the possible two-number factorizations of 24 and observe the sums of these numbers.

For simplicity, the table doesn't include signs of the integers. The signs are determined according to the **Table of Signs**.

product = 24 (pairs of factors of 24)	**sum = −10** (sum of factors)
1 · 24	25
2 · 12	14
3 · 8	11
4 · 6	10

Bingo!

Since the product is positive and the sum is negative, both integers must be negative. So, we take -4 and -6.

Thus, $x^2 - 10x + 24 = \boldsymbol{(x-4)(x-6)}$. The reader is encouraged to check this factorization by multiplying the obtained binomials.

b. To factor the trinomial $x^2 + 9x - 36$, we look for two integers with a product of -36 and a sum of 9. So, let us list the possible factorizations of 36 into two numbers and observe the differences of these numbers.

product = −36 (pairs of factors of 36)	**sum = 9** (difference of factors)
1 · 36	35
2 · 18	16
3 · 12	9
4 · 9	5
6 · 6	0

This row contains the solution, so there is no need to list any of the subsequent rows.

Since the product is negative and the sum is positive, the integers are of different signs and the one with the larger absolute value assumes the sign of the sum, which is positive. So, we take 12 and -3.

Thus, $x^2 + 9x - 36 = \boldsymbol{(x+12)(x-3)}$. Again, the reader is encouraged to check this factorization by multiplying the obtained binomials.

c. To factor the trinomial $x^2 - 39xy - 40y^2$, we look for two binomials of the form $(x + ?\,y)(x + ?\,y)$ where the question marks are two integers with a product of -40 and a sum of 39. Since the two integers are of different signs and the absolute values of these integers differ by 39, the two integers must be -40 and 1.

Therefore, $x^2 - 39xy - 40y^2 = \boldsymbol{(x-40y)(x+y)}$.

Suggestion: Create a table of pairs of factors only if guessing the two integers with the given product and sum becomes too difficult.

d. When attempting to factor the trinomial $x^2 + 7x + 9$, we look for a pair of integers that would multiply to 9 and add to 7. There are only two possible factorizations of 9: $9 \cdot 1$ and $3 \cdot 3$. However, neither of the sums, $9 + 1$ or $3 + 3$, are equal to 7. So, there is no possible way of factoring $x^2 + 7x + 9$ into two linear binomials with integral coefficients. Therefore, if we admit only integral coefficients, this polynomial is **not factorable**.

Factorization of Quadratic Trinomials $ax^2 + bx + c$ with $a \neq 0$

Before discussing factoring quadratic trinomials with a leading coefficient different than 1, let us observe the multiplication process of two linear binomials with integral coefficients.

$$(mx + p)(nx + q) = mnx^2 + mqx + npx + pq = \underbrace{a}_{mn}\, x^2 + \underbrace{b}_{(mq+np)}\, x + \underbrace{c}_{pq}$$

To reverse this process, notice that this time, we are looking for four integers m, n, p, and q that satisfy the conditions

$$mn = a,\ \ pq = c,\ \ mq + np = b,$$

where a, b, c are the coefficients of the quadratic trinomial that needs to be factored. This produces a lot more possibilities to consider than in the guessing method used in the case of the leading coefficient equal to 1. However, if at least one of the outside coefficients, a or c, are prime, the guessing method still works reasonably well.

GUESSING METHOD

For example, consider $2x^2 + x - 6$. Since the coefficient $a = 2 = mn$ is a prime number, there is only one factorization of a, which is $1 \cdot 2$. So, we can assume that $m = 2$ and $n = 1$. Therefore,

$$2x^2 + x - 6 = (2x \pm |p|)(x \mp |q|)$$

Since the constant term $c = -6 = pq$ is negative, the binomial factors have different signs in the middle. Also, since pq is negative, we search for such p and q that the inside and outside products **differ** by the middle term $b = x$, up to its sign. The only factorizations of 6 are $1 \cdot 6$ and $2 \cdot 3$. So we try

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{1)(x}_{x} \mp 6}_{12x})$$

differs by $11x$ → too much

Observe that these two trials can be disregarded at once as 2 is not a common factor of all the terms of the trinomial, while it is a common factor of the terms of one of the binomials.

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{6)(x}_{6x} \mp 1}_{2x})$$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{2)(x}_{2x} \mp 3}_{6x})$$

differs by $4x$ → still too much

$$2x^2 + x - 6 = (\underbrace{2x \pm \underbrace{3)(x}_{3x} \mp 2}_{4x})$$

differs by x → perfect!

Then, since the difference between the inner and outer products should be positive, the larger product must be positive and the smaller product must be negative. So, we distribute the signs as below.

$$2x^2 + x - 6 = (\underbrace{2x - \underbrace{3)(x}_{-3x} + 2}_{4x})$$

In the end, it is a good idea to multiply the product to check if it results in the original polynomial. We leave this task to the reader.

What if the outside coefficients of the quadratic trinomial are both composite? Checking all possible distributions of coefficients m, n, p, and q might be too cumbersome. Luckily, there is another method of factoring, called **decomposition**.

The decomposition method is based on the reverse FOIL process.
Suppose the polynomial $6x^2 + 19x + 15$ factors into $(mx + p)(nx + q)$. Observe that the FOIL multiplication of these two binomials results in the four term polynomial,

$$mnx^2 + mqx + npx + pq,$$

which after combining the two middle terms gives us the original trinomial. So, reversing these steps would lead us to the factored form of $6x^2 + 19x + 15$.

To reverse the FOIL process, we would like to:

This product is often referred to as the **master product** or the ***ac*-product**.

- Express the middle term, $19x$, as a sum of two terms, mqx and npx, such that the product of their coefficients, $mnpq$, is equal to the product of the outside coefficients $ac = 6 \cdot 15 = 90$.
- Then, factor the four-term polynomial by grouping.

Thus, we are looking for two integers with the product of 90 and the sum of 19. One can check that 9 and 10 satisfy these conditions. Therefore,

DECOMPOSITION METHOD

$$\begin{aligned} &6x^2 + 19x + 15 \\ &= 6x^2 + 9x + 10x + 15 \\ &= 3x(2x+3) + 5(2x+3) \\ &= (2x+3)(3x+5) \end{aligned}$$

Example 2 ► **Factoring Trinomials with the Leading Coefficient Different than 1**

Factor completely each trinomial.

a. $6x^3 + 14x^2 + 4x$ **b.** $-6y^2 - 10 + 19y$
c. $18a^2 - 19ab - 12b^2$ **d.** $2(x+3)^2 + 5(x+3) - 12$

Solution ► **a.** First, we factor out the GCF, which is $2x$. This gives us

$$6x^3 + 14x^2 + 4x = 2x(3x^2 + 7x + 2)$$

The outside coefficients of the remaining trinomial are prime, so we can apply the guessing method to factor it further. The first terms of the possible binomial factors must be $3x$ and x while the last terms must be 2 and 1. Since both signs in the trinomial are positive, the signs used in the binomial factors must be both positive as well. So, we are ready to give it a try:

$$2x(\underbrace{3x + \underbrace{2\,)(x}_{2x} + 1}_{3x}\,) \quad \text{or} \quad 2x(\underbrace{3x + \underbrace{1\,)(x}_{x} + 2}_{6x}\,)$$

The first distribution of coefficients does not work as it would give us $2x + 3x = 5x$ for the middle term. However, the second distribution works as $x + 6x = 7x$, which matches the middle term of the trinomial. So,

$$6x^3 + 14x^2 + 4x = \mathbf{2x(3x+1)(x+2)}$$

b. Notice that the trinomial is not arranged in decreasing order of powers of y. So, first, we rearrange the last two terms to achieve the decreasing order. Also, we factor out the -1, so that the leading term of the remaining trinomial is positive.

$$-6y^2 - 10 + 19y = -6y^2 + 19y - 10 = -(6y^2 - 19y + 10)$$

Then, since the outside coefficients are composite, we will use the decomposition method of factoring. The ac-product equals to 60 and the middle coefficient equals to -19. So, we are looking for two integers that multiply to 60 and add to -19. The integers that satisfy these conditions are -15 and -4. Hence, we factor

$$\begin{aligned} &-(6y^2 - 19y + 10)\\ &= -(6y^2 - 15y - 4y + 10)\\ &= -[3y(2y-5) - 2(2y-5)]\\ &= -\mathbf{(2y-5)(3y-2)} \end{aligned}$$

the **square bracket** is essential because of the negative sign outside

remember to **reverse the sign!**

c. There is no common factor to take out of the polynomial $18a^2 - 19ab - 12b^2$. So, we will attempt to factor it into two binomials of the type $(ma \pm pb)(na \mp qb)$, using the decomposition method. The ac-product equals $-12 \cdot 18 = -2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ and the middle coefficient equals -19. To find the two integers that multiply to the ac-product and add to -19, it is convenient to group the factors of the product

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

in such a way that the products of each group differ by 19. It turns out that grouping all the 2's and all the 3's satisfy this condition, as 8 and 27 differ by 19. Thus, the desired integers are -27 and 8, as the sum of them must be -19. So, we factor

$$\begin{aligned} &18a^2 - 19ab - 12b^2\\ &= 18a^2 - 27ab + 8ab - 12b^2\\ &= 9a(2a-3b) + 4b(2a-3b)\\ &= \mathbf{(2a-3b)(9a+4b)} \end{aligned}$$

d. To factor $2(x+3)^2 + 5(x+3) - 12$, first, we notice that treating the group $(x+3)$ as another variable, say a, simplifies the problem to factoring the quadratic trinomial

$$2a^2 + 5a - 12$$

This can be done by the guessing method. Since

$$2a^2 + 5a - 12 = (2a-3)(a+4),$$

$$-3a$$

$$8a$$

then

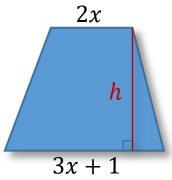
$$\begin{aligned} 2(x+3)^2 + 5(x+3) - 12 &= [2(x+3) - 3][(x+3) + 4]\\ &= (2x + 6 - 3)(x + 3 + 4)\\ &= \mathbf{(2x+3)(x+7)} \end{aligned}$$

Note 1: Polynomials that can be written in the form $\boldsymbol{a}(\quad)^2 + \boldsymbol{b}(\quad) + \boldsymbol{c}$, where $a \neq 0$ and $(\quad)$ represents any nonconstant polynomial expression, are referred to as **quadratic in form**. To factor such polynomials, it is convenient to **replace** the expression in the bracket by **a single variable**, different than the original one. This was illustrated in *Example 2d* by substituting a for $(x+3)$. However, when using this **substitution method**, we must **remember to leave the final answer in terms of the original variable**. So, after factoring, we replace a back with $(x+3)$, and then simplify each factor.

Note 2: Some students may feel comfortable factoring polynomials quadratic in form directly, without using substitution.

Example 3 ► **Application of Factoring in Geometry Problems**

Suppose that the area in square meters of a trapezoid is given by the polynomial $5x^2 - 9x - 2$. If the two bases are $2x$ and $(3x + 1)$ meters long, then what polynomial represents the height of the trapezoid?



Solution ► Using the formula for the area of a trapezoid, we write the equation

$$\frac{1}{2}h(a+b) = 5x^2 - 9x - 2$$

Since $a + b = 2x + (3x + 1) = 5x + 1$, then we have

$$\frac{1}{2}h(5x+1) = 5x^2 - 9x - 2,$$

which after factoring the right-hand side gives us

$$\frac{1}{2}h(5x+1) = (5x+1)(x-2).$$

To find h, it is enough to divide the above equation by the common factor $(5x + 1)$ and then multiply it by 2. So,

$$h = 2(x-2) = \mathbf{2x - 4}.$$

F.2 Exercises

1. If $ax^2 + bx + c$ has no monomial factor, can either of the possible binomial factors have a monomial factor?

2. Is $(2x+5)(2x-4)$ a complete factorization of the polynomial $4x^2 + 2x - 20$?

3. When factoring the polynomial $-2x^2 - 7x + 15$, students obtained the following answers:

$$(-2x+3)(x+5),\ (2x-3)(-x-5),\ \text{or}\ -(2x-3)(x+5)$$

Which of the above factorizations are correct?

4. Is the polynomial $x^2 - x + 2$ factorable or is it prime?

Fill in the missing factor.

5. $x^2 - 4x + 3 = (\qquad)(x-1)$

6. $x^2 + 3x - 10 = (\qquad)(x-2)$

7. $x^2 - xy - 20y^2 = (x+4y)(\qquad)$

8. $x^2 + 12xy + 35y^2 = (x+5y)(\qquad)$

Factor, if possible.

9. $x^2 + 7x + 12$

10. $x^2 - 12x + 35$

11. $y^2 + 2y - 48$

12. $a^2 - a - 42$

13. $x^2 + 2x + 3$

14. $p^2 - 12p - 27$

15. $m^2 - 15m + 56$

16. $y^2 + 3y - 28$

17. $18 - 7n - n^2$

18. $20 + 8p - p^2$

19. $x^2 - 5xy + 6y^2$

20. $p^2 + 9pq + 20q^2$

Factor completely.

21. $-x^2 + 4x + 21$

22. $-y^2 + 14y + 32$

23. $n^4 - 13n^3 - 30n^2$

24. $y^3 - 15y^2 + 54y$

25. $-2x^2 + 28x - 80$

26. $-3x^2 - 33x - 72$

27. $x^4y + 7x^2y - 60y$

28. $24ab^2 + 6a^2b^2 - 3a^3b^2$

29. $40 - 35t^{15} - 5t^{30}$

30. $x^4y^2 + 11x^2y + 30$

31. $64n - 12n^5 - n^9$

32. $24 - 5x^a - x^{2a}$

33. If a polynomial $x^2 + \boxed{}x + 36$ with an unknown coefficient *b* by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of *b*?

Fill in the missing factor.

34. $2x^2 + 7x + 3 = (\qquad)(x+3)$

35. $3x^2 - 10x + 8 = (\qquad)(x-2)$

36. $4x^2 + 8x - 5 = (2x-1)(\qquad)$

37. $6x^2 - x - 15 = (2x+3)(\qquad)$

Factor completely.

38. $2x^2 - 5x - 3$

39. $6y^2 - y - 2$

40. $4m^2 + 17m + 4$

41. $6t^2 - 13t + 6$

42. $10x^2 + 23x - 5$

43. $42n^2 + 5n - 25$

44. $3p^2 - 27p + 24$

45. $-12x^2 - 2x + 30$

46. $6x^2 + 41xy - 7y^2$

47. $18x^2 + 27xy + 10y^2$

48. $8 - 13a + 6a^2$

49. $15 - 14n - 8n^2$

50. $30x^4 + 3x^3 - 9x^2$

51. $10x^3 - 6x^2 + 4x^4$

52. $2y^6 + 7xy^3 + 6x^2$

53. $9x^2y^2 - 4 + 5xy$

54. $16x^2y^3 + 3y - 16xy^2$

55. $4p^4 - 28p^2q + 49q^2$

56. $4(x-1)^2 - 12(x-1) + 9$

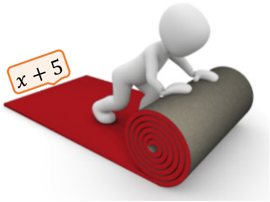
57. $2(a+2)^2 + 11(a+2) + 15$

58. $4x^{2a} - 4x^a - 3$

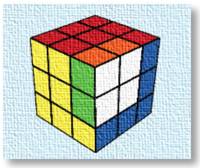
59. If a polynomial $3x^2 + \boxed{}x - 20$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of $\boxed{b}$?

60. The volume of a case of apples is $2x^3 - 3x^2 - 2x$ cubic feet, and the height of the case is $(x-2)$ feet. Find a polynomial representing the area of the bottom of the case?

61. Suppose the width of a rectangular runner carpet is $(x+5)$ feet. If the area of the carpet is $(3x^2 + 17x + 10)$ square feet, find the polynomial that represents the length of the carpet.



F3 Special Factoring and a General Strategy of Factoring



Recall that in *Section P2*, we considered formulas that provide a shortcut for finding special products, such as a product of two **conjugate** binomials,

$$(a+b)(a-b)=\boldsymbol{a^2-b^2},$$

or the **perfect square** of a binomial,

$$(a\pm b)^2=\boldsymbol{a^2\pm 2ab+b^2}.$$

Since factoring reverses the multiplication process, these formulas can be used as shortcuts in factoring binomials of the form $\boldsymbol{a^2-b^2}$ **(difference of squares)**, and trinomials of the form $\boldsymbol{a^2\pm 2ab+b^2}$ **(perfect square)**. In this section, we will also introduce a formula for factoring binomials of the form $\boldsymbol{a^3\pm b^3}$ **(sum or difference of cubes)**. These special product factoring techniques are very useful in simplifying expressions or solving equations, as they allow for more efficient algebraic manipulations.

At the end of this section, we give a summary of all the factoring strategies shown in this chapter.

Difference of Squares

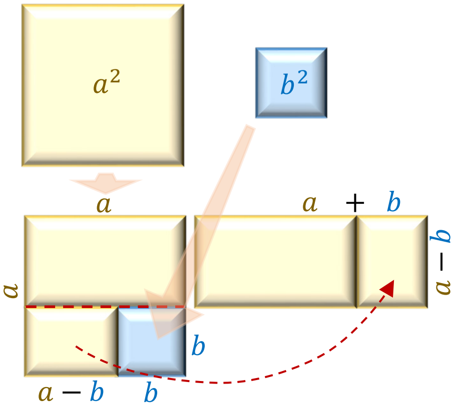


Figure 3.1

Out of the special factoring formulas, the easiest one to use is the difference of squares,

$$\boxed{\boldsymbol{a^2-b^2=(a+b)(a-b)}}$$

Figure 3.1 shows a geometric interpretation of this formula. The area of the yellow square, a^2, diminished by the area of the blue square, b^2, can be rearranged to a rectangle with the length of $(a+b)$ and the width of $(a-b)$.

To factor a difference of squares $\boldsymbol{a^2-b^2}$, first, identify $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared, and then, form two factors, the sum $(\boldsymbol{a+b})$, and the difference $(\boldsymbol{a-b})$, as illustrated in the example below.

Example 1 ► **Factoring Differences of Squares**

Factor each polynomial completely.

a. $25x^2-1$ **b.** $3.6x^4-0.9y^6$

c. x^4-81 **d.** $16-(a-2)^2$

Solution ► **a.** First, we rewrite each term of $25x^2-1$ as a perfect square of an expression.

$$25x^2-1=(\overset{\overset{a}{\downarrow}}{5x})^2-\overset{\overset{b}{\downarrow}}{1}{}^2$$

Then, treating $5x$ as the a and 1 as the b in the difference of squares formula $a^2-b^2=(a+b)(a-b)$, we factor:

$$\begin{array}{c} a^2 - b^2 = (a + b)(a - b) \\ \downarrow \quad \downarrow \quad\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ 25x^2 - 1 = (5x)^2 - 1^2 = (\mathbf{5x + 1})(\mathbf{5x - 1}) \end{array}$$

b. First, we factor out 0.9 to leave the coefficients in a perfect square form. So,

$$3.6x^4 - 0.9y^6 = 0.9(4x^4 - y^6)$$

Then, after writing the terms of $4x^4 - y^6$ as perfect squares of expressions that correspond to a and b in the difference of squares formula $a^2 - b^2 = (a + b)(a - b)$, we factor

$$0.9(4x^4 - y^6) = 0.9[(\overset{\overset{a}{\downarrow}}{2x^2})^2 - (\overset{\overset{b}{\downarrow}}{y^3})^2] = \mathbf{0.9(2x^2 + y^3)(2x^2 - y^3)}$$

c. Similarly as in the previous two examples, $x^4 - 81$ can be factored by following the difference of squares pattern. So,

$$x^4 - 81 = (x^2)^2 - (9)^2 = (x^2 + 9)(x^2 - 9)$$

However, this factorization is not complete yet. Notice that $x^2 - 9$ is also a difference of squares, so the original polynomial can be factored further. Thus,

$$x^4 - 81 = (x^2 + 9)(x^2 - 9) = \mathbf{(x^2 + 9)(x + 3)(x - 3)}$$

Recall that
$a^2 + b^2 \neq (a + b)^2$

Attention: The sum of squares, $x^2 + 9$, cannot be factored using real coefficients.

Generally, except for a common factor, a quadratic binomial of the form $\boldsymbol{a^2 + b^2}$ is **not factorable** over the real numbers.

d. Following the difference of squares formula, we have

$$\begin{aligned} 16 - (a - 2)^2 &= 4^2 - (a - 2)^2 \\ &= [4 + (a - 2)][4 - (a - 2)] \\ &= (4 + a - 2)(4 - a + 2) && \text{work out the inner brackets} \\ &= \mathbf{(2 + a)(6 - a)} && \text{combine like terms} \end{aligned}$$

Remember to use brackets after the negative sign!

Perfect Squares

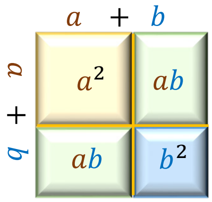


Figure 3.2

Another frequently used special factoring formula is the **perfect square** of a sum or a difference.

$$\boldsymbol{a^2 + 2ab + b^2 = (a + b)^2}$$

or

$$\boldsymbol{a^2 - 2ab + b^2 = (a - b)^2}$$

Figure 3.2 shows the geometric interpretation of the perfect square of a sum. We encourage the reader to come up with a similar interpretation of the perfect square of a difference.

To factor a perfect square trinomial $\boldsymbol{a^2 \pm 2ab + b^2}$, we find $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared. Then, depending on the middle sign, we use $\boldsymbol{a}$ and $\boldsymbol{b}$ to form the perfect square of the sum $(\boldsymbol{a + b})^{\boldsymbol{2}}$, or the perfect square of the difference $(\boldsymbol{a - b})^{\boldsymbol{2}}$.

Example 2 ► Identifying Perfect Square Trinomials

Decide whether the given polynomial is a perfect square.

a. $9x^2 + 6x + 4$

b. $9x^2 + 4y^2 - 12xy$

c. $25p^4 + 40p^2 - 16$

d. $49y^6 + 84xy^3 + 36x^2$

Solution ►

a. Observe that the outside terms of the trinomial $9x^2 + 6x + 4$ are perfect squares, as $9x^2 = (3x)^2$ and $4 = 2^2$. So, the trinomial would be a perfect square if the middle terms would equal $2 \cdot 3x \cdot 2 = 12x$. Since this is not the case, our trinomial is **not a perfect square**.

Attention: Except for a common factor, trinomials of the type $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable** over the real numbers!

b. First, we arrange the trinomial in decreasing order of the powers of x. So, we obtain $9x^2 - 12xy + 4y^2$. Then, since $9x^2 = (3x)^2$, $4y^2 = (2y)^2$, and the middle term (except for the sign) equals $2 \cdot 3x \cdot 2y = 12xy$, we claim that the trinomial **is a perfect square**. Since the middle term is negative, this is the perfect square of a difference. So, the trinomial $9x^2 - 12xy + 4y^2$ can be seen as

$$\begin{array}{ccccccccccccccc} a^2 & - & 2 & & a & & b & + & b^2 & = & (& a & - & b &)^2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\ (3x)^2 & - & 2 & \cdot & 3x & \cdot & 2y & + & (2y)^2 & = & (& 3x & - & 2y &)^2 \end{array}$$

c. Even though the coefficients of the trinomial $25p^4 + 40p^2 - 16$ and the distribution of powers seem to follow the pattern of a perfect square, the last term is negative, which makes it **not a perfect square**.

d. Since $49y^6 = (7y^3)^2$, $36x^2 = (6x)^2$, and the middle term equals $2 \cdot 7y^3 \cdot 6x = 84xy^3$, we claim that the trinomial **is a perfect square**. Since the middle term is positive, this is the perfect square of a sum. So, the trinomial $49y^6 + 84xy^3 + 36x^2$ can be seen as

$$\begin{array}{ccccccccccccccc} a^2 & + & 2 & & a & & b & + & b^2 & = & (& a & + & b &)^2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\ (7y^3)^2 & + & 2 & \cdot & 7y^3 & \cdot & 6x & + & (6x)^2 & = & (& 7y^3 & + & 6x &)^2 \end{array}$$

Example 3 ► Factoring Perfect Square Trinomials

Factor each polynomial completely.

a. $25x^2 + 10x + 1$

b. $a^2 - 12ab + 36b^2$

c. $m^2 - 8m + 16 - 49n^2$

d. $-4y^2 - 144y^8 + 48y^5$

Solution ►

a. The outside terms of the trinomial $25x^2 + 10x + 1$ are perfect squares of $5a$ and 1, and the middle term equals $2 \cdot 5x \cdot 1 = 10x$, so we can follow the perfect square formula. Therefore,

$$25x^2 + 10x + 1 = (\boldsymbol{5x + 1})^{\boldsymbol{2}}$$

b. The outside terms of the trinomial $a^2 - 12ab + 36b^2$ are perfect squares of a and $6b$, and the middle term (disregarding the sign) equals $2 \cdot a \cdot 6b = 12ab$, so we can follow the perfect square formula. Therefore,

$$a^2 - 12ab + 36b^2 = (\boldsymbol{a - 6b})^{\boldsymbol{2}}$$

c. Observe that the first three terms of the polynomial $m^2 - 8m + 16 - 49n^2$ form a perfect square of $m - 4$ and the last term is a perfect square of $7n$. So, we can write

$$m^2 - 8m + 16 - 49n^2 = (m - 4)^2 - (7n)^2$$

This is not in factored form yet!

Notice that this way we have formed a difference of squares. So we can factor it by following the difference of squares formula

$$(m - 4)^2 - (7n)^2 = (\boldsymbol{m - 4 - 7n})(\boldsymbol{m - 4 + 7n})$$

d. As in any factoring problem, first we check the polynomial $-4y^2 - 144y^8 + 48y^5$ for a common factor, which is $4y^2$. To leave the leading term of this polynomial positive, we factor out $-4y^2$. So, we obtain

$$-4y^2 - 144y^8 + 48y^5$$

$$= -4y^2\,(1 + 36y^6 - 12y^3)$$

$$= -4y^2\,(36y^6 - 12y^3 + 1)$$

arrange the polynomial in decreasing powers

$$= \boldsymbol{-4y^2\,(6y^3 - 1)^2}$$

fold to the perfect square form

Sum or Difference of Cubes

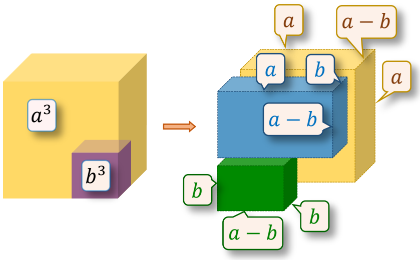
The last special factoring formula to discuss in this section is the **sum or difference of cubes**.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

or

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

The reader is encouraged to confirm these formulas by multiplying the factors in the right-hand side of each equation. In addition, we offer a geometric visualization of one of these formulas, the difference of cubes, as shown in *Figure 3.3*.



$a^3 - b^3 = a^2(a - b) + ab(a - b) + b^2(a - b)$

$= (a - b)(a^2 + ab + b^2)$

Figure 3.3

Hints for memorization of the sum or difference of cubes formulas:

- The binomial factor is a copy of the sum or difference of the terms that were originally cubed.
- The trinomial factor follows the pattern of a perfect square, except that the **middle term is single**, not doubled.
- The signs in the factored form follow the pattern ***S****ame*-***O****pposite*-***P****ositive* (**SOP**).

Example 4 ► Factoring Sums or Differences of Cubes

Factor each polynomial completely.

a. $8x^3+1$

b. $27x^7y-125xy^4$

c. $2n^6-128$

d. $(p-2)^3+q^3$

Solution ►

a. First, we rewrite each term of $8x^3+1$ as a perfect cube of an expression.

$$8x^3+1=(\overset{\overset{a}{\downarrow}}{2x})^3+\overset{\overset{b}{\downarrow}}{1}{}^3$$

Then, treating $2x$ as the a and 1 as the b in the sum of cubes formula $a^3+b^3=(a+b)(a^2-ab+b^2)$, we factor:

$$\overset{\overset{a^3}{\downarrow}}{8x^3}+\overset{\overset{b^3}{\downarrow}}{1}\ \ 8x^3+1=(2x)^3+1^3=(\overset{\overset{a}{\downarrow}}{2x}+\overset{\overset{b}{\downarrow}}{1})(\overset{\overset{a^2}{\downarrow}}{(2x)^2}-\overset{\overset{a}{\downarrow}}{2x}\cdot\overset{\overset{b}{\downarrow}}{1}+\overset{\overset{b^2}{\downarrow}}{1^2})$$

$$=\mathbf{(2x+1)(4x^2-2x+1)}$$

Quadratic trinomials of the form $a^2\pm ab+b^2$ are **not factorable**!

Notice that the trinomial $4x^2-2x+1$ is not factorable anymore.

b. Since the two terms of the polynomial $27x^7y-125xy^4$ contain the common factor xy, we factor it out and obtain

$$27x^7y-125xy^4=xy(27x^6-125y^3)$$

Observe that the remaining polynomial is a difference of cubes, $(3x^2)^3-(5y)^3$. So, we factor,

$$27x^7y-125xy^4=xy[(3x^2)^3-(5y)^3]$$

$$=xy(\overset{\overset{a}{\downarrow}}{3x^2}-\overset{\overset{b}{\downarrow}}{5y})[\overset{\overset{a^2}{\downarrow}}{(3x^2)^2}+\overset{\overset{a}{\downarrow}}{3x^2}\cdot\overset{\overset{b}{\downarrow}}{5y}+\overset{\overset{b^2}{\downarrow}}{(5y)^2}]$$

$$=\boldsymbol{xy(3x^2-5y)(9x^4+15x^2y+25y^2)}$$

c. After factoring out the common factor 2, we obtain

$$2n^6-128=2(n^6-64)$$

Difference of squares or **difference of cubes?**

Notice that $n^6 - 64$ can be seen either as a difference of squares, $(n^3)^2 - 8^2$, or as a difference of cubes, $(n^2)^3 - 4^3$. It turns out that applying the **difference of squares** formula first **leads us to a complete factorization** while starting with the difference of cubes does not work so well here. See the two approaches below.

$$(n^3)^2 - 8^2$$
$$= (n^3 + 8)(n^3 - 8)$$
$$= (n + 2)(n^2 - 2n + 4)(n - 2)(n^2 + 2n + 4)$$

4 prime factors, so the factorization is complete

$$(n^2)^3 - 4^3$$
$$= (n^2 - 4)(n^4 + 4n^2 + 16)$$
$$= (n + 2)(n - 2)(n^4 + 4n^2 + 16)$$

There is no easy way of factoring this trinomial!

Therefore, the original polynomial should be factored as follows:

$$2n^6 - 128 = 2(n^6 - 64) = 2[(n^3)^2 - 8^2] = 2(n^3 + 8)(n^3 - 8)$$
$$= \boldsymbol{2(n + 2)(n^2 - 2n + 4)(n - 2)(n^2 + 2n + 4)}$$

d. To factor $(p - 2)^3 + q^3$, we follow the sum of cubes formula $(a + b)(a^2 - ab + b^2)$ by assuming $a = p - 2$ and $b = q$. So, we have

$$(p - 2)^3 + q^3 = (p - 2 + q)\,[(p - 2)^2 - (p - 2)q + q^2]$$
$$= (p - 2 + q)\,[p^2 - 4p + 4 - pq + 2q + q^2]$$
$$= \boldsymbol{(p + q - 2)\,[p^2 - pq + q^2 - 4p + 2q + 4]}$$

General Strategy of Factoring

Recall that a polynomial with integral coefficients is factored completely if all of its factors are prime over the integers.

How to Factorize Polynomials Completely?

1. Factor out all **common factors**. Leave the remaining polynomial with a positive leading term and integral coefficients, if possible.

2. Check the number of terms. If the polynomial has

- **more than three terms**, try to factor by **grouping**; a four term polynomial may require 2-2, 3-1, or 1-3 types of grouping.
- **three terms**, factor by **guessing**, **decomposition**, or follow the **perfect square** formula, if applicable.
- **two terms**, follow the **difference of squares**, or **sum or difference of cubes** formula, if applicable. Remember that sum of squares, $\boldsymbol{a^2 + b^2}$, is **not factorable** over the real numbers, except for possibly a common factor.

3. Keep in mind the **special factoring formulas**:

Difference of Squares	$\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$
Perfect Square of a Sum	$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$
Perfect Square of a Difference	$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$
Sum of Cubes	$\boldsymbol{a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)}$
Difference of Cubes	$\boldsymbol{a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)}$

4. **Keep factoring** each of the obtained factors until all of them are **prime** over the integers.

Example 5 ► Multiple-step Factorization

Factor each polynomial completely.

a. $80x^5 - 5x$

b. $4a^2 - 4a + 1 - b^2$

c. $(5r+8)^2 - 6(5r+8) + 9$

d. $(p-2q)^3 + (p+2q)^3$

Solution ►

a. First, we factor out the GCF of $80x^5$ and $-5x$, which equals to $5x$. So, we obtain

$$80x^5 - 5x = 5x(16x^4 - 1)$$

repeated difference of squares

Then, we notice that $16x^4 - 1$ can be seen as the difference of squares $(4x^2)^2 - 1^2$. So, we factor further

$$80x^5 - 5x = 5x(4x^2+1)(4x^2-1)$$

The first binomial factor, $4x^2 + 1$, cannot be factored any further using integral coefficients as it is the sum of squares, $(2x)^2 + 1^2$. However, the second binomial factor, $4x^2 - 1$, is still factorable as a difference of squares, $(2x)^2 - 1^2$. Therefore,

$$80x^5 - 5x = \boldsymbol{5x\left(4x^2+1\right)(2x+1)(2x-1)}$$

This is a complete factorization as all the factors are prime over the integers.

3-1 type of grouping

b. The polynomial $4a^2 - 4a + 1 - b^2$ consists of four terms, so we might be able to factor it by grouping. Observe that the 2-2 type of grouping has no chance to succeed, as the first two terms involve only the variable a while the second two terms involve only the variable b. This means that after factoring out the common factor in each group, the remaining binomials would not be the same. So, the 2-2 grouping would not lead us to a factorization. However, the 3-1 type of grouping should help. This is because the first three terms form the perfect square, $(2a-1)^2$, and there is a subtraction before the last term b^2, which is also a perfect square. So, in the end, we can follow the difference of squares formula to complete the factoring process.

$$\begin{aligned} \underbrace{4a^2 - 4a + 1} - \underbrace{b^2} &= (2a-1)^2 - b^2 \\ &= \boldsymbol{(2a-1-b)(2a-1+b)} \end{aligned}$$

factoring by substitution

c. To factor $(5r+8)^2 - 6(5r+8) + 9$, it is convenient to substitute a new variable, say $\boldsymbol{a}$, for the expression $5r+8$. Then,

$$\begin{aligned}(5r+8)^2 - 6(5r+8) + 9 &= \boldsymbol{a}^2 - 6\boldsymbol{a} + 9\\ &= (\boldsymbol{a}-3)^2\\ &= (5r+8-3)^2\\ &= (5r+5)^2\end{aligned}$$

perfect square!

go back to the original variable

Remember to represent the new variable by a **different letter** than the original variable!

Notice that $5r+5$ can still be factored by taking the 5 out. So, for a complete factorization, we factor further

$$(5r+5)^2 = \left(5(r+1)\right)^2 = \boldsymbol{25(r+1)^2}$$

multiple special formulas and simplifying

d. To factor $(p-2q)^3 + (p+2q)^3$, we follow the sum of cubes formula $(a+b)(a^2 - ab + b^2)$ by assuming $a = p - 2q$ and $b = p + 2q$. So, we have

$$(p-2q)^3 + (p+2q)^3$$

$$= (p - \cancel{2q} + p + \cancel{2q})\,[(p-2q)^2 - (p-2q)(p+2q) + (p+2q)^2]$$

$$= 2p\,[p^2 - \cancel{4pq} + 4q^2 - (p^2 - 4q^2) + p^2 + \cancel{4pq} + 4q^2]$$

$$= 2p\,(2p^2 + 8q^2 - p^2 + 4q^2) = \boldsymbol{2p\left(p^2 + 12q^2\right)}$$

F.3 Exercises

Determine whether each polynomial in problems 7-18 is a perfect square, a difference of squares, a sum or difference of cubes, or neither.

1. $0.25x^2 - 0.16y^2$

2. $x^2 - 14x + 49$

3. $9x^4 + 4x^2 + 1$

4. $4x^2 - (x+4)^2$

5. $125x^3 - 64$

6. $y^{12} + 0.008x^3$

7. $-y^4 + 16x^4$

8. $64 + 48x^3 + 9x^6$

9. $25x^6 - 10x^3y^2 + y^4$

10. $-4x^6 - y^6$

11. $-8x^3 + 27y^6$

12. $81x^2 - 16x$

13. Generally, the sum of squares is not factorable. For example, $x^2 + 9$ cannot be factored in integral coefficients. However, some sums of squares can be factored. For example, the binomial $25x^2 + 100$ can be factored. Factor the above example and discuss what makes a sum of two squares factorable.

14. Insert the correct signs into the blanks.

a. $8 + a^3 = (2 \underline{\quad} a)(4 \underline{\quad} 2a \underline{\quad} a^2)$

b. $b^3 - 1 = (b \underline{\quad} 1)(b^2 \underline{\quad} b \underline{\quad} 1)$

Factor each polynomial completely, if possible.

15. $x^2 - y^2$

16. $x^2 + 2xy + y^2$

17. $x^3 - y^3$

18. $16x^2 - 100$

19. $4z^2 - 4z + 1$

20. $x^3 + 27$

21. $4z^2 + 25$

22. $y^2 + 18y + 81$

23. $125 - y^3$

24. $144x^2 - 64y^2$

25. $n^2 + 20nm + 100m^2$

26. $27a^3b^6 + 1$

27. $9a^4 - 25b^6$

28. $25 - 40x + 16x^2$

29. $p^6 - 64q^3$

30. $16x^2z^2 - 100y^2$

31. $4 + 49p^2 + 28p$

32. $x^{12} + 0.008y^3$

33. $r^4 - 9r^2$

34. $9a^2 - 12ab - 4b^2$

35. $\frac{1}{8} - a^3$

36. $0.04x^2 - 0.09y^2$

37. $x^4 + 8x^2 + 1$

38. $-\frac{1}{27} + t^3$

39. $16x^6 - 121x^2y^4$

40. $9 + 60pq + 100p^2q^2$

41. $-a^3b^3 - 125c^6$

42. $36n^{2t} - 1$

43. $9a^8 - 48a^4b + 64b^2$

44. $9x^3 + 8$

45. $(x+1)^2 - 49$

46. $\frac{1}{4}u^2 - uv + v^2$

47. $2t^4 - 128t$

48. $81 - (n+3)^2$

49. $x^{2n} + 6x^n + 9$

50. $8 - (a+2)^3$

51. $16z^4 - 1$

52. $5c^3 + 20c^2 + 20c$

53. $(x+5)^3 - x^3$

54. $a^4 - 81b^4$

55. $0.25z^2 - 0.7z + 0.49$

56. $(x-1)^3 + (x+1)^3$

57. $(x-2y)^2 - (x+y)^2$

58. $0.81p^8 + 9p^4 + 25$

59. $(x+2)^3 - (x-2)^3$

Factor each polynomial completely.

60. $3y^3 - 12x^2y$

61. $2x^2 + 50a^2 - 20ax$

62. $x^3 - xy^2 + x^2y - y^3$

63. $y^2 - 9a^2 + 12y + 36$

64. $64u^6 - 1$

65. $7m^3 + m^6 - 8$

66. $-7n^2 + 2n^3 + 4n - 14$

67. $a^8 - b^8$

68. $y^9 - y$

69. $(x^2-2)^2 - 4(x^2-2) - 21$

70. $8(p-3)^2 - 64(p-3) + 128$

71. $a^2 - b^2 - 6b - 9$

72. $25(2a-b)^2 - 9$

73. $3x^2y^2z + 25xyz^2 + 28z^3$

74. $x^{8a} - y^2$

75. $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$

76. $4x^2y^4 - 9y^4 - 4x^2z^4 + 9z^4$

77. $c^{2w+1} + 2c^{w+1} + c$

F4 Solving Polynomial Equations and Applications of Factoring

Many application problems involve solving polynomial equations. In Chapter L, we studied methods for solving linear, or first-degree, equations. Solving higher degree polynomial equations requires other methods, which often involve factoring. In this chapter, we study solving polynomial equations using the zero-product property, graphical connections between roots of an equation and zeros of the corresponding function, and some application problems involving polynomial equations or formulas that can be solved by factoring.

Zero-Product Property

Recall that to solve a linear equation, for example $2x+1=0$, it is enough to isolate the variable on one side of the equation by applying reverse operations. Unfortunately, this method usually does not work when solving higher degree polynomial equations. For example, we would not be able to solve the equation $x^2-x=0$ through the reverse operation process, because the variable x appears in different powers.

So … how else can we solve it?

In this particular example, it is possible to guess the solutions. They are $x=0$ and $x=1$.

But how can we solve it algebraically?

It turns out that factoring the left-hand side of the equation $x^2-x=0$ helps. Indeed, $x(x-1)=0$ tells us that the product of x and $x-1$ is 0. Since the product of two quantities is 0, at least one of them must be 0. So, either $x=0$ or $x-1=0$, which solves to $x=1$.

The equation discussed above is an example of a second degree polynomial equation, more commonly known as a quadratic equation.

Definition 4.1 ► A **quadratic equation** is a second degree polynomial equation in one variable that can be written in the form,

$$\boldsymbol{ax^2+bx+c=0},$$

where a, b, and c are real numbers and $a \neq 0$. This form is called **standard form**.

One of the methods of solving such equations involves factoring and the zero-product property that is stated below.

Zero-Product Property ► For any real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{ab=0} \text{ if and only if } \boldsymbol{a=0} \text{ or } \boldsymbol{b=0}$$

This means that any product containing a factor of 0 is equal to 0, and conversely, if a product is equal to 0, then at least one of its factors is equal to 0.

Proof ► The implication "if $\boldsymbol{a=0}$ or $\boldsymbol{b=0}$, then $\boldsymbol{ab=0}$" is true by the multiplicative property of zero.

To prove the implication "if $\boldsymbol{ab=0}$, then $\boldsymbol{a=0}$ or $\boldsymbol{b=0}$", let us assume first that $a \neq 0$. (As, if $a=0$, then the implication is already proven.)

Since $a \neq 0$, then $\frac{1}{a}$ exists. Therefore, both sides of $ab = 0$ can be multiplied by $\frac{1}{a}$ and we obtain

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$

$$b = 0,$$

which concludes the proof.

Attention: The zero-product property works only for a product equal to $\mathbf{0}$. For example, the fact that $\boldsymbol{ab} = \mathbf{1}$ does not mean that either a or b equals to 1.

Example 1 ► **Using the Zero-Product Property to Solve Polynomial Equations**

Solve each equation.

a. $(x-3)(2x+5) = 0$

b. $2x(x-5)^2 = 0$

Solution ►

a. Since the product of $x-3$ and $2x+5$ is equal to zero, then by the zero-product property at least one of these expressions must equal to zero. So,

$$x - 3 = 0 \quad \text{or} \quad 2x + 5 = 0$$

which results in

$$x = 3 \quad \text{or} \quad 2x = -5$$

$$x = -\frac{5}{2}$$

Thus, $\left\{-\frac{5}{2}, 3\right\}$ is the solution set of the given equation.

b. Since the product $2x(x-5)^2$ is zero, then either $x = 0$ or $x - 5 = 0$, which solves to $x = 5$. Thus, the solution set is equal to $\{\mathbf{0}, \mathbf{5}\}$.

Note 1: The factor of 2 does not produce any solution, as 2 is never equal to 0.

Note 2: The perfect square $(x-5)^2$ equals to 0 if and only if the base $x - 5$ equals to 0.

Solving Polynomial Equations by Factoring

To solve polynomial equations of second or higher degree by factoring, we

- **arrange** the polynomial **in decreasing order** of powers **on one side** of the equation,
- keep the **other side** of the equation **equal to 0**,
- **factor** the polynomial **completely**,
- use the zero-product property to **form linear equations for each factor**,
- **solve** the linear equations to find the roots (solutions) to the original equation.

Example 2 ► **Solving Quadratic Equations by Factoring**

Solve each equation by factoring.

a. $x^2 + 9 = 6x$

b. $15x^2 - 12x = 0$

c. $(x + 2)(x - 1) = 4(3 - x) - 8$

d. $(x - 3)^2 = 36x^2$

Solution ►

a. To solve $x^2 + 9 = 6x$ by factoring we need one side of this equation equal to 0. So, first, we move the $6x$ term to the left side of the equation,

$$x^2 + 9 - 6x = 0,$$

and arrange the terms in decreasing order of powers of x,

$$x^2 - 6x + 9 = 0.$$

Then, by observing that the resulting trinomial forms a perfect square of $x - 3$, we factor

$$(x - 3)^2 = 0,$$

which is equivalent to

$$x - 3 = 0,$$

and finally

$$x = 3.$$

So, the solution is $x = \mathbf{3}$**.**

b. After factoring the left side of the equation $15x^2 - 12x = 0$,

$$3x(5x - 4) = 0,$$

we use the zero-product property. Since 3 is never zero, the solutions come from the equations

$$x = \mathbf{0} \quad \text{or} \quad 5x - 4 = 0.$$

Solving the second equation for x, we obtain

$$5x = 4,$$

and finally

$$x = \mathbf{\frac{4}{5}}.$$

So, the solution set consists of **0** and $\mathbf{\frac{4}{5}}$.

c. To solve $(x + 2)(x - 1) = 4(3 - x) - 8$ by factoring, first, we work out the brackets and arrange the polynomial in decreasing order of exponents on the left side of the equation. So, we obtain

$$x^2 + x - 2 = 12 - 4x - 8$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

Now, we can read the solutions from each bracket, that is, $x = \boldsymbol{-6}$ and $x = \boldsymbol{1}$.

Observation: In the process of solving a linear equation of the form $ax + b = 0$, first we subtract b and then we divide by a. So the solution, sometimes referred to as the root, is $x = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$. This allows us to read the solution directly from the equation. For example, the solution to $x - 1 = 0$ is $x = 1$ and the solution to $2x - 1 = 0$ is $x = \frac{1}{2}$.

d. To solve $(x-3)^2 = 36x^2$, we bring all the terms to one side and factor the obtained difference of squares, following the formula $a^2 - b^2 = (a+b)(a-b)$. So, we have

$$(x-3)^2 - 36x^2 = 0$$
$$(x-3+6x)(x-3-6x) = 0$$
$$(7x-3)(-5x-3) = 0$$

Then, by the zero-product property,

$$7x - 3 = 0 \quad \text{or} \quad -5x - 3 = 0,$$

which results in

$$x = \frac{\boldsymbol{3}}{\boldsymbol{7}} \quad \text{or} \quad x = -\frac{\boldsymbol{3}}{\boldsymbol{5}}.$$

Example 3 ► Solving Polynomial Equations by Factoring

Solve each equation by factoring.

a. $2x^3 - 2x^2 = 12x$

b. $x^4 + 36 = 13x^2$

Solution ►

a. First, we bring all the terms to one side of the equation and then factor the resulting polynomial.

$$2x^3 - 2x^2 = 12x$$
$$2x^3 - 2x^2 - 12x = 0$$
$$2x(x^2 - x - 6) = 0$$
$$2x(x-3)(x+2) = 0$$

By the zero-product property, the factors $x, (x-3)$ and $(x+2)$, give us the corresponding solutions, $0, 3,$ and -2. So, the solution set of the given equation is $\{\boldsymbol{0}, \boldsymbol{3}, \boldsymbol{-2}\}$.

b. Similarly as in the previous examples, we solve $x^4 + 36 = 13x^2$ by factoring and using the zero-product property. Since

$$x^4 - 13x^2 + 36 = 0$$

$$(x^2-4)(x^2-9)=0$$
$$(x+2)(x-2)(x+3)(x-3)=0,$$

then, the solution set of the original equation is $\{\boldsymbol{-2,2,-3,3}\}$

Observation: n-th degree polynomial equations may have up to n roots (solutions).

Factoring in Applied Problems

Factoring is a useful strategy when solving applied problems. For example, factoring is often used in **solving formulas** for a variable, in **finding roots** of a polynomial function, and generally, in any problem involving **polynomial equations** that can be solved by factoring.

Example 4 ► **Solving Formulas with the Use of Factoring**

Solve each formula for the specified variable.

a. $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$, for $\boldsymbol{h}$ **b.** $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solution ► **a.** To solve $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$ for $\boldsymbol{h}$, we want to keep both terms containing $\boldsymbol{h}$ on the same side of the equation and bring the remaining terms to the other side. Here is an equivalent equation,

$$A-2wl=2\boldsymbol{h}w+2l\boldsymbol{h},$$

which, for convenience, could be written starting with h-terms:

$$2\boldsymbol{h}w+2l\boldsymbol{h}=A-2wl$$

Now, factoring $\boldsymbol{h}$ out causes $\boldsymbol{h}$ to appear in only one place, which is what we need to isolate it. So,

$$(2w+2l)\boldsymbol{h}=A-2wl$$
$$\boldsymbol{h}=\frac{\boldsymbol{A-2wl}}{\boldsymbol{2w+2l}}$$

Notice: In the above formula, there is nothing that can be simplified. Trying to reduce 2 or $2w$ or l would be an error, as there is no essential common factor that can be carried out of the numerator.

b. When solving $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$ for $\boldsymbol{t}$, our goal is to, firstly, keep the variable $\boldsymbol{t}$ in the numerator and secondly, to keep it in a single place. So, we have

$$s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$$
$$s\boldsymbol{t}=2\boldsymbol{t}+3$$

factor t

$$\boldsymbol{st} - 2\boldsymbol{t} = 3$$

$$\boldsymbol{t}(s-2) = 3$$

$$\boldsymbol{t} = \frac{\mathbf{3}}{\boldsymbol{s-2}}.$$

Example 5 ► **Finding Roots of a Polynomial Function**

A toy-rocket is launched vertically with an initial velocity of 40 meters per second. If its height in meters after t seconds is given by the function

$$h(t) = -5t^2 + 40t,$$

in how many seconds will the rocket hit the ground?

Solution ► The rocket hits the ground when its height is 0. So, we need to find the time t for which $h(t) = 0$. Therefore, we solve the equation

$$-5t^2 + 40t = 0$$

for t. From the factored form

$$-5t(t-8) = 0$$

we conclude that the rocket is on the ground at times 0 and 8 seconds. So, the rocket hits the ground **8 seconds** after it was launched.

Example 6 ► **Solving an Application Problem with the Use of Factoring**

The height of a triangle is 1 meter less than twice the length of the base. If the area of the triangle is 14 m^2, how long are the base and the height?

Solution ► Let b and h represent the base and the height of the triangle, correspondingly. The first sentence states that h is 1 less than 2 times b. So, we record

$$h = 2b - 1.$$

Using the formula for area of a triangle, $A = \frac{1}{2}bh$, and the fact that $A = 14$, we obtain

$$14 = \frac{1}{2}b(2b-1).$$

Since this is a one-variable quadratic equation, we will attempt to solve it by factoring, after bringing all the terms to one side of the equation. So, we have

to clear the fraction, multiply each term by 2 before working out the bracket

$$0 = \frac{1}{2}b(2b-1) - 14$$

$$0 = b(2b-1) - 28$$

$$0 = 2b^2 - b - 28$$

$$0 = (2b + 7)(b - 4),$$

which by the zero-product property gives us $b = -\frac{7}{2}$ or $b = 4$. Since b represents the length of the base, it must be positive. So, the base is **4** meters long and the height is $h = 2b - 1 = 2 \cdot 4 - 1 = \mathbf{7}$ meters long.

F.4 Exercises

True or false.

1. If $xy = 0$ then $x = 0$ or $y = 0$.

2. If $ab = 1$ then $a = 1$ or $b = 1$.

3. If $x + y = 0$ then $x = 0$ or $y = 0$.

4. If $a^2 = 0$ then $a = 0$.

5. If $x^2 = 1$ then $x = 1$.

6. Which of the following equations is **not** in proper form for using the zero-product property.

a. $x(x-1) + 3(x-1) = 0$

b. $(x+3)(x-1) = 0$

c. $x(x-1) = 3(x-1)$

d. $(x+3)(x-1) = -3$

Solve each equation.

7. $3(x-1)(x+4) = 0$

8. $2(x+5)(x-7) = 0$

9. $(3x+1)(5x+4) = 0$

10. $(2x-3)(4x-1) = 0$

11. $x^2 + 9x + 18 = 0$

12. $x^2 - 18x + 80 = 0$

13. $2x^2 = 7 - 5x$

14. $3k^2 = 14k - 8$

15. $x^2 + 6x = 0$

16. $6y^2 - 3y = 0$

17. $(4-a)^2 = 0$

18. $(2b+5)^2 = 0$

19. $0 = 4n^2 - 20n + 25$

20. $0 = 16x^2 + 8x + 1$

21. $p^2 - 32 = -4p$

22. $19a + 36 = 6a^2$

23. $x^2 + 3 = 10x - 2x^2$

24. $3x^2 + 9x + 30 = 2x^2 - 2x$

25. $(3x+4)(3x-4) = -10x$

26. $(5x+1)(x+3) = -2(5x+1)$

27. $4(y-3)^2 - 36 = 0$

28. $3(a+5)^2 - 27 = 0$

29. $(x-3)(x+5)=-7$

30. $(x+8)(x-2)=-21$

31. $(2x-1)(x-3)=x^2-x-2$

32. $4x^2+x-10=(x-2)(x+1)$

33. $4(2x+3)^2-(2x+3)-3=0$

34. $5(3x-1)^2+3=-16(3x-1)$

35. $x^3+2x^2-15x=0$

36. $6x^3-13x^2-5x=0$

37. $25x^3=64x$

38. $9x^3=49x$

39. $y^4-26y^2+25=0$

40. $n^4-50n^2+49=0$

41. $x^3-6x^2=-8x$

42. $x^3-2x^2=3x$

43. $a^3+a^2-9a-9=0$

44. $2x^3-x^2-2x+1=0$

45. $5x^3+2x^2-20x-8=0$

46. $2x^3+3x^2-18x-27=0$

47. Discuss the validity of the following solution:

$$\begin{gathered}x^3=9x\\x^2=9\\x=3\end{gathered}$$

How many solutions should we expect? What is the solution set of the original equation? What went wrong in the above procedure?

48. Given that $f(x)=x^2+14x+50$, find all values of x such that $f(x)=5$.

49. Given that $g(x)=2x^2-15x$, find all values of x such that $g(x)=-7$.

50. Given that $f(x)=2x^2+3x$ and $g(x)=-6x+5$, find all values of x such that $f(x)=g(x)$.

51. Given that $g(x)=2x^2+11x-16$ and $h(x)=5+9x-x^2$, find all values of x such that $g(x)=h(x)$.

Solve each equation for the specified variable.

52. $\boldsymbol{P}rt=A-\boldsymbol{P}$, for $\boldsymbol{P}$

53. $3\boldsymbol{s}+2p=5-r\boldsymbol{s}$, for $\boldsymbol{s}$

54. $5a+b\boldsymbol{r}=\boldsymbol{r}-2c$, for $\boldsymbol{r}$

55. $E=\frac{R+\boldsymbol{r}}{\boldsymbol{r}}$, for $\boldsymbol{r}$

56. $z=\frac{x+2\boldsymbol{y}}{\boldsymbol{y}}$, for $\boldsymbol{y}$

57. $c=\frac{-2\boldsymbol{t}+4}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solve each problem.

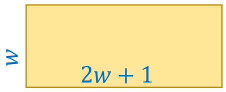
58. Bartek threw down a small rock from the top of a 120 m high observation tower. Suppose the distance travelled by the rock, in meters, is modelled by the function $d(t)=vt+4t^2$, where v is the initial velocity in m/s, and t is the time in seconds. In how many seconds will the rock hit the ground if it was thrown with the initial velocity of 4 m/s?

59. A camera is dropped from a hot-air balloon 320 meters above the ground. Suppose the height of the camera above the ground, in meters, is given by the function $h(t)=320-5t^2$, where t is the time in seconds. How long will it take for the camera to hit the ground?

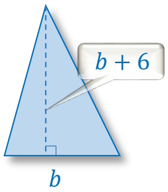
60. The sum of squares of two consecutive numbers is 85. Find the smaller number.

61. The difference between a number and its square is -156. Find the number.

62. The length of a rectangle is 1 centimeter more than twice the width. If the area of this rectangle is 105 cm^2, find its width and length.

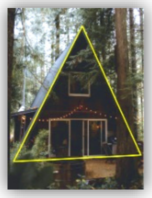


63. A postcard is 7 cm longer than it is wide. The area of this postcard is 144 cm^2. Find its length and width.



64. A triangle with the area of 80 cm^2 is 6 cm taller than the length of its base. Find the dimensions of the triangle.

65. A triangular house is 3 m taller than it is wide. If the cross-sectional area (see the accompanying picture) of the house is 35 m^2, what are the width and the height of this house?



66. Amira designs a rectangular flower bed with a pathway of uniform width around it. She has 42 square meters of ground available for the whole project (including the path). If the flower bed is planned to be 3 meters by 4 meters, how wide would be the pathway around it?

67. Suppose a rectangular flower bed is 5 m longer than it is wide. What are the dimensions of the flower bed if its area is 84 m^2 ?

68. Suppose a picture frame measures 10 cm by 18 cm, and it frames a picture with 48 cm^2 of area. How wide is the frame?

69. When 187 cm^2 picture is framed, its outside dimensions become 15 cm by 21 cm. How wide is the frame?

70. After lengthening each side of a square by 4 cm, the area of the enlarged square turns out to be 225 cm^2. How long is the side of the original square?

71. A square piece of drywall was used to fix a hole in a wall. The sides of the piece of drywall had to be shortened by 2 inches in order to cover the required area of 49 in^2. What were the dimensions of the original piece of drywall?

Attributions

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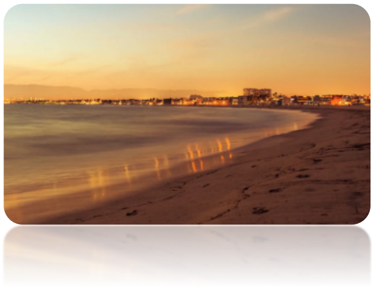
Rational Expressions and Functions



In the previous two chapters we discussed algebraic expressions, equations, and functions related to polynomials. In this chapter, we will examine a broader category of algebraic expressions, *rational expressions*, also referred to as *algebraic fractions*. Similarly as in arithmetic, where a rational number is a quotient of two integers with a denominator that is different than zero, a rational expression is a quotient of two polynomials, also with a denominator that is different than zero.

We start by introducing the related topic of integral exponents, including scientific notation. Then, we discuss operations on algebraic fractions, solving rational equations, and properties and graphs of rational functions with an emphasis on such features as domain, range, and asymptotes. At the end of this chapter, we show examples of applied problems, including work problems, that require solving rational equations.

RT1 Integral Exponents and Scientific Notation

Integral Exponents

In *Section P2*, we discussed the following power rules, using whole numbers for the exponents.

product rule	$\boldsymbol{a^m \cdot a^n = a^{m+n}}$	$\boldsymbol{(ab)^n = a^n b^n}$
quotient rule	$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$	$\boldsymbol{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$
power rule	$\boldsymbol{(a^m)^n = a^{mn}}$	$\boldsymbol{a^0 = 1}$ for $\boldsymbol{a \neq 0}$ $\boldsymbol{0^0}$ is undefined

Observe that these rules gives us the following result.

$$\boldsymbol{a^{-1}} = a^{n-(n+1)} = \frac{a^n}{a^{n+1}} = \frac{\cancel{a^n}}{\cancel{a^n}\cdot a} = \boldsymbol{\frac{1}{a}}$$

quotient rule product rule

Consequantly, $\boldsymbol{a^{-n}} = (a^n)^{-1} = \boldsymbol{\frac{1}{a^n}}$.

power rule

Since $\boxed{\boldsymbol{a^{-n} = \frac{1}{a^n}}}$, then the expression a^n is meaningful for any integral exponent n and a nonzero real base a. So, the above rules of exponents can be extended to include integral exponents.

In practice, to work out the negative sign of an exponent, take the **reciprocal of the base**, or equivalently, "**change the level**" of the power. For example,

$$3^{-2} = \left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9} \quad \text{and} \quad \frac{2^{-3}}{3^{-1}} = \frac{3^1}{2^3} = \frac{3}{8}.$$

Attention! Exponents apply only to the number, letter, or expression in a bracket immediately to the left of the exponent. For example,

$$\boldsymbol{x^{-2} = \frac{1}{x^2}}, \quad \boldsymbol{(-x)^{-2}} = \frac{1}{(-x)^2} = \boldsymbol{\frac{1}{x^2}}, \quad but \quad \boldsymbol{-x^{-2} = -\frac{1}{x^2}}.$$

Example 1 Evaluating Expressions with Integral Exponents

Evaluate each expression.

a. $3^{-1} + 2^{-1}$

b. $\frac{5^{-2}}{2^{-5}}$

c. $\frac{-2^2}{2^{-7}}$

d. $\frac{-2^{-2}}{3\cdot 2^{-3}}$

Solution

a. $3^{-1} + 2^{-1} = \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \boldsymbol{\frac{5}{6}}$

Caution! $3^{-1} + 2^{-1} \neq (3+2)^{-1}$, because the value of $3^{-1} + 2^{-1}$ is $\frac{5}{6}$, as shown in the example, while the value of $(3+2)^{-1}$ is $\frac{1}{5}$.

b. $\frac{5^{-2}}{2^{-5}} = \frac{2^5}{5^2} = \boldsymbol{\frac{32}{25}}$

Note: To work out the negative exponent, move the power from the numerator to the denominator or vice versa.

c. $\frac{-2^2}{2^{-7}} = -2^2 \cdot 2^7 = \boldsymbol{-2^9}$

Attention! The role of a negative sign in front of a base number or in front of an exponent is different. To work out the negative in 2^{-7}, we either take the reciprocal of the base, or we change the position of the power to a different level in the fraction. So, $2^{-7} = \left(\frac{1}{2}\right)^7$ or $2^{-7} = \frac{1}{2^7}$. However, the negative sign in -2^2 just means that the number is negative. So, $-2^2 = -4$. ***Caution!*** $-2^2 \neq \frac{1}{4}$

d. $\frac{-2^{-2}}{3\cdot 2^{-3}} = \frac{-2^{\overset{1}{\cancel{3}}}}{3\cdot \cancel{2^2}} = \boldsymbol{-\frac{2}{3}}$

Note: Exponential expressions can be simplified in many ways. For example, to simplify $\frac{2^{-2}}{2^{-3}}$, we can work out the negative exponents first by moving the powers to a different level, $\frac{2^3}{2^2}$, and then reduce the common factors as shown in the example; or we can employ the quotient rule of powers to obtain

$$\frac{2^{-2}}{2^{-3}} = 2^{-2-(-3)} = 2^{-2+3} = 2^1 = 2.$$

Example 2 ► **Simplifying Exponential Expressions Involving Negative Exponents**

Simplify the given expression. Leave the answer with only positive exponents.

a. $4x^{-5}$ **b.** $(x+y)^{-1}$

c. $x^{-1}+y^{-1}$ **d.** $(-2^3x^{-2})^{-2}$

e. $\dfrac{x^{-4}y^2}{x^2y^{-5}}$ **f.** $\left(\dfrac{-4m^5n^3}{24mn^{-6}}\right)^{-2}$

Solution ►

a. $4x^{-5}=\dfrac{4}{x^5}$

exponent -5 refers to x only!

b. $(x+y)^{-1}=\dfrac{1}{x+y}$

these expressions are NOT equivalent!

c. $x^{-1}+y^{-1}=\dfrac{1}{x}+\dfrac{1}{y}$

d. $$(-2^3x^{-2})^{-2}=\left(\frac{-2^3}{x^2}\right)^{-2}=\left(\frac{x^2}{-2^3}\right)^2=\frac{(x^2)^2}{(-1)^2(2^3)^2}=\frac{x^4}{2^6}$$

work out the negative exponents inside the bracket

work out the negative exponents outside the bracket

a "–" sign can be treated as a factor of -1

power rule – multiply exponents

e. $$\frac{x^{-4}y^2}{x^2y^{-5}}=\frac{y^2y^5}{x^2x^4}=\frac{y^7}{x^6}$$

product rule – add exponents

f. $$\left(\frac{-\cancel{4}m^{\overset{4}{\cancel{5}}}n^3}{\underset{6}{\cancel{24}}\cancel{m}n^{-6}}\right)^{-2}=\left(\frac{-m^4n^3n^6}{6}\right)^{-2}=\left(\frac{(-1)m^4n^9}{6}\right)^{-2}=\left(\frac{6}{(-1)m^4n^9}\right)^2=\frac{36}{m^8n^{18}}$$

$(-1)^2=1$

Scientific Notation

Integral exponents allow us to record numbers with a very large or very small absolute value in a shorter, more convenient form.

For example, the average distance from the Sun to the Saturn is 1,430,000,000 km, which can be recorded as $1.43\cdot 1{,}000{,}000{,}000$ or more concisely as $1.43\cdot 10^9$.

Similarly, the mass of an electron is 0.0000000000000000000000000009 grams, which can be recorded as $9\cdot 0.0000000000000000000000000001$, or more concisely as $9\cdot 10^{-28}$.

This more concise representation of numbers is called **scientific notation** and it is frequently used in sciences and engineering.

Definition 1.1 ► A real number $\boldsymbol{x}$ is written in **scientific notation** iff $\boldsymbol{x=a\cdot 10^n}$,
where the coefficient $\boldsymbol{a}$ is such that $|\boldsymbol{a}|\in[\mathbf{1},\mathbf{10})$, and the exponent $\boldsymbol{n}$ is an integer.

Example 3 ► **Converting Numbers to Scientific Notation**

Convert each number to scientific notation.

a. $520{,}000$ **b.** -0.000102 **c.** $12.5 \cdot 10^3$

Solution ►

an integer has its **decimal** dot **after the last digit**

a. To represent 520,000 in scientific notation, we place a decimal point after the first nonzero digit,

$$5\,.\,\underbrace{2\;0\;0\;0\;0}$$

and then count the number of decimal places needed for the decimal point to move to its original position, which by default was after the last digit. In our example the number of places we need to move the decimal place is 5. This means that 5.2 needs to be multiplied by 10^5 in order to represent the value of 520,000. So, $\mathbf{520,000 = 5.2 \cdot 10^5}$.

Note: To comply with the scientific notation format, we always place the decimal point after the first nonzero digit of the given number. This will guarantee that the coefficient $\boldsymbol{a}$ satisfies the condition $\mathbf{1 \le |a| < 10}$.

b. As in the previous example, to represent -0.000102 in scientific notation, we place a decimal point after the first nonzero digit,

$$-\,0\,.\,\underbrace{0\;0\;0\;1}\,.\,0\;2$$

and then count the number of decimal places needed for the decimal point to move to its original position. In this example, we move the decimal 4 places to the left. So the number 1.02 needs to be divided by 10^4, or equivalently, multiplied by 10^{-4} in order to represent the value of -0.000102. So, $\mathbf{-0.000102 = -1.02 \cdot 10^{-4}}$.

Observation: Notice that moving the decimal to the **right** corresponds to using a **positive** exponent, as in *Example 3a*, while moving the decimal to the **left** corresponds to using a **negative** exponent, as in *Example 3b*.

c. Notice that $12.5 \cdot 10^3$ is not in scientific notation as the coefficient 12.5 is <u>not</u> smaller than 10. To convert $12.5 \cdot 10^3$ to scientific notation, first, convert 12.5 to scientific notation and then multiply the powers of 10. So,

$$12.5 \cdot 10^3 = 1.25 \cdot 10 \cdot 10^3 = \mathbf{1.25 \cdot 10^4}$$

multiply powers by adding exponents

Example 4 ► Converting from Scientific to Decimal Notation

Convert each number to decimal notation.

a. $-6.57 \cdot 10^6$ **b.** $4.6 \cdot 10^{-7}$

Solution ► **a.** The exponent 6 indicates that the decimal point needs to be moved 6 places to the right. So,

$$-6.57 \cdot 10^6 = -6\,.\underbrace{5\,7\,\_\,\_\,\_\,\_}_{\text{fill the empty places by zeros}}. = \mathbf{-6,570,000}$$

b. The exponent -7 indicates that the decimal point needs to be moved 7 places to the left. So,

$$4.6 \cdot 10^{-7} = 0.\underbrace{\_\,\_\,\_\,\_\,\_\,\_\,4}_{\text{fill the empty places by zeros}}.6 = \mathbf{0.00000046}$$

Example 5 ► Using Scientific Notation in Computations

Evaluate. Leave the answer in scientific notation.

a. $6.5 \cdot 10^7 \cdot 3 \cdot 10^5$ **b.** $\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}}$

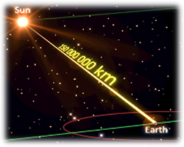
Solution ► **a.** Since the product of the coefficients $6.5 \cdot 3 = 19.5$ is larger than 10, we convert it to scientific notation and then multiply the remaining powers of 10. So,

$$6.5 \cdot 10^7 \cdot 3 \cdot 10^5 = 19.5 \cdot 10^7 \cdot 10^5 = \boxed{1.95 \cdot 10} \cdot 10^{12} = \mathbf{1.95 \cdot 10^{13}}$$

b. Similarly as in the previous example, since the quotient $\frac{3.6}{9} = 0.4$ is smaller than 1, we convert it to scientific notation and then work out the remaining powers of 10. So,

$$\underset{\text{divide powers by subtracting exponents}}{\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}}} = 0.4 \cdot 10^{-11} = \boxed{4 \cdot 10^{-1}} \cdot 10^{-11} = \mathbf{4 \cdot 10^{-12}}$$

Example 6 ► Using Scientific Notation to Solve Problems



Earth is approximately $1.5 \cdot 10^8$ kilometers from the Sun. Estimate the time in days needed for a space probe moving at an average rate of $2.4 \cdot 10^4$ km/h to reach the Sun? *Assume that the probe moves along a straight line.*

Solution ► To find time T needed for the space probe travelling at the rate $R = 2.4 \cdot 10^4$ km/h to reach the Sun that is at the distance $D = 1.5 \cdot 10^8$ km from Earth, first, we solve the motion formula $R \cdot T = D$ for T. Since $T = \frac{D}{R}$, we calculate,

$$T = \frac{1.5 \cdot 10^8}{2.4 \cdot 10^4} = 0.625 \cdot 10^4 = 6.25 \cdot 10^3$$

So, it will take $6.25 \cdot 10^3$ hours $= \frac{6250}{24}$ days $\cong$ **260.4 days** for the space probe to reach the Sun.

RT.1 Exercises

True or false.

1. $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

2. $10^{-4} = 0.00001$

3. $(0.25)^{-1} = 4$

4. $-4^5 = \frac{1}{4^5}$

5. $(-2)^{-10} = 4^{-5}$

6. $2 \cdot 2 \cdot 2^{-1} = \frac{1}{8}$

7. $3x^{-2} = \frac{1}{3x^2}$

8. $-2^{-2} = -\frac{1}{4}$

9. $\frac{5^{10}}{5^{-12}} = 5^{-2}$

10. The number $0.68 \cdot 10^{-5}$ is written in scientific notation.

11. $98.6 \cdot 10^7 = 9.86 \cdot 10^6$

12. *Match each expression in Row I with the equivalent expression(s) in Row II, if possible.*

a. 5^{-2} **b.** -5^{-2} **c.** $(-5)^{-2}$ **d.** $-(-5)^{-2}$ **e.** $-5 \cdot 5^{-2}$

A. 25 **B.** $\frac{1}{25}$ **C.** -25 **D.** $-\frac{1}{5}$ **E.** $-\frac{1}{25}$

Evaluate each expression.

13. $4^{-6} \cdot 4^3$

14. $-9^3 \cdot 9^{-5}$

15. $\frac{2^{-3}}{2^6}$

16. $\frac{2^{-7}}{2^{-5}}$

17. $\frac{-3^{-4}}{5^{-3}}$

18. $-\left(\frac{3}{2}\right)^{-2}$

19. $2^{-2} + 2^{-3}$

20. $(2^{-1} - 3^{-1})^{-1}$

Simplify each expression, if possible. Leave the answer with only ***positive exponents****. Assume that all variables represent nonzero real numbers. Keep large numerical coefficients as powers of prime numbers, if possible.*

21. $(-2x^{-3})(7x^{-8})$

22. $(5x^{-2}y^3)(-4x^{-7}y^{-2})$

23. $(9x^{-4n})(-4x^{-8n})$

24. $(-3y^{-4a})(-5y^{-3a})$

25. $-4x^{-3}$

26. $\frac{x^{-4n}}{x^{6n}}$

27. $\frac{3n^5}{nm^{-2}}$

28. $\frac{14a^{-4}b^{-3}}{-8a^8b^{-5}}$

29. $\frac{-18x^{-3}y^3}{-12x^{-5}y^5}$

30. $(2^{-1}p^{-7}q)^{-4}$

31. $(-3a^2b^{-5})^{-3}$

32. $\left(\frac{5x^{-2}}{y^3}\right)^{-3}$

33. $\left(\frac{2x^3y^{-2}}{3y^{-3}}\right)^{-3}$

34. $\left(\frac{-4x^{-3}}{5x^{-1}y^4}\right)^{-4}$

35. $\left(\frac{125x^2y^{-3}}{5x^4y^{-2}}\right)^{-5}$

36. $\left(\frac{-200x^3y^{-5}}{8x^5y^{-7}}\right)^{-4}$

37. $[(-2x^{-4}y^{-2})^{-3}]^{-2}$

38. $\frac{12a^{-2}(a^{-3})^{-2}}{6a^7}$

39. $\frac{(-2k)^2m^{-5}}{(km)^{-3}}$

40. $\left(\frac{2p}{q^2}\right)^3\left(\frac{3p^4}{q^{-4}}\right)^{-1}$

41. $\left(\frac{-3x^4y^6}{15x^{-6}y^7}\right)^{-3}$

42. $\left(\frac{-4a^3b^2}{12a^6b^{-5}}\right)^{-3}$

43. $\left(\frac{-9^{-2}x^{-4}y}{3^{-3}x^{-3}y^2}\right)^8$

44. $(4^{-x})^{2y}$

45. $(5^a)^{-a}$

46. x^ax^{-a}

47. $\frac{9n^{2-x}}{3n^{2-2x}}$

48. $\frac{12x^{a+1}}{-4x^{2-a}}$

49. $\left(x^{b-1}\right)^3\left(x^{b-4}\right)^{-2}$

50. $\frac{25x^{a+b}y^{b-a}}{-5x^{a-b}y^{b+a}}$

Convert each number to scientific notation.

51. 26,000,000,000

52. −0.000132

53. 0.0000000105

54. 705.6

Convert each number to decimal notation.

55. $6.7 \cdot 10^8$

56. $5.072 \cdot 10^{-5}$

57. $2 \cdot 10^{12}$

58. $9.05 \cdot 10^{-9}$

59. One megabyte of computer memory equals 2^{20} bytes. Using decimal notation, write the number of bytes in 1 megabyte. Then, using scientific notation, approximate this number by rounding the scientific notation coefficient to two decimals places.

Evaluate. State your answer in scientific notation.

60. $(6.5 \cdot 10^3)(5.2 \cdot 10^{-8})$

61. $(2.34 \cdot 10^{-5})(5.7 \cdot 10^{-6})$

62. $(3.26 \cdot 10^{-6})(5.2 \cdot 10^{-8})$

63. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

64. $\frac{7.5 \cdot 10^9}{2.5 \cdot 10^4}$

65. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

66. $\frac{0.05 \cdot 16000}{0.0004}$

67. $\frac{0.003 \cdot 40{,}000}{0.00012 \cdot 600}$

Solve each problem. State your answer in ***scientific notation***.

68. A *light-year* is an astronomical unit measuring the distance that light travels in one year. If light travels approximately $3 \cdot 10^5$ kilometers per second, how long is a light-year in kilometers?

69. In 2018, the national debt in Canada was about $6.7 \cdot 10^{11}$ dollars. If the Canadian population in 2018 was approximately $3.7 \cdot 10^7$, what was the share of this debt per person?

70. One of the brightest stars in the night sky, Vega, is about $2.365 \cdot 10^{14}$ kilometers from Earth. If one light-year is approximately $9.46 \cdot 10^{12}$ kilometers, how many light-years is it from Earth to Vega?

71. The Columbia River discharges its water to the Pacific Ocean at approximately 265,000 ft^3/sec. What is the supply of water that comes from the Columbia River in one minute? in one day? *State the answer in scientific notation.*

72. Assuming the current trends continue, the population P of Canada, in millions, can be modelled by the equation $P = 34(1.011)^x$, where x is the number of years passed after the year 2010. According to this model, what is the predicted Canadian population for the years 2025 and 2030?

73. The mass of the Moon is $7.348 \cdot 10^{22}$ kg while the mass of Earth is $5.976 \cdot 10^{24}$ kg. How many times heavier is Earth than the Moon?

74. Most calculators cannot handle operations on numbers outside of the interval $(10^{-100}, 10^{100})$. How can we compute $(5 \cdot 10^{120})^3$ without the use of a calculator?

RT2 Rational Expressions and Functions; Multiplication and Division of Rational Expressions

In arithmetic, a rational number is a quotient of two integers with denominator different than zero. In algebra, a *rational expression*, offten called an *algebraic fraction*, is a quotient of two polynomials, also with denominator different than zero. In this section, we will examine rational expressions and functions, paying attention to their domains. Then, we will simplify, multiply, and divide rational expressions, employing the factoring skills developed in *Chapter P*.

Rational Expressions and Functions

Here are some examples of rational expressions:

$$-\frac{x^2}{2xy}, \quad x^{-1}, \quad \frac{x^2-4}{x-2}, \quad \frac{8x^2+6x-5}{4x^2+5x}, \quad \frac{x-3}{3-x}, \quad x^2-25, \quad 3x(x-1)^{-2}$$

Definition 2.1 ► A **rational expression** (**algebraic fraction**) is a quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$. Since division by zero is not permitted, a rational expression is defined only for the x-values that make the denominator of the expression different than zero. The set of such x-values is referred to as the **domain** of the expression.

Note 1: Negative exponents indicate hidden fractions and therefore represent rational expressions. For instance, $x^{-1} = \frac{1}{x}$.

Note 2: A single polynomial can also be seen as a rational expression because it can be considered as a fraction with a denominator of 1.
For instance, $x^2 - 25 = \frac{x^2-25}{1}$.

Definition 2.2 ► A **rational function** is a function defined by a rational expression,

$$f(x) = \frac{P(x)}{Q(x)}.$$

The **domain** of such function consists of all real numbers except for the x-values that make the denominator $Q(x)$ equal to 0. So, the domain $D = \mathbb{R} \setminus \{x | Q(x) = 0\}$

For example, the domain of the rational function $f(x) = \frac{1}{x-3}$ is the set of all real numbers except for 3 because 3 would make the denominator equal to 0. So, we write $D = \mathbb{R} \setminus \{3\}$. Sometimes, to make it clear that we refer to function f, we might denote the domain of f by D_f, rather than just D.

Figure 1 shows a graph of the function $f(x) = \frac{1}{x-3}$. Notice that the graph does not cross the dashed vertical line whose equation is $x = 3$. This is because $f(3)$ is not defined. A closer look at the graphs of rational functions will be given in *Section RT5*.

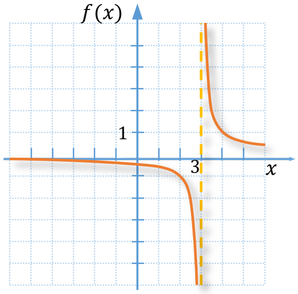


Figure 1

Example 1 ► Evaluating Rational Expressions or Functions

Evaluate the given expression or function for $x = -1, 0, 1$. If the value cannot be calculated, write ***undefined***.

a. $3x(x-1)^{-2}$

b. $f(x) = \frac{x}{x^2+x}$

Solution ►

a. If $x = -1$, then $3x(x-1)^{-2} = 3(-1)(-1-1)^{-2} = -3(-2)^{-2} = \frac{-3}{(-2)^2} = -\mathbf{\frac{3}{4}}$.

If $x = 0$, then $3x(x-1)^{-2} = 3(0)(0-1)^{-2} = \mathbf{0}$.

If $x = 1$, then $3x(x-1)^{-2} = 3(1)(1-1)^{-2} = 3 \cdot 0^{-2} = \boldsymbol{undefined}$, as division by zero is not permitted.

Note: Since the expression $3x(x-1)^{-2}$ cannot be evaluated at $x = 1$, the number 1 does not belong to its domain.

b. $f(-1) = \frac{-1}{(-1)^2+(-1)} = \frac{-1}{1-1} = \boldsymbol{undefined}$.

$f(0) = \frac{0}{(0)^2+(0)} = \frac{0}{0} = \boldsymbol{undefined}$.

$f(1) = \frac{1}{(1)^2+(1)} = \mathbf{\frac{1}{2}}$.

Observation: Function $f(x) = \frac{x}{x^2+x}$ is undefined at $x = 0$ and $x = -1$. This is because the denominator $x^2 + x = x(x+1)$ becomes zero when the x-value is 0 or -1.

Example 2 ► Finding Domains of Rational Expressions or Functions

Find the domain of each expression or function.

a. $\frac{4}{2x+5}$

b. $\frac{x-2}{x^2-2x}$

c. $f(x) = \frac{x^2-4}{x^2+4}$

d. $g(x) = \frac{2x-1}{x^2-4x-5}$

Solution ►

a. The domain of $\frac{4}{2x+5}$ consists of all real numbers except for those that would make the denominator $2x + 5$ equal to zero. To find these numbers, we solve the equation

$$\begin{aligned} 2x + 5 &= 0 \\ 2x &= -5 \\ x &= -\mathbf{\frac{5}{2}} \end{aligned}$$

So, the domain of $\frac{4}{2x+5}$ is the set of all real numbers except for $-\frac{5}{2}$. This can be recorded in set notation as $\mathbb{R} \setminus \left\{-\frac{5}{2}\right\}$, or in set-builder notation as $\left\{\boldsymbol{x} \middle| \boldsymbol{x} \neq -\frac{5}{2}\right\}$, or in interval notation as $\left(-\infty, -\frac{5}{2}\right) \cup \left(-\frac{5}{2}, \infty\right)$.

DOMAIN

b. To find the domain of $\frac{x-2}{x^2-2x}$, we want to exclude from the set of real numbers all the x-values that would make the denominator $x^2 - 2x$ equal to zero. After solving the equation

$$x^2 - 2x = 0$$

via factoring

$$x(x-2) = 0$$

and zero-product property

$$x = \mathbf{0} \quad \text{or} \quad x = \mathbf{2},$$

we conclude that the domain is the set of all real numbers except for 0 and 2, which can be recorded as $\mathbb{R} \setminus \{\mathbf{0}, \mathbf{2}\}$. This is because the x-values of 0 or 2 make the denominator of the expression $\frac{x-2}{x^2-2x}$ equal to zero.

c. To find the domain of the function $f(x) = \frac{x^2-4}{x^2+4}$, we first look for all the x-values that make the denominator $x^2 + 4$ equal to zero. However, $x^2 + 4$, as a sum of squares, is never equal to 0. So, the domain of function f is the set of all real numbers $\mathbb{R}$.

d. To find the domain of the function $g(x) = \frac{2x-1}{x^2-4x-5}$, we first solve the equation $x^2 - 4x - 5 = 0$ to find which x-values make the denominator equal to zero. After factoring, we obtain

$$(x-5)(x+1) = 0$$

which results in $\qquad x = 5 \text{ and } x = -1$

Thus, the domain of g equals to $\boldsymbol{D_g} = \mathbb{R} \setminus \{-\mathbf{1}, \mathbf{5}\}$.

Equivalent Expressions

Definition 2.3 ► Two expressions are **equivalent** in the **common domain** iff (if and only if) they produce the same values for every input from the domain.

Consider the expression $\frac{x-2}{x^2-2x}$ from *Example 2b.* Notice that this expression can be simplified to $\frac{\cancel{x-2}}{x\cancel{(x-2)}} = \frac{1}{x}$ by reducing common factors in the numerator and the denominator. However, the domain of the simplified fraction, $\frac{1}{x}$, is the set $\mathbb{R} \setminus \{0\}$, which is different than the domain of the original fraction, $\mathbb{R} \setminus \{0,2\}$. Notice that for $x = 2$, the expression $\frac{x-2}{x^2-2x}$ is undefined while the value of the expression $\frac{1}{x}$ is $\frac{1}{2}$. So, the two expressions are not equivalent in the set of real numbers. However, if the domain of $\frac{1}{x}$ is

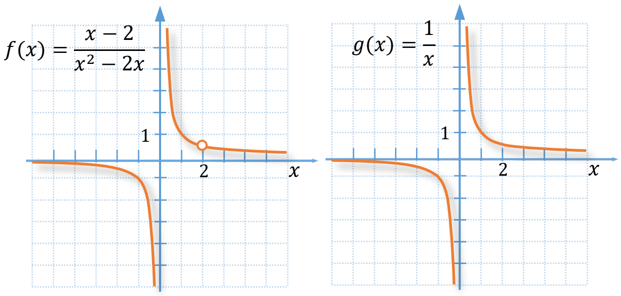


Figure 2

restricted to the set $\mathbb{R} \setminus \{0,2\}$, then the two expressions produce the same values and as such, they are equivalent. We say that the two expressions are **equivalent** in the **common domain**.

The above situation can be illustrated by graphing the related functions, $f(x) = \frac{x-2}{x^2-2x}$ and $g(x) = \frac{1}{x}$, as in *Figure 2*. The graphs of both functions are exactly the same except for the hole in the graph of f at the point $\left(2, \frac{1}{2}\right)$.

So, from now on, when writing statements like $\frac{x-2}{x^2-2x} = \frac{1}{x}$, we keep in mind that they apply only to real numbers which make both denominators different than zero. Thus, by saying in short that two **expressions are equivalent**, we really mean that they are **equivalent in the common domain**.

Note: The **domain** of $f(x) = \frac{x-2}{x^2-2x} = \frac{\cancel{x-2}}{x(\cancel{x-2})} = \frac{1}{x}$ is still $\mathbb{R} \setminus \{\mathbf{0}, \mathbf{2}\}$, even though the $(x-2)$ term was simplified.

The process of simplifying expressions involves creating equivalent expressions. In the case of rational expressions, equivalent expressions can be obtained by multiplying or dividing the numerator and denominator of the expression by the same nonzero polynomial. For example,

$$\frac{\boldsymbol{-x-3}}{\boldsymbol{-5x}} = \frac{(-x-3)\cdot(-1)}{(-5x)\cdot(-1)} = \frac{\boldsymbol{x+3}}{\boldsymbol{5x}}$$

$$\frac{\boldsymbol{x-3}}{\boldsymbol{3-x}} = \frac{\cancel{(x-3)}}{-1\cancel{(x-3)}} = \frac{1}{-1} = \boldsymbol{-1}$$

To simplify a rational expression:

- **Factor** the numerator and denominator **completely**.
- **Eliminate all common factors** by following the property of multiplicative identity. *Do not eliminate common terms - they must be factors!*

Example 3 ► **Simplifying Rational Expressions**

Simplify each expression.

a. $\frac{7a^2b^2}{21a^3b-14a^3b^2}$ **b.** $\frac{x^2-9}{x^2-6x+9}$ **c.** $\frac{20x-15x^2}{15x^3-5x^2-20x}$

Solution ► **a.** First, we factor the denominator and then reduce the common factors. So,

$$\frac{7a^2b^2}{21a^3b-14a^3b^2} = \frac{\cancel{7}\cancel{a^2}b^{\overset{1}{\cancel{2}}}}{\cancel{7}a^{\overset{1}{\cancel{3}}}\cancel{b}(3-2b)} = \frac{\boldsymbol{b}}{\boldsymbol{a(3-2b)}}$$

b. As before, we factor and then reduce. So,

$$\frac{x^2-9}{x^2-6x+9}=\frac{\cancel{(x-3)}(x+3)}{(x-3)^{\cancel{2}\,1}}=\frac{\boldsymbol{x+3}}{\boldsymbol{x-3}}$$

Neither x nor 3 can be reduced, as they are NOT factors !

c. Factoring and reducing the numerator and denominator gives us

$$\frac{20x-15x^2}{15x^3-5x^2-20x}=\frac{\cancel{5x}(4-3x)}{\cancel{5x}(3x^2-x-4)}=\frac{4-3x}{(3x-4)(x+1)}$$

Since $\frac{4-3x}{3x-4}=\frac{-(3x-4)}{3x-4}=-1$, the above expression can be reduced further to

$$\frac{\cancel{4-3x}^{\,-1}}{\cancel{(3x-4)}(x+1)}=\frac{\boldsymbol{-1}}{\boldsymbol{x+1}}$$

Notice: An opposite expression in the numerator and denominator can be reduced to -1. For example, since $a-b$ is opposite to $b-a$, then

$$\frac{\boldsymbol{a-b}}{\boldsymbol{b-a}}=\boldsymbol{-1}, \text{ as long as } a\neq b.$$

Caution: Note that $a-b$ is NOT opposite to $a+b$!

Multiplication and Division of Rational Expressions

Recall that to multiply common fractions, we multiply their numerators and denominators, and then simplify the resulting fraction. Multiplication of algebraic fractions is performed in a similar way.

To multiply rational expressions:

- **factor** each numerator and denominator **completely**,
- **reduce all common factors** in any of the numerators and denominators,
- **multiply** the remaining expressions by writing the product of their numerators over the product of their denominators.

For instance,

$$\frac{3x}{x^2+5x}\cdot\frac{3x+15}{6x}=\frac{3\cancel{x}}{\cancel{x}\cancel{(x+5)}}\cdot\frac{\cancel{3}\cancel{(x+5)}}{\underset{2}{\cancel{6x}}}=\frac{3}{2x}$$

Example 4 ► **Multiplying Algebraic Fractions**

Multiply and simplify. Assume nonzero denominators.

a. $\frac{2x^2y^3}{3xy^2}\cdot\frac{(2x^3y)^2}{2(xy)^3}$

b. $\frac{x^3-y^3}{x+y}\cdot\frac{3x+3y}{x^2-y^2}$

Solution ► **a.** To multiply the two algebraic fractions, we use appropriate rules of powers to simplify each fraction, and then reduce all the remaining common factors. So,

$$\frac{2x^{\overset{1}{\cancel{2}}}y^{\overset{1}{\cancel{3}}}}{3\cancel{x}\cancel{y^2}}\cdot\frac{(2x^3y)^2}{2(xy)^3}=\frac{2xy}{3}\cdot\frac{\overset{2}{\cancel{4}}x^{\overset{3}{\cancel{6}}}\cancel{y^2}}{\cancel{2}\cancel{x^3}y^{\cancel{3}\,1}}=\frac{2x\cancel{y}\cdot 2x^3}{3\cdot\cancel{y}}=\boldsymbol{\frac{4x^4}{3}}=\boldsymbol{\frac{4}{3}x^4}$$

equivalent answers

b. After factoring and simplifying, we have

$$\frac{x^3-y^3}{x+y}\cdot\frac{3x+3y}{x^2-y^2}=\frac{\cancel{(x-y)}(x^2+xy+y^2)}{x+y}\cdot\frac{3\cancel{(x+y)}}{\cancel{(x-y)}\cancel{(x+y)}}=\boldsymbol{\frac{3(x^2+xy+y^2)}{x+y}}$$

Recall: $x^3-y^3=(x-y)(x^2+xy+y^2)$

$x^2-y^2=(x+y)(x-y)$

To divide rational expressions, **multiply** the first, the *dividend*, **by the reciprocal** of the second, the *divisor*.

For instance,

$$\frac{5x-10}{3x}\div\frac{3x-6}{2x^2}=\frac{5x-10}{3x}\cdot\frac{2x^2}{3x-6}=\frac{5\cancel{(x-2)}}{3\cancel{x}}\cdot\frac{2x^{\overset{1}{\cancel{2}}}}{3\cancel{(x-2)}}=\frac{10x}{9}$$

multiply by the **reciprocal**

follow multiplication rules

Example 5 ► **Dividing Algebraic Fractions**

Perform operations and simplify. Assume nonzero denominators.

a. $\frac{2x^2+2x}{x-1}\div(x+1)$ **b.** $\frac{x^2-25}{x^2+5x+4}\div\frac{x^2-10x+25}{2x^2+8x}\cdot\frac{x^2+x}{4x^2}$

Solution ► **a.** To divide by $(x+1)$ we multiply by the reciprocal $\frac{1}{(x+1)}$. So,

$$\frac{2x^2+2x}{x-1}\div(x+1)=\frac{2x\cancel{(x+1)}}{x-1}\cdot\frac{1}{\cancel{(x+1)}}=\frac{2x}{x-1}$$

b. The order of operations indicates to perform the division first. To do this, we convert the division into multiplication by the reciprocal of the middle expression. Therefore,

$$\frac{x^2-25}{x^2+5x+4}\div\frac{x^2-10x+25}{2x^2+8x}\cdot\frac{x^2+x}{4x^2}$$

$$=\frac{(x-5)(x+5)}{(x+4)\cancel{(x+1)}}\cdot\frac{2x^2+8x}{x^2-10x+25}\cdot\frac{\cancel{x}\cancel{(x+1)}}{4x^{\cancel{2}\,1}}$$

$$=\frac{\cancel{(x-5)}(x+5)}{\cancel{(x+4)}}\cdot\frac{\cancel{2}\cancel{x}\cancel{(x+4)}}{(x-5)^{\cancel{2}\,1}}\cdot\frac{1}{\underset{2}{\cancel{4}}\cancel{x}}=\boldsymbol{\frac{(x+5)}{2(x-5)}}$$

Reduction of common factors can be done gradually, especially if there are many common factors to divide out.

RT.2 Exercises

True or false.

1. $f(x)=\frac{4}{\sqrt{x}-4}$ is a rational function.

2. The domain of $f(x)=\frac{x-2}{4}$ is the set of all real numbers.

3. $\frac{x-3}{4-x}$ is equivalent to $-\frac{x-3}{x-4}$.

4. $\frac{n^2+1}{n^2-1}$ is equivalent to $\frac{n+1}{n-1}$.

Given the rational function f, find $f(-1)$, $f(0)$, and $f(2)$.

5. $f(x)=\frac{x}{x-2}$

6. $f(x)=\frac{5x}{3x-x^2}$

7. $f(x)=\frac{x-2}{x^2+x-6}$

For each rational function, find all numbers that are <u>not</u> in the domain. Then give the ***domain****, using both* ***set notation*** *and* ***interval notation****.*

8. $f(x)=\frac{x}{x+2}$

9. $g(x)=\frac{x}{x-6}$

10. $h(x)=\frac{2x-1}{3x+7}$

11. $f(x)=\frac{3x+2}{5x-4}$

12. $g(x)=\frac{x+2}{x^2-4}$

13. $h(x)=\frac{x-2}{x^2+4}$

14. $f(x)=\frac{5}{3x-x^2}$

15. $g(x)=\frac{x^2+x-6}{x^2+12x+35}$

16. $h(x)=\frac{7}{|4x-3|}$

17. *Which rational expressions are equivalent and what is their simplest form?*

a. $\frac{2x+3}{2x-3}$ **b.** $\frac{2x-3}{3-2x}$ **c.** $\frac{2x+3}{3+2x}$ **d.** $\frac{2x+3}{-2x-3}$ **e.** $\frac{3-2x}{2x-3}$

18. *Which rational expressions can be simplified?*

a. $\frac{x^2+2}{x^2}$ **b.** $\frac{x^2+2}{2}$ **c.** $\frac{x^2-x}{x^2}$ **d.** $\frac{x^2-y^2}{y^2}$ **e.** $\frac{x}{x^2-x}$

Simplify each expression, if possible.

19. $\frac{24a^3b}{3ab^3}$

20. $\frac{-18x^2y^3}{8x^3y}$

21. $\frac{7-x}{x-7}$

22. $\frac{x+2}{x-2}$

23. $\frac{a-5}{-5+a}$

24. $\frac{(3-y)(x+1)}{(y-3)(x-1)}$

25. $\frac{12x-15}{21}$

26. $\frac{18a-2}{22}$

27. $\frac{4y-12}{4y+12}$

28. $\frac{7x+14}{7x-14}$

29. $\frac{6m+18}{7m+21}$

30. $\frac{3z^2+z}{18z+6}$

31. $\frac{m^2-25}{20-4m}$

32. $\frac{9n^2-3}{4-12n^2}$

33. $\frac{t^2-25}{t^2-10t+25}$

34. $\frac{p^2-36}{p^2+12t+36}$

35. $\frac{x^2-9x+8}{x^2+3x-4}$

36. $\frac{p^2+8p-9}{p^2-5p+4}$

37. $\frac{x^3-y^3}{x^2-y^2}$

38. $\frac{b^2-a^2}{a^3-b^3}$

Perform operations and simplify. Assume nonzero denominators.

39. $\frac{18a^4}{5b^2} \cdot \frac{25b^4}{9a^3}$

40. $\frac{28}{xy} \div \frac{63x^3}{2y^2}$

41. $\frac{12x}{49(xy^2)^3} \cdot \frac{(7xy)^2}{8}$

42. $\frac{x+1}{2x-3} \cdot \frac{2x-3}{2x}$

43. $\frac{10a}{6a-12} \cdot \frac{20a-40}{30a^3}$

44. $\frac{a^2-1}{4a} \cdot \frac{2}{1-a}$

45. $\frac{y^2-25}{4y} \cdot \frac{2}{5-y}$

46. $(8x-16) \div \frac{3x-6}{10}$

47. $(y^2-4) \div \frac{2-y}{8y}$

48. $\frac{3n-9}{n^2-9} \cdot (n^3+27)$

49. $\frac{x^2-16}{x^2} \cdot \frac{x^2-4x}{x^2-x-12}$

50. $\frac{y^2+10y+25}{y^2-9} \cdot \frac{y^2-3y}{y+5}$

51. $\frac{b-3}{b^2-4b+3} \div \frac{b^2-b}{b-1}$

52. $\frac{x^2-6x+9}{x^2+3x} \div \frac{x^2-9}{x}$

53. $\frac{x^2-2x}{3x^2-5x-2} \cdot \frac{9x^2-4}{9x^2-12x+4}$

54. $\frac{t^2-49}{t^2+4t-21} \cdot \frac{t^2+8t+15}{t^2-2t-35}$

55. $\frac{a^3-b^3}{a^2-b^2} \div \frac{2a-2b}{2a+2b}$

56. $\frac{64x^3+1}{4x^2-100} \cdot \frac{4x+20}{64x^2-16x+4}$

57. $\frac{x^3y-64y}{x^3y+64y} \div \frac{x^2y^2-16y^2}{x^2y^2-4xy^2+16y^2}$

58. $\frac{p^3-27q^3}{p^2+pq-12q^2} \cdot \frac{p^2-2pq-24q^2}{p^2-5pq-6q^2}$

59. $\frac{4x^2-9y^2}{8x^3-27y^3} \cdot \frac{4x^2+6xy+9y^2}{4x^2+12xy+9y^2}$

60. $\frac{2x^2+x-1}{6x^2+x-2} \div \frac{2x^2+5x+3}{6x^2+13x+6}$

61. $\frac{6x^2-13x+6}{14x^2-25x+6} \div \frac{14-21x}{49x^2+7x-6}$

62. $\frac{4y^2-12y+36}{27-3y^2} \div (y^3+27)$

63. $\frac{3y}{x^2} \div \frac{y^2}{x} \div \frac{y}{5x}$

64. $\frac{x+1}{y-2} \div \frac{2x+2}{y-2} \div \frac{x}{y}$

65. $\frac{a^2-4b^2}{a+2b} \div (a+2b) \cdot \frac{2b}{a-2b}$

66. $\frac{9x^2}{x^2-16y^2} \div \frac{1}{x^2+4xy} \cdot \frac{x-4y}{3x}$

67. $\frac{x^2-25}{x-4} \div \frac{x^2-2x-15}{x^2-10x+24} \cdot \frac{x+3}{x^2+10x+25}$

68. $\frac{y-3}{y^2-8y+16} \cdot \frac{y^2-16}{y+4} \div \frac{y^2+3y-18}{y^2+11y+30}$

Given $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$ and $f(x) \div g(x)$.

69. $f(x) = \frac{x-4}{x^2+x}$ and $g(x) = \frac{2x}{x+1}$

70. $f(x) = \frac{x^3-3x^2}{x+5}$ and $g(x) = \frac{4x^2}{x-3}$

71. $f(x) = \frac{x^2-7x+12}{x+3}$ and $g(x) = \frac{9-x^2}{x-4}$

72. $f(x) = \frac{x+6}{4-x^2}$ and $g(x) = \frac{2-x}{x^2+8x+12}$

RT3 Addition and Subtraction of Rational Expressions

Many real-world applications involve adding or subtracting algebraic fractions. Like in the case of common fractions, to add or subtract algebraic fractions, we first need to change them equivalently to fractions with the same denominator. Thus, we begin by discussing the techniques of finding the least common denominator.

Least Common Denominator

The **least common denominator (LCD)** for fractions with given denominators is the same as the **least common multiple (LCM)** of these denominators. The methods of finding the LCD for fractions with numerical denominators were reviewed in *Section R3*. For example,

$$LCD(4,6,8) = 24,$$

because 24 is a multiple of 4, 6, and 8, and there is no smaller natural number that would be divisible by all three numbers, 4, 6, and 8.

Suppose the denominators of three algebraic fractions are $4(x^2 - y^2)$, $-6(x+y)^2$, and $8x$. The numerical factor of the least common multiple is 24. The variable part of the LCM is built by taking the product of all the different variable factors from each expression, with each factor raised to the **greatest** exponent that occurs in any of the expressions. In our example, since $4(x^2 - y^2) = 4(x+y)(x-y)$, then

$$LCD(\ 4(x+y)(x-y),\ \ -6(x+y)^2,\ \ 8x\ ) = \mathbf{24}\boldsymbol{x}(\boldsymbol{x}+\boldsymbol{y})^{\mathbf{2}}(\boldsymbol{x}-\boldsymbol{y})$$

Notice that we do not worry about the negative sign of the middle expression. This is because a negative sign can always be written in front of a fraction or in the numerator rather than in the denominator. For example,

$$\frac{1}{-6(x+y)^2} = -\frac{1}{6(x+y)^2} = \frac{-1}{6(x+y)^2}$$

In summary, to find the LCD for algebraic fractions, follow the steps:

- **Factor** each denominator **completely**.
- Build the LCD for the denominators by including the following as factors:
  - **LCD of all numerical coefficients**,
  - all of the **different factors** from each denominator, with each factor **raised to the greatest exponent** that occurs in any of the denominators.
    Note: Disregard any factor of -1.

Example 1 ► **Determining the LCM for the Given Expressions**

Find the LCM for the given expressions.

a. $12x^3y$ and $15xy^2(x-1)$ **b.** $x^2 - 2x - 8$ and $x^2 + 3x + 2$

c. $y^2 - x^2$, $2x^2 - 2xy$, and $x^2 + 2xy + y^2$

Solution ► **a.** Notice that both expressions, $12x^3y$ and $15xy^2(x-1)$, are already in factored form. The $LCM(12,15) = 60$, as

divide by 3

$$\begin{array}{c|cc} 3 & 12 & 15 \\ \hline \cdot & 4 & \cdot\, 5 \end{array} = \mathbf{60}$$

no more common factors, so we multiply the numbers in the letter L

The highest power of x is 3, the highest power of y is 2, and $(x-1)$ appears in the first power. Therefore,

$$LCM\left(12x^3y,\ \ 15xy^2(x-1)\right) = \mathbf{60x^3y^2(x-1)}$$

b. To find the LCM of $x^2 - 2x - 8$ and $x^2 + 3x + 2$, we factor each expression first:

$$x^2 - 2x - 8 = (x-4)(x+2)$$
$$x^2 + 3x + 2 = (x+1)(x+2)$$

There are three different factors in these expressions, $(x-4)$, $(x+2)$, and $(x+1)$. All of these factors appear in the first power, so

$$LCM(\,x^2 - 2x - 8,\ \ x^2 + 3x + 2\,) = \mathbf{(x-4)(x+2)(x+1)}$$

notice that $(x+2)$ is taken only once!

c. As before, to find the LCM of $y^2 - x^2$, $2x^2 - 2xy$, and $x^2 + 2xy + y^2$, we factor each expression first:

$$y^2 - x^2 = (y+x)(y-x) = -(x+y)(x-y)$$
$$2x^2 - 2xy = 2x(x-y)$$
$$x^2 + 2xy + y^2 = (x+y)^2$$

as $\boldsymbol{y - x = -(x-y)}$
and $\boldsymbol{y + x = x + y}$

Since the factor of -1 can be disregarded when finding the LCM, the opposite factors can be treated as the same by factoring the -1 out of one of the expressions. So, there are four different factors to consider, 2, x, $(x+y)$, and $(x-y)$. The highest power of $(x+y)$ is 2 and the other factors appear in the first power. Therefore,

$$LCM(\,y^2 - x^2,\ \ 2x^2 - 2xy,\ \ x^2 + 2xy + y^2\,) = \mathbf{2x(x-y)(x+y)^2}$$

Addition and Subtraction of Rational Expressions

Observe addition and subtraction of common fractions, as review in *Section R3*.

work out the numerator

$$\frac{1}{2} + \frac{2}{3} - \frac{5}{6} = \frac{1 \cdot 3 + 2 \cdot 2 - 5}{6} = \frac{3 + 4 - 5}{6} = \frac{2}{6} = \mathbf{\frac{1}{3}}$$

convert fractions to the lowest common denominator

simplify, if possible

To add or subtract algebraic fractions, follow the steps:

- **Factor** the denominators of all algebraic fractions **completely**.
- **Find the LCD** of all the denominators.
- **Convert each algebraic fraction to the lowest common denominator** found in the previous step and write the sum (or difference) as a single fraction.
- **Simplify** the numerator and the whole fraction, if possible.

Example 2 ► Adding and Subtracting Rational Expressions

Perform the operations and simplify if possible.

a. $\frac{a}{5}-\frac{3b}{2a}$

b. $\frac{x}{x-y}+\frac{y}{y-x}$

c. $\frac{3x^2+3xy}{x^2-y^2}-\frac{2-3x}{x-y}$

d. $\frac{y+1}{y^2-7y+6}+\frac{y-1}{y^2-5y-6}$

e. $\frac{2x}{x^2-4}+\frac{5}{2-x}-\frac{1}{2+x}$

f. $(2x-1)^{-2}+(2x-1)^{-1}$

Solution ►

Multiplying the numerator and denominator of a fraction by the same factor is equivalent to multiplying the whole fraction by 1, which does not change the value of the fraction.

a. Since $LCM(5,2a)=10a$, we would like to rewrite expressions, $\frac{a}{5}$ and $\frac{3b}{2a}$, so that they have a denominator of $10a$. This can be done by multiplying the numerator and denominator of each expression by the factors of $10a$ that are missing in each denominator. So, we obtain

$$\frac{a}{5}-\frac{3b}{2a}=\frac{a}{5}\cdot\frac{2a}{2a}-\frac{3b}{2a}\cdot\frac{5}{5}=\frac{\mathbf{2a^2-15b}}{\mathbf{10a}}$$

b. Notice that the two denominators, $x-y$ and $y-x$, are opposite expressions. If we write $y-x$ as $-(x-y)$, then

$$\frac{x}{x-y}+\frac{y}{y-x}=\frac{x}{x-y}+\frac{y}{-(x-y)}=\frac{x}{x-y}-\frac{y}{x-y}=\frac{x-y}{x-y}=\mathbf{1}$$

combine the signs

c. To find the LCD, we begin by factoring $x^2-y^2=(x-y)(x+y)$. Since this expression includes the second denominator as a factor, the LCD of the two fractions is $(x-y)(x+y)$. So, we calculate

keep the bracket after a "–" sign

$$\frac{3x^2+3xy}{x^2-y^2}-\frac{2-3x}{x-y}=\frac{(3x^2+3xy)\cdot 1-(2-3x)\cdot(x+y)}{(x-y)(x+y)}=$$

$$\frac{3x^2+3xy-(2x+2y-3x^2-3xy)}{(x-y)(x+y)}=\frac{3x^2+3xy-2x-2y+3x^2+3xy}{(x-y)(x+y)}=$$

$$\frac{\mathbf{6x^2+6xy-2x-2y}}{(x-y)(x+y)}=\frac{2(3x^2+3xy-x-y)}{(x-y)(x+y)}=\frac{2\big(3x(x+y)-(x+y)\big)}{(x-y)(x+y)}$$

$$=\frac{2\cancel{(x+y)}(3x-1)}{(x-y)\cancel{(x+y)}}=\frac{\mathbf{2(3x-1)}}{\mathbf{(x-y)}}$$

d. To find the LCD, we first factor each denominator. Since

$$y^2-7y+6=(y-6)(y-1)\ \text{ and }\ y^2-5y-6=(y-6)(y+1),$$

then $LCD=(y-6)(y-1)(y+1)$ and we calculate

multiply by the missing bracket

$$\frac{y+1}{y^2-7y+6}+\frac{y-1}{y^2-5y-6}=\frac{y+1}{(y-6)(y-1)}+\frac{y-1}{(y-6)(y+1)}=$$

$$\frac{(y+1)\cdot(y+1)+(y-1)\cdot(y-1)}{(y-6)(y-1)(y+1)}=\frac{y^2+2y+1+(y^2-2y+1)}{(y-6)(y-1)(y+1)}=$$

$$\frac{2y^2+2}{(y-6)(y-1)(y+1)}=\frac{2(y^2+1)}{(y-6)(y-1)(y+1)}$$

e. As in the previous examples, we first factor the denominators, including factoring out a negative from any opposite expression. So,

$$\frac{2x}{x^2-4}+\frac{5}{2-x}-\frac{1}{2+x}=\frac{2x}{(x-2)(x+2)}+\frac{5}{-(x-2)}-\frac{1}{x+2}=$$

LCD $=(x-2)(x+2)$

$$\frac{2x-5(x+2)-1(x-2)}{(x-2)(x+2)}=\frac{2x-5x-10-x+2}{(x-2)(x+2)}=$$

$$\frac{-4x-8}{(x-2)(x+2)}=\frac{-4\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}=\frac{\mathbf{-4}}{\mathbf{x-2}}$$

f. Recall that a negative exponent really represents a hidden fraction. So, we may choose to rewrite the powers with negative exponents as fractions, and then add them using techniques as shown in previous examples.

$$(2x-1)^{-2}+(2x-1)^{-1}=\frac{1}{(2x-1)^2}+\frac{1}{2x-1}=\frac{1+1\cdot(2x-1)}{(2x-1)^2}=$$

$$\frac{1+2x-1}{(2x-1)^2}=\frac{2x}{(2x-1)^2}=\frac{\mathbf{2x}}{\mathbf{(2x-1)^2}}$$

nothing to simplify this time

Note: Since addition (or subtraction) of rational expressions results in a rational expression, from now on the term "rational expression" will include sums of rational expressions as well.

Example 3 ► **Adding Rational Expressions in Application Problems**

Assume that a boat travels n kilometers up the river and then returns back to the starting point. If the water in the river flows with a constant current of c km/h, the total time for the round-trip can be calculated via the expression $\frac{n}{r+c}+\frac{n}{r-c}$, where r is the speed of the boat in still water in kilometers per hour. Write a single rational expression representing the total time of this trip.

Solution ► To find a single rational expression representing the total time, we perform the addition using $(r+c)(r-c)$ as the lowest common denominator. So,

$$\frac{n}{r+c}+\frac{n}{r-c}=\frac{n(r-c)+n(r+c)}{(r+c)(r-c)}=\frac{nr-nc+nr+nc}{(r+c)(r-c)}=\boldsymbol{\frac{2nr}{r^2-c^2}}$$

Example 4 ► **Adding and Subtracting Rational Functions**

Given $f(x)=\frac{1}{x^2+10x+24}$ and $g(x)=\frac{2}{x^2+4x}$, find

a. $(f+g)(x)$

b. $(f-g)(x)$.

Solution ► **a.** $$(\boldsymbol{f}+\boldsymbol{g})(\boldsymbol{x})=f(x)+g(x)=\frac{1}{x^2+10x+24}+\frac{2}{x^2+4x}$$

$$=\frac{1}{(x+6)(x+4)}+\frac{2}{x(x+4)}=\frac{1\cdot x+2(x+6)}{x(x+6)(x+4)}=\frac{x+2x+12}{x(x+6)(x+4)}$$

$$=\frac{3x+12}{x(x+6)(x+4)}=\frac{3(x+4)}{x(x+6)(x+4)}=\boldsymbol{\frac{3}{x(x+6)}}$$

b. $$(\boldsymbol{f}-\boldsymbol{g})(\boldsymbol{x})=f(x)-g(x)=\frac{1}{x^2+10x+24}-\frac{2}{x^2+4x}$$

$$=\frac{1}{(x+6)(x+4)}-\frac{2}{x(x+4)}=\frac{1\cdot x-2(x+6)}{x(x+6)(x+4)}=\frac{x-2x-12}{x(x+6)(x+4)}$$

$$=\boldsymbol{\frac{-x-12}{x(x+6)(x+4)}}$$

RT.3 Exercises

1. **a.** What is the LCM for 6 and 9? **b.** What is the LCD for $\frac{1}{6}$ and $\frac{1}{9}$?

2. **a.** What is the LCM for $x^2 - 25$ and $x + 5$? **b.** What is the LCD for $\frac{1}{x^2-25}$ and $\frac{1}{x+5}$?

Find the LCD and then perform the indicated operations. Simplify the resulting fraction.

3. $\frac{5}{12} + \frac{13}{18}$

4. $\frac{11}{30} - \frac{19}{75}$

5. $\frac{3}{4} + \frac{7}{30} - \frac{1}{16}$

6. $\frac{5}{8} - \frac{7}{12} + \frac{11}{40}$

Find the ***least common multiple (LCM)*** *for each group of expressions.*

7. $24a^3b^4,\ 18a^5b^2$

8. $6x^2y^2,\ 9x^3y,\ 15y^3$

9. $x^2 - 4,\ x^2 + 2x$

10. $10x^2,\ 25(x^2 - x)$

11. $(x-1)^2,\ 1 - x$

12. $y^2 - 25,\ 5 - y$

13. $x^2 - y^2,\ xy + y^2$

14. $5a - 15,\ a^2 - 6a + 9$

15. $x^2 + 2x + 1,\ x^2 - 4x - 1$

16. $n^2 - 7n + 10,\ n^2 - 8n + 15$

17. $2x^2 - 5x - 3,\ 2x^2 - x - 1,\ x^2 - 6x + 9$

18. $1 - 2x,\ 2x + 1,\ 4x^2 - 1$

19. $x^5 - 4x^4 + 4x^3,\ 12 - 3x^2,\ 2x + 4$

True or false? If true, explain why. If false, correct it.

20. $\frac{1}{2x} + \frac{1}{3x} = \frac{1}{5x}$

21. $\frac{1}{x-3} + \frac{1}{3-x} = 0$

22. $\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$

23. $\frac{3}{4} + \frac{x}{5} = \frac{3+x}{20}$

Perform the indicated operations and simplify if possible.

24. $\frac{x-2y}{x+y} + \frac{3y}{x+y}$

25. $\frac{a+3}{a+1} - \frac{a-5}{a+1}$

26. $\frac{4a+3}{a-3} - 1$

27. $\frac{n+1}{n-2} + 2$

28. $\frac{x^2}{x-y} + \frac{y^2}{y-x}$

29. $\frac{4a-2}{a^2-49} + \frac{5+3a}{49-a^2}$

30. $\frac{2y-3}{y^2-1} - \frac{4-y}{1-y^2}$

31. $\frac{a^3}{a-b} + \frac{b^3}{b-a}$

32. $\frac{1}{x+h} - \frac{1}{x}$

33. $\frac{x-2}{x+3} + \frac{x+2}{x-4}$

34. $\frac{x-1}{3x+1} + \frac{2}{x-3}$

35. $\frac{4xy}{x^2-y^2} + \frac{x-y}{x+y}$

36. $\frac{x-1}{3x+15} - \frac{x+3}{5x+25}$

37. $\frac{y-2}{4y+8} - \frac{y+6}{5y+10}$

38. $\frac{4x}{x-1} - \frac{2}{x+1} - \frac{4}{x^2-1}$

39. $\frac{-2}{y+2} + \frac{5}{y-2} + \frac{y+3}{y^2-4}$

40. $\frac{y}{y^2-y-20} + \frac{2}{y+4}$

41. $\frac{5x}{x^2-6x+8} - \frac{3x}{x^2-x-12}$

42. $\frac{9x+2}{3x^2-2x-8} + \frac{7}{3x^2+x-4}$

43. $\frac{3y+2}{2y^2-y-10} + \frac{8}{2y^2-7y+5}$

44. $\frac{6}{y^2+6y+9} + \frac{5}{y^2-9}$

45. $\frac{3x-1}{x^2+2x-3} - \frac{x+4}{x^2-9}$

46. $\frac{1}{x+1} - \frac{x}{x-2} + \frac{x^2+2}{x^2-x-2}$

47. $\frac{2}{y+3} - \frac{y}{y-1} + \frac{y^2+2}{y^2+2y-3}$

48. $\frac{4x}{x^2-1}+\frac{3x}{1-x}-\frac{4}{x-1}$

49. $\frac{5y}{1-2y}-\frac{2y}{2y+1}+\frac{3}{4y^2-1}$

50. $\frac{x+5}{x-3}-\frac{x+2}{x+1}-\frac{6x+10}{x^2-2x-3}$

Perform the indicated operations and simplify if possible.

51. $2x^{-3}+(3x)^{-1}$

52. $(x^2-9)^{-1}+2(x-3)^{-1}$

53. $\left(\frac{x+1}{3}\right)^{-1}-\left(\frac{x-4}{2}\right)^{-1}$

54. $\left(\frac{a-3}{a^2}-\frac{a-3}{9}\right)\div\frac{a^2-9}{3a}$

55. $\frac{x^2-4x+4}{2x+1}\cdot\frac{2x^2+x}{x^3-4x}-\frac{3x-2}{x+1}$

56. $\frac{2}{x-3}-\frac{x}{x^2-x-6}\cdot\frac{x^2-2x-3}{x^2-x}$

Given $f(x)$ and $g(x)$, find $(f+g)(x)$ and $(f-g)(x)$. Leave the answer in simplified single fraction form.

57. $f(x)=\frac{x}{x+2},\ g(x)=\frac{4}{x-3}$

58. $f(x)=\frac{x}{x^2-4},\ g(x)=\frac{1}{x^2+4x+4}$

59. $f(x)=\frac{3x}{x^2+2x-3},\ g(x)=\frac{1}{x^2-2x+1}$

60. $f(x)=x+\frac{1}{x-1},\ g(x)=\frac{1}{x+1}$

Solve each problem.

61. There are two part-time waitresses at a restaurant. One waitress works every fourth day, and the other one works every sixth day. Both waitresses were hired and start working on the same day. How often do they both work on the same day?

62. A cylindrical water tank is being filled and drained at the same time. To find the rate of change of the water level one could use the expression $\frac{H}{T_{in}}-\frac{H}{T_{out}}$, where H is the height of the water in the full tank while T_{in} and T_{out} represent the time needed to fill and empty the tank, respectively. Write the rate of change of the water level as a single algebraic fraction.

63. To determine the Canadian population percent growth over the past year, one could use the expression $100\left(\frac{P_1}{P_0}-1\right)$, where P_1 represents the current population and P_0 represents the last year's population. Write this expression as a single algebraic fraction.

64. A boat travels k kilometers against a c km/h current. Assuming the current remains constant, one could calculate the total time, in hours, needed for the entire trip via the expression $\frac{k}{s-c}+\frac{k}{s+c}$, where s represents the speed of the boat in calm water. Write this expression as a single algebraic fraction.

RT4 Complex Fractions

When working with algebraic expressions, sometimes we come across needing to simplify expressions like these:

$$\frac{\frac{x^2-9}{x+1}}{\frac{x+3}{x^2-1}},\quad \frac{1+\frac{1}{x}}{1-\frac{1}{y}},\quad \frac{\frac{1}{x+2}-\frac{1}{x+h+2}}{h},\quad \frac{1}{\frac{1}{a}-\frac{1}{b}}$$

A complex fraction is a quotient of rational expressions (including sums of rational expressions) where at least one of these expressions contains a fraction itself. In this section, we will examine two methods of simplifying such fractions.

Simplifying Complex Fractions

Definition 4.1 ► A **complex fraction** is a **quotient** of rational expressions (including their sums) that result in a fraction with more than two levels. For example, $\frac{\frac{1}{2}}{3}$ has three levels while $\frac{\frac{1}{2x}}{\frac{3}{4x}}$ has four levels. Such fractions can be **simplified to a single fraction** with only two levels. For example,

$$\frac{\frac{1}{2}}{3}=\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6},\qquad or \qquad \frac{\frac{1}{2x}}{\frac{3}{4x^2}}=\frac{1}{2\cancel{x}}\cdot\frac{4x^{\overset{1}{\cancel{2}}}}{3}=\frac{2x}{3}$$

There are two common methods of simplifying complex fractions.

Method I (***multiplying by the reciprocal of the denominator***)

Replace the main division in the complex fraction with a multiplication of the numerator fraction by the reciprocal of the denominator fraction. We then simplify the resulting fraction if possible. Both examples given in *Definition 4.1* were simplified using this strategy.

Method I is the most convenient to use when both the numerator and the denominator of a complex fraction consist of single fractions. However, if either the numerator or the denominator of a complex fraction contains addition or subtraction of fractions, it is usually easier to use the method shown below.

Method II (***multiplying by LCD***)

Multiply the numerator and denominator of a complex fraction by the least common denominator of all the fractions appearing in the numerator or in the denominator of the complex fraction. Then, simplify the resulting fraction if possible. For example, to simplify $\frac{y+\frac{1}{x}}{x+\frac{1}{y}}$, multiply the numerator $y+\frac{1}{x}$ and the denominator $x+\frac{1}{y}$ by the $LCD\left(\frac{1}{x},\frac{1}{y}\right)=xy$. So,

$$\frac{\left(y+\frac{1}{x}\right)}{\left(x+\frac{1}{y}\right)}\cdot\frac{xy}{xy}=\frac{xy^2+y}{x^2y+x}=\frac{y\cancel{(xy+1)}}{x\cancel{(xy+1)}}=\boldsymbol{\frac{y}{x}}$$

Example 1 ► Simplifying Complex Fractions

Use a method of your choice to simplify each complex fraction.

a. $\frac{\frac{x^2-2x-8}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-4x-21}}$

b. $\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}$

c. $\frac{x+\frac{1}{5}}{x-\frac{1}{3}}$

d. $\frac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}$

Solution ►

a. Since the expression $\frac{\frac{x^2-2x-8}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-4x-21}}$ contains a single fraction in both the numerator and denominator, we will simplify it using method I, as below.

$$\frac{\frac{x^2-2x-8}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-4x-21}} = \frac{(x-4)\cancel{(x+2)}}{(x-5)\cancel{(x+3)}}\cdot\frac{(x-7)\cancel{(x+3)}}{(x+6)\cancel{(x+2)}} = \frac{\boldsymbol{(x-4)(x-7)}}{\boldsymbol{(x-5)(x+6)}}$$

factor and multiply by the reciprocal

b. $\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}$ can be simplified in the following two ways:

Method I

$$\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}} = \frac{a+b}{\frac{b^3+a^3}{a^3b^3}} = \frac{(a+b)a^3b^3}{a^3+b^3}$$

$$= \frac{\cancel{(a+b)}a^3b^3}{\cancel{(a+b)}(a^2-ab+b^2)} = \frac{\boldsymbol{a^3b^3}}{\boldsymbol{a^2-ab+b^2}}$$

Method II

$$\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}\cdot\frac{a^3b^3}{a^3b^3} = \frac{(a+b)a^3b^3}{b^3+a^3}$$

$$= \frac{\cancel{(a+b)}a^3b^3}{\cancel{(a+b)}(a^2-ab+b^2)} = \frac{\boldsymbol{a^3b^3}}{\boldsymbol{a^2-ab+b^2}}$$

Caution: In Method II, the factor that we multiply the complex fraction by **must be equal to 1**. This means that **the numerator and denominator of this factor must be exactly the same**.

c. To simplify $\frac{x+\frac{1}{5}}{x-\frac{1}{3}}$, we will use method II. Multiplying the numerator and denominator by the $LCD\left(\frac{1}{5},\frac{1}{3}\right)=15$, we obtain

$$\frac{x+\frac{1}{5}}{x-\frac{1}{3}}\cdot\frac{15}{15} = \frac{\boldsymbol{15x+3}}{\boldsymbol{15x-5}}$$

d. Again, to simplify $\frac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}$, we will use method II. Notice that the lowest common multiple of the denominators in blue is $(x+2)(x-2)$. So, after multiplying the numerator and denominator of the whole expression by the LCD, we obtain

$$\frac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}\cdot\frac{(x+2)(x-2)}{(x+2)(x-2)}=\frac{6-5(x-2)}{7-4(x+2)}=\frac{6-5x+10}{7-4x-8}$$

$$=\frac{-5x+16}{-4x-1}=\boldsymbol{\frac{5x-16}{4x+1}}$$

Example 2 ► Simplifying Rational Expressions with Negative Exponents

Simplify each expression. Leave the answer with only positive exponents.

a. $\frac{x^{-2}-y^{-1}}{y-x}$ **b.** $\frac{a^{-3}}{a^{-1}-b^{-1}}$

Solution ► **a.** If we write the expression with no negative exponents, it becomes a complex fraction, which can be simplified as in *Example 1*. So,

$$\frac{x^{-2}-y^{-1}}{y-x}=\frac{\frac{1}{x}-\frac{1}{y}}{y-x}\cdot\frac{xy}{xy}=\frac{y-x}{xy(y-x)}=\boldsymbol{\frac{1}{xy}}$$

b. As above, first, we rewrite the expression with only positive exponents and then simplify as any other complex fraction.

Remember! This factor must be = 1

$$\frac{a^{-3}}{a^{-1}-b^{-1}}=\frac{\frac{1}{a^3}}{\frac{1}{a}-\frac{1}{b}}\cdot\frac{a^3b}{a^3b}=\frac{b}{a^2b-a^3}=\boldsymbol{\frac{b}{a^2(b-a)}}$$

Example 3 ► Simplifying the Difference Quotient for a Rational Function

Find and simplify the expression $\frac{f(a+h)-f(a)}{h}$ for the function $f(x)=\frac{1}{x+1}$.

Solution ► Since $f(a+h)=\frac{1}{a+h+1}$ and $f(a)=\frac{1}{a+1}$, then

$$\frac{f(a+h)-f(a)}{h}=\frac{\frac{1}{a+h+1}-\frac{1}{a+1}}{h}$$

To simplify this expression, we can multiply the numerator and denominator by the lowest common denominator, which is $(a+h+1)(a+1)$. Thus,

This bracket is essential!

$$\frac{\frac{1}{a+h+1}-\frac{1}{a+1}}{h}\cdot\frac{(a+h+1)(a+1)}{(a+h+1)(a+1)}=\frac{a+1-(a+h+1)}{h(a+h+1)(a+1)}$$

$$=\frac{\cancel{a}+\cancel{1}-\cancel{a}-h-\cancel{1}}{h(a+h+1)(a+1)}=\frac{-\cancel{h}}{\cancel{h}(a+h+1)(a+1)}=\frac{\mathbf{-1}}{\boldsymbol{(a+h+1)(a+1)}}$$

keep the denominator in a factored form

RT.4 Exercises

Simplify each complex fraction.

1. $\dfrac{2-\frac{1}{3}}{3+\frac{7}{3}}$

2. $\dfrac{5-\frac{3}{4}}{4+\frac{1}{2}}$

3. $\dfrac{\frac{3}{8}-5}{\frac{2}{3}+6}$

4. $\dfrac{\frac{2}{3}+\frac{4}{5}}{\frac{3}{4}-\frac{1}{2}}$

Simplify each complex rational expression.

5. $\dfrac{\frac{x^3}{y}}{\frac{x^2}{y^3}}$

6. $\dfrac{\frac{n-5}{6n}}{\frac{n-5}{8n^2}}$

7. $\dfrac{1-\frac{1}{a}}{4+\frac{1}{a}}$

8. $\dfrac{\frac{2}{n}+3}{\frac{5}{n}-6}$

9. $\dfrac{\frac{9-3x}{4x+12}}{\frac{x-3}{6x-24}}$

10. $\dfrac{\frac{9}{y}}{\frac{15}{y}-6}$

11. $\dfrac{\frac{4}{x}-\frac{2}{y}}{\frac{4}{x}+\frac{2}{y}}$

12. $\dfrac{\frac{3}{a}+\frac{4}{b}}{\frac{4}{a}-\frac{3}{b}}$

13. $\dfrac{a-\frac{3a}{b}}{b-\frac{b}{a}}$

14. $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{x^2-y^2}{xy}}$

15. $\dfrac{\frac{4}{y}-\frac{y}{x^2}}{\frac{1}{x}-\frac{2}{y}}$

16. $\dfrac{\frac{5}{p}-\frac{1}{q}}{\frac{1}{5q^2}-\frac{5}{p^2}}$

17. $\dfrac{\frac{n-12}{n}+n}{n+4}$

18. $\dfrac{2t-1}{\frac{3t-2}{t}+2t}$

19. $\dfrac{\frac{1}{a-h}-\frac{1}{a}}{h}$

20. $\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

21. $\dfrac{4+\frac{12}{2x-3}}{5+\frac{15}{2x-3}}$

22. $\dfrac{1+\frac{3}{x+2}}{1+\frac{6}{x-1}}$

23. $\dfrac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}}$

24. $\dfrac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$

25. $\dfrac{\frac{x+3}{x}-\frac{4}{x-1}}{\frac{x}{x-1}+\frac{1}{x}}$

26. $\dfrac{\frac{3}{x^2+6x+9}+\frac{3}{x+3}}{\frac{6}{x^2-9}+\frac{6}{3-x}}$

27. $\dfrac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^3}+\frac{1}{b^3}}$

28. $\dfrac{\frac{4p^2-12p+9}{2p^2+7p-15}}{\frac{2p^2-15p+18}{p^2-p-30}}$

29. Are the expressions $\dfrac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$ and $\dfrac{x+y}{x^2+y^2}$ equivalent? Explain why or why not.

Simplify each expression. Leave the answer with only ***positive exponents****.*

30. $\frac{1}{a^{-2}-b^{-2}}$ **31.** $\frac{x^{-1}+x^{-2}}{3x^{-1}}$ **32.** $\frac{x^{-2}}{y^{-3}-x^{-3}}$ **33.** $\frac{1-(2n+1)^{-1}}{1+(2n+1)^{-1}}$

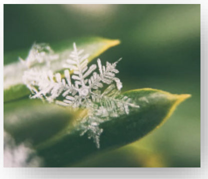
Find and simplify the ***difference quotient*** $\frac{f(a+h)-f(a)}{h}$ *for the given function.*

34. $f(x)=\frac{5}{x}$ **35.** $f(x)=\frac{2}{x^2}$ **36.** $f(x)=\frac{1}{1-x}$ **37.** $f(x)=-\frac{1}{x-2}$

Simplify each ***continued fraction****.*

38. $a-\frac{a}{1-\frac{a}{1-a}}$ **39.** $3-\frac{2}{1-\frac{2}{3-\frac{2}{x}}}$ **40.** $a+\frac{a}{2+\frac{1}{1-\frac{2}{a}}}$

RT5 Rational Equations and Graphs



In previous sections of this chapter, we worked with rational expressions. If two rational expressions are equated, a *rational equation* arises. Such equations often appear when solving application problems that involve rates of work or amounts of time considered in motion problems. In this section, we will discuss how to solve rational equations, with close attention to their domains. We will also take a look at the graphs of reciprocal functions, their properties and transformations.

Rational Equations

Definition 5.1 ► A **rational equation** is an equation involving only rational expressions and containing at least one fractional expression.

Here are some examples of rational equations:

$$\frac{x}{2}-\frac{12}{x}=-1, \qquad \frac{x^2}{x-5}=\frac{25}{x-5}, \qquad \frac{2x}{x-3}-\frac{6}{x}=\frac{18}{x^2-3x}$$

Attention! A rational equation contains an *equals* sign, while a rational expression does not. An equation can be solved for a given variable, while an expression can only be simplified or evaluated. For example, $\boldsymbol{\frac{x}{2}-\frac{12}{x}}$ is an **expression** to simplify, while $\boldsymbol{\frac{x}{2}=\frac{12}{x}}$ is an **equation** to solve.
When working with algebraic structures, it is essential to identify whether they are equations or expressions before applying appropriate strategies.

By *Definition 5.1*, rational equations contain one or more denominators. Since division by zero is not allowed, we need to pay special attention to the variable values that would make any of these denominators equal to zero. Such values would have to be excluded from the set of possible solutions. For example, neither 0 nor 3 can be solutions to the equation

$$\frac{2x}{x-3}-\frac{6}{x}=\frac{18}{x^2-3x},$$

as it is impossible to evaluate either of its sides for $x=0$ or 3. So, when solving a rational equation, it is important to find its domain first.

Definition 5.2 ► The **domain** of the variable(s) of a rational equation (in short, the **domain of a rational equation**) is the **intersection of the domains** of all rational expressions within the equation.

As stated in *Definition 2.1*, the domain of each single algebraic fraction is the set of all real numbers except for the **zeros of the denominator** (the variable values that would make the denominator equal to zero). Therefore, the **domain of a rational equation is** the set of **all real numbers except for the zeros of all the denominators** appearing in this equation.

Example 1 ► Determining Domains of Rational Equations

Find the domain of the variable in each of the given equations.

a. $\frac{x}{2}-\frac{12}{x}=-1$

b. $\frac{2x}{x-2}=\frac{-3}{x}+\frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8}-\frac{4}{y^2+6y+8}=\frac{2}{y^2-16}$

Solution ►

a. The equation $\frac{x}{2}-\frac{12}{x}=-1$ contains two denominators, 2 and x. 2 is never equal to zero and x becomes zero when $x=0$. Thus, the domain of this equation is $\mathbb{R}\setminus\{\mathbf{0}\}$.

b. The equation $\frac{2x}{x-2}=\frac{-3}{x}+\frac{4}{x-2}$ contains two types of denominators, $x-2$ and x. The $x-2$ becomes zero when $x=2$, and x becomes zero when $x=0$. Thus, the domain of this equation is $\mathbb{R}\setminus\{\mathbf{0},\mathbf{2}\}$.

c. The equation $\frac{2}{y^2-2y-8}-\frac{4}{y^2+6y+8}=\frac{2}{y^2-16}$ contains three different denominators. To find the zeros of these denominators, we solve the following equations by factoring:

$y^2-2y-8=0$	$y^2+6y+8=0$	$y^2-16=0$
$(y-4)(y+2)=0$	$(y+4)(y+2)=0$	$(y-4)(y+4)=0$
$y=4$ or $y=-2$	$y=-4$ or $y=-2$	$y=4$ or $y=-4$

So, -4, -2, and 4 must be excluded from the domain of this equation. Therefore, the domain $D=\mathbb{R}\setminus\{\mathbf{-4},\mathbf{-2},\mathbf{4}\}$.

To solve a rational equation, it is convenient to clear all the fractions first and then solve the resulting polynomial equation. This can be achieved by multiplying all the terms of the equation by the least common denominator.

Caution! Only **equations**, not expressions, can be changed equivalently by **multiplying** both of their sides by the **LCD**.
Multiplying **expressions** by any number other than 1 creates expressions that are **NOT equivalent** to the original ones. So, avoid multiplying rational expressions by the LCD.

Example 2 ► Solving Rational Equations

Solve each equation.

a. $\frac{x}{2}-\frac{12}{x}=-1$

b. $\frac{2x}{x-2}=\frac{-3}{x}+\frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8}-\frac{4}{y^2+6y+8}=\frac{2}{y^2-16}$

d. $\frac{x-1}{x-3}=\frac{2}{x-3}$

Solution ► **a.** The domain of the equation $\frac{x}{2}-\frac{12}{x}=-1$ is the set $\mathbb{R}\setminus\{0\}$, as discussed in *Example 1a*. The $LCM(2,x)=2x$, so we calculate

$$\frac{x}{2}-\frac{12}{x}=-1$$

$$\cancel{2}x\cdot\frac{x}{\cancel{2}}-2\cancel{x}\cdot\frac{12}{\cancel{x}}=-1\cdot 2x$$

multiply each term by the **LCD**

$$x^2-24=-2x$$

$$x^2+2x-24=0$$

factor to find the **possible roots**

$$(x+6)(x-4)=0$$

$$x=-6 \text{ or } x=4$$

Since both of these numbers belong to the domain, the solution set of the original equation is $\{\mathbf{-6,4}\}$.

b. The domain of the equation $\frac{2x}{x-2}=\frac{-3}{x}+\frac{4}{x-2}$ is the set $\mathbb{R}\setminus\{0,2\}$, as discussed in *Example 1b*. The $LCM(x-2,x)=x(x-2)$, so we calculate

$$\frac{2x}{x-2}=\frac{-3}{x}+\frac{4}{x-2}$$

$$x\cancel{(x-2)}\cdot\frac{2x}{\cancel{x-2}}=\frac{-3}{\cancel{x}}\cdot\cancel{x}(x-2)+\frac{4}{\cancel{x-2}}\cdot x\cancel{(x-2)}$$

$$2x^2=-3(x-2)+4x$$

$$2x^2=-3x+6+4x$$

expand the bracket,
collect like terms, and
bring the terms over to one side

$$2x^2-x-6=0$$

$$(2x+3)(x-2)=0$$

factor to find the **possible roots**

$$x=-\frac{3}{2} \text{ or } x=2$$

Since 2 is excluded from the domain, there is only one solution to the original equation, $x=-\frac{\mathbf{3}}{\mathbf{2}}$.

c. The domain of the equation $\frac{2}{y^2-2y-8}-\frac{4}{y^2+6y+8}=\frac{2}{y^2-16}$ is the set $\mathbb{R}\setminus\{-4,-2,4\}$, as discussed in *Example 1c*. To find the LCD, it is useful to factor the denominators first. Since
$y^2-2y-8=(y-4)(y+2)$,
$y^2+6y+8=(y+4)(y+2)$, and
$y^2-16=(y-4)(y+4)$, then the LCD needed to clear the fractions in the original equation is $(y-4)(y+4)(y+2)$. So, we calculate

$$\frac{2}{(y-4)(y+2)}-\frac{4}{(y+4)(y+2)}=\frac{2}{(y-4)(y+4)}$$

$$\frac{\cancel{(y-4)}(y+4)\cancel{(y+2)}\cdot 2}{\cancel{(y-4)}\cancel{(y+2)}} - \frac{(y-4)\cancel{(y+4)}\cancel{(y+2)}\cdot 4}{\cancel{(y+4)}\cancel{(y+2)}} = \frac{2\cdot\cancel{(y-4)}\cancel{(y+4)}(y+2)}{\cancel{(y-4)}\cancel{(y+4)}}$$

$$2(y+4) - 4(y-4) = 2(y+2)$$

$$2y + 8 - 4y + 16 = 2y + 4$$

$$20 = 4y$$

$$y = 5$$

Since 5 is in the domain, this is the true solution.

d. First, we notice that the domain of the equation $\frac{x-1}{x-3} = \frac{2}{x-3}$ is the set $\mathbb{R} \setminus \{3\}$. To solve this equation, we can multiply it by the $LCD = x - 3$, as in the previous examples, or we can apply the method of cross-multiplication, as the equation is a proportion. Here, we show both methods.

Use the method of your choice – either one is fine.

Multiplication by LCD:

$$\frac{x-1}{x-3} = \frac{2}{x-3}$$

multiply both sides by $x-3$

$$x - 1 = 2$$

$$x = 3$$

this multiplication is permitted as $x - 3 \neq 0$

Cross-multiplication:

$$\frac{x-1}{x-3} \times \frac{2}{x-3}$$

$$(x-1)(x-3) = 2(x-3)$$

divide both sides by $(x-3)$

$$x - 1 = 2$$

$$x = 3$$

this division is permitted as $x - 3 \neq 0$

Since 3 is excluded from the domain, there is **no solution** to the original equation.

Summary of Solving Rational Equations in One Variable

- **Determine the domain** of the variable.
- **Clear** all the **fractions by multiplying** both sides of the equation **by the LCD** of these fractions.
- **Find possible solutions** by solving the resulting equation.
- **Check** the possible solutions **against the domain**. The solution set consists of only these possible solutions that belong to the domain.

Graphs of Basic Rational Functions

So far, we discussed operations on rational expressions and solving rational equations. Now, we will look at rational functions, such as

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{-2}{x+3}, \qquad or \qquad h(x) = \frac{x-3}{x-2}.$$

Definition 5.3 ► A **rational function** is any function that can be written in the form

$$\boldsymbol{f(x) = \frac{P(x)}{Q(x)}},$$

where P and Q are polynomials and Q is not a zero polynomial.

The **domain** $\boldsymbol{D_f}$ of such function f includes all x-values for which $Q(x) \neq 0$.

Example 3 ► **Finding the Domain of a Rational Function**

Find the domain of each function.

a. $g(x) = \frac{-2}{x+3}$ **b.** $h(x) = \frac{x-3}{x-2}$

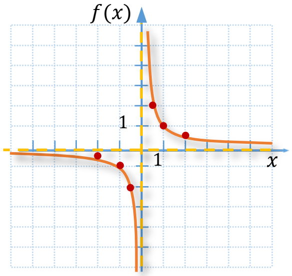
Solution ► **a.** Since $x + 3 = 0$ for $x = -3$, the domain of g is the set of all real numbers except for -3. So, the domain $\boldsymbol{D_g} = \mathbb{R} \setminus \{\boldsymbol{-3}\}$.

b. Since $x - 2 = 0$ for $x = 2$, the domain of h is the set of all real numbers except for 2. So, the domain $\boldsymbol{D_h} = \mathbb{R} \setminus \{\boldsymbol{2}\}$.

Note: The subindex f in the notation D_f indicates that the domain is of function f.

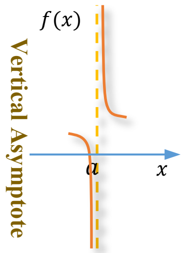
To graph a rational function, we usually start by making a table of values. Because the graphs of rational functions are typically nonlinear, it is a good idea to plot at least 3 points on each side of each x-value where the function is undefined. For example, to graph the basic rational function, $f(x) = \frac{1}{x}$, called the ***reciprocal function***, we evaluate f for a few points to the right of zero and to the left of zero. This is because f is undefined at $x = 0$, which means that the graph of f does not cross the y-axis. After plotting the obtained points, we connect them within each group, to the right of zero and to the left of zero, creating two disjoint curves. To see the shape of each curve clearly, we might need to evaluate f at some additional points.

x	$f(x)$
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
0	*undefined*
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$

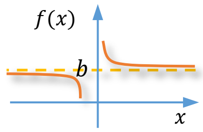


The **domain** of the reciprocal function $f(x) = \frac{1}{x}$ is $\mathbb{R} \setminus \{\boldsymbol{0}\}$, as the denominator x must be different than zero. Projecting the graph of this function onto the y-axis helps us determine the **range**, which is also $\mathbb{R} \setminus \{\boldsymbol{0}\}$.

There is another interesting feature of the graph of the reciprocal function $f(x) = \frac{1}{x}$. Observe that the graph approaches two lines, $y = 0$, the x-axis, and $x = 0$, the y-axis. These lines are called **asymptotes**. They effect the shape of the graph, but they themselves do not belong to the graph. To indicate the fact that asymptotes do not belong to the graph, we use a dashed line when graphing them.



Vertical Asymptote



Horizontal Asymptote

In general, if the y-values of a rational function approach ∞ or $-\infty$ as the x-values approach a real number a, the vertical line $x = a$ is a vertical asymptote of the graph. This can be recorded with the use of arrows, as follows:

read: approaches

$$x = a \text{ is a } \textbf{vertical asymptote} \Leftrightarrow y \to \infty \text{ (or } -\infty) \text{ when } x \to a.$$

Also, if the y-values approach a real number b as x-values approach ∞ or $-\infty$, the horizontal line $y = b$ is a horizontal asymptote of the graph. Again, using arrows, we can record this statement as:

$$y = a \text{ is a } \textbf{horizontal asymptote} \Leftrightarrow y \to b \text{ when } x \to \infty \text{ (or } -\infty).$$

Example 4 ► Graphing and Analysing the Graphs of Basic Rational Functions

For each function, state its domain and the equation of the vertical asymptote, graph it, and then state its range and the equation of the horizontal asymptote.

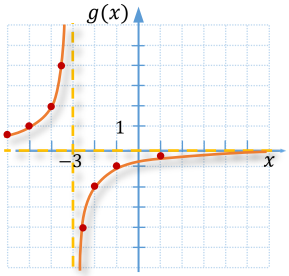
a. $g(x) = \frac{-2}{x+3}$

b. $h(x) = \frac{x-3}{x-2}$

Solution ►

a. The domain of function $g(x) = \frac{-2}{x+3}$ is $\boldsymbol{D_g} = \mathbb{R} \setminus \{\boldsymbol{-3}\}$, as discussed in *Example 3a*. Since -3 is excluded from the domain, we expect the vertical asymptote to be at $\boldsymbol{x = -3}$.

x	$g(x)$
$-\frac{5}{2}$	-4
-2	-2
-1	-1
1	$-\frac{1}{2}$
-3	*undefined*
$-\frac{7}{2}$	4
-4	2
-5	1
-6	$\frac{2}{3}$

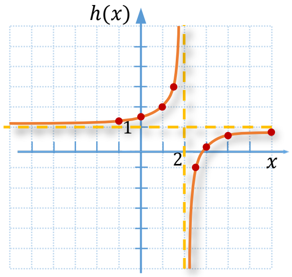


To graph function g, we evaluate it at some points to the right and to the left of -3. The reader is encouraged to check the values given in the table. Then, we draw the vertical asymptote $x = -3$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the x-axis. Indeed, the value of zero cannot be attained by the function $g(x) = \frac{-2}{x+3}$, as in order for a fraction to become zero, its numerator would have to be zero. So, the range of function g is $\mathbb{R} \setminus \{0\}$ and $\boldsymbol{y = 0}$ is the equation of the horizontal asymptote.

b. The domain of function $h(x) = \frac{x-3}{x-2}$ is $\boldsymbol{D_h} = \mathbb{R} \setminus \{\boldsymbol{2}\}$, as discussed in *Example 3b*. Since 2 is excluded from the domain, we expect the vertical asymptote to be at $\boldsymbol{x = 2}$.

x	$h(x)$
-1	$\frac{4}{3}$
0	$\frac{3}{2}$
1	2
$\frac{3}{2}$	3
2	*undefined*
$\frac{5}{2}$	-1
3	0
4	$\frac{1}{2}$
6	$\frac{3}{4}$

As before, to graph function h, we evaluate it at some points to the right and to the left of 2. Then, we draw the vertical asymptote $x = 2$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the line $\boldsymbol{y = 1}$. Thus, the range of function h is $\mathbb{R} \setminus \{\mathbf{1}\}$.



Notice that $\frac{x-3}{x-2} = \frac{x-2-1}{x-2} = \frac{x-2}{x-2} - \frac{1}{x-2} = 1 - \frac{1}{x-2}$. Since $\frac{1}{x-2}$ is never equal to zero, $1 - \frac{1}{x-2}$ is never equal to 1. This confirms the range and the horizontal asymptote stated above.

Example 5 ► Connecting the Algebraic and Graphical Solutions of Rational Equations

Given that $f(x) = \frac{x+2}{x-1}$, find all the x-values for which $f(x) = 2$. Illustrate the situation with a graph.

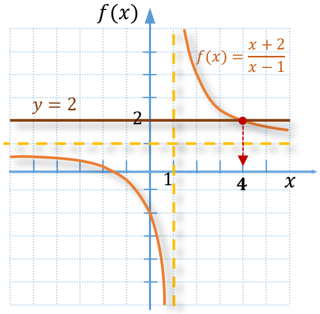
Solution ► To find all the x-values for which $f(x) = 2$, we replace $f(x)$ in the equation $f(x) = \frac{x+2}{x-1}$ with 2 and solve the resulting equation. So, we have

$$2 = \frac{x+2}{x-1}$$

$$2x - 2 = x + 2$$

$$x = 4$$

Thus, $f(x) = 2$ for $\boldsymbol{x = 4}$.



The geometrical connection can be observed by graphing the function $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$ and the line $y = 2$ on the same grid, as illustrated by the accompanying graph. The x-coordinate of the intersection of the two graphs is the solution to the equation $2 = \frac{x+2}{x-1}$. This also means that $f(4) = \frac{4+2}{4-1} = 2$. So, we can say that $f(4) = 2$.

Example 6 ► Graphing the Reciprocal of a Linear Function

Suppose $f(x) = 2x - 3$.

a. Determine the reciprocal function $g(x) = \frac{1}{f(x)}$ and its domain D_g.

b. Determine the equation of the vertical asymptote of the reciprocal function g.

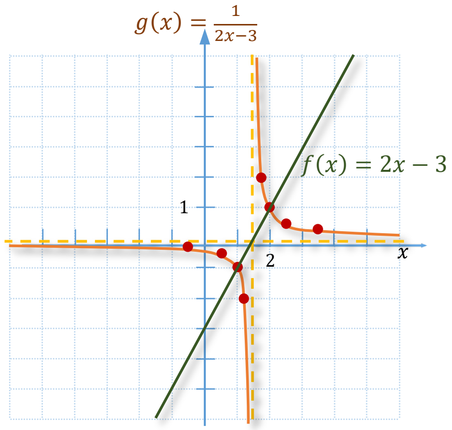
c. Graph the function f and its reciprocal function g on the same grid. Then, describe the relations between the two graphs.

Solution ► **a.** The reciprocal of $f(x) = 2x - 3$ is the function $\boldsymbol{g(x)} = \frac{\mathbf{1}}{\mathbf{2x-3}}$. Since $2x - 3 = 0$ for $x = \frac{3}{2}$, then the domain $\boldsymbol{D_g} = \mathbb{R} \setminus \left\{\frac{\mathbf{3}}{\mathbf{2}}\right\}$.

b. A vertical asymptote of a rational function in simplified form is a vertical line passing through any of the x-values that are excluded from the domain of such a function. So, the equation of the vertical asymptote of function $g(x) = \frac{1}{2x-3}$ is $\boldsymbol{x} = \frac{\mathbf{3}}{\mathbf{2}}$.

c. To graph functions f and g, we can use a table of values as below.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$	$\boldsymbol{g(x)}$
$-\frac{1}{2}$	-4	$-\frac{1}{4}$
$\frac{1}{2}$	-2	$-\frac{1}{2}$
1	-1	-1
$\frac{5}{4}$	$-\frac{1}{2}$	-2
$\frac{3}{2}$	0	*undefined*
$\frac{7}{4}$	$\frac{1}{2}$	2
2	1	1
$\frac{5}{2}$	2	$\frac{1}{2}$
$\frac{7}{2}$	4	$\frac{1}{4}$



Notice that the vertical asymptote of the reciprocal function comes through the zero of the linear function. Also, the values of both functions are positive to the right of $\frac{3}{2}$ and negative to the left of $\frac{3}{2}$. In addition, $f(2) = g(2) = 1$ and $f(1) = g(1) = -1$. This is because the reciprocal of 1 is 1 and the reciprocal of -1 is -1. For the rest of the values, observe that the values of the linear function that are very close to zero become very large in the reciprocal function and conversely, the values of the linear function that are very far from zero become very close to zero in the reciprocal function. This suggests the horizontal asymptote at zero.

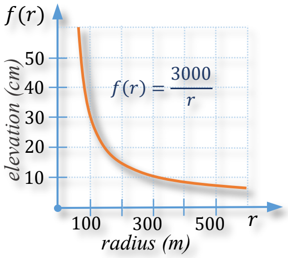
Example 7 ► Using Properties of a Rational Function in an Application Problem

Elevating the outer rail of a track allows for a safer turn of a train on a circular curve. The elevation depends on the allowable speed of the train and the radius of the curve. Suppose that a circular curve with a radius of r meters is being designed for a train travelling 100 kilometers per hour. Assume that the function $f(r) = \frac{3000}{r}$ can be used to calculate the proper elevation $y = f(r)$, in centimeters, for the outer rail.

a. Evaluate $f(300)$ and interpret the result.

b. Suppose that the outer rail for a curve is elevated 12 centimeters. Find the radius of this curve.

c. Observe the accompanying graph of the function f and discuss how the elevation of the outer rail changes as the radius r increases.



Solution

a. $f(300) = \frac{3000}{300} = 10$. Thus, the outer rail on a curve with a 300-meter radius should be elevated **10 centimeters** for a train to travel through it at 100 km/hr safely.

b. Since the elevation $y = f(r) = 12$ centimeters, to find the corresponding value of r, we need to solve the equation

$$12 = \frac{3000}{r}.$$

After multiplying this equation by r and dividing it by 12, we obtain

$$r = \frac{3000}{12} = 250$$

So, the radius of this curve should be **250 meters**.

c. As the radius increases, the outer rail needs less elevation.

RT.5 Exercises

State the ***domain*** *for each equation. There is no need to solve it.*

1. $\frac{x+5}{4} - \frac{x+3}{3} = \frac{x}{6}$

2. $\frac{5}{6a} - \frac{a}{4} = \frac{8}{2a}$

3. $\frac{3}{x+4} = \frac{2}{x-9}$

4. $\frac{4}{3x-5} + \frac{2}{x} = \frac{9}{4x+7}$

5. $\frac{4}{y^2-25} - \frac{1}{y+5} = \frac{2}{y-7}$

6. $\frac{x}{2x-6} - \frac{3}{x^2-6x+9} = \frac{x-2}{3x-9}$

Solve each equation.

7. $\frac{3}{8} + \frac{1}{3} = \frac{x}{12}$

8. $\frac{1}{4} - \frac{5}{6} = \frac{1}{y}$

9. $x + \frac{8}{x} = -9$

10. $\frac{4}{3a} - \frac{3}{a} = \frac{10}{3}$

11. $\frac{r}{8} + \frac{r-4}{12} = \frac{r}{24}$

12. $\frac{n-2}{2} - \frac{n}{6} = \frac{4n}{9}$

13. $\frac{5}{r+20}=\frac{3}{r}$

14. $\frac{5}{a+4}=\frac{3}{a-2}$

15. $\frac{y+2}{y}=\frac{5}{3}$

16. $\frac{x-4}{x+6}=\frac{2x+3}{2x-1}$

17. $\frac{x}{x-1}-\frac{x^2}{x-1}=5$

18. $3-\frac{12}{x^2}=\frac{5}{x}$

19. $\frac{1}{3}-\frac{x-1}{x}=\frac{x}{3}$

20. $\frac{1}{x}+\frac{2}{x+10}=\frac{x}{x+10}$

21. $\frac{1}{y-1}+\frac{5}{12}=\frac{-2}{3y-3}$

22. $\frac{7}{6x+3}-\frac{1}{3}=\frac{2}{2x+1}$

23. $\frac{8}{3k+9}-\frac{8}{15}=\frac{2}{5k+15}$

24. $\frac{6}{m-4}+\frac{5}{m}=\frac{2}{m^2-4m}$

25. $\frac{3}{y-2}+\frac{2y}{4-y^2}=\frac{5}{y+2}$

26. $\frac{x}{x-2}+\frac{x}{x^2-4}=\frac{x+3}{x+2}$

27. $\frac{1}{2x+10}=\frac{8}{x^2-25}-\frac{2}{x-5}$

28. $\frac{5}{y+3}=\frac{1}{4y^2-36}+\frac{2}{y-3}$

29. $\frac{6}{x^2-4x+3}-\frac{1}{x-3}=\frac{1}{4x-4}$

30. $\frac{7}{x-2}-\frac{8}{x+5}=\frac{1}{2x^2+6x-20}$

31. $\frac{5}{x-4}-\frac{3}{x-1}=\frac{x^2-1}{x^2-5x+4}$

32. $\frac{y}{y+1}+\frac{3y+5}{y^2+4y+3}=\frac{2}{y+3}$

33. $\frac{3x}{x+2}+\frac{72}{x^3+8}=\frac{24}{x^2-2x+4}$

34. $\frac{4}{x+3}+\frac{7}{x^2-3x+9}=\frac{108}{x^3+27}$

35. $\frac{x}{2x-9}-3x=\frac{10}{9-2x}$

36. $\frac{-2}{x^2+2x-3}-\frac{5}{3-3x}=\frac{4}{3x+9}$

For the given rational function f, find all values of x for which $f(x)$ has the indicated value.

37. $f(x)=2x-\frac{15}{x};\ \ f(x)=1$

38. $f(x)=\frac{x-5}{x+1};\ \ f(x)=\frac{3}{5}$

39. $g(x)=\frac{-3x}{x+3}+x;\ \ g(x)=4$

40. $g(x)=\frac{4}{x}+\frac{1}{x-2};\ \ g(x)=3$

*Graph each rational function. State its **domain**, **range** and the equations of the **vertical** and **horizontal asymptotes**.*

41. $f(x)=\frac{2}{x}$

42. $g(x)=-\frac{1}{x}$

43. $h(x)=\frac{2}{x-3}$

44. $f(x)=\frac{-1}{x+1}$

45. $g(x)=\frac{x-1}{x+2}$

46. $h(x)=\frac{x+2}{x-3}$

*For each function f, find its reciprocal function $g(x)=\frac{1}{f(x)}$ and graph both functions on the same grid. Then, state the equations of the **vertical** and **horizontal asymptotes** of function g.*

47. $f(x)=\frac{1}{2}x+1$

48. $f(x)=-x+2$

49. $f(x)=-2x-3$

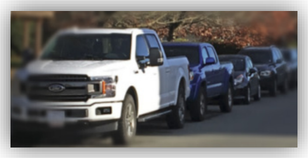
Solve each equation.

50. $\frac{x}{1+\frac{1}{x+1}} = x - 3$

51. $\frac{2-\frac{1}{x}}{4-\frac{1}{x^2}} = 1$

Solve each problem.

52. Suppose that the number of vehicles searching for a parking place at UFV parking lot is modelled by the function



$$f(x) = \frac{x^2}{2(1-x)},$$

where $0 \le x < 1$ is a quantity known as ***traffic intensity***.

a. For each traffic intensity, find the number of vehicles searching for a parking place. *Round your answer to the nearest one.*

i. 0.2 **ii.** 0.8 **iii.** 0.98

b. Observing answers to part (**a**), conclude how does the number of vehicles searching for a parking place changes when the traffic intensity get closer to 1.

53. Suppose that the percent of deaths caused by smoking, called the ***incidence rate,*** is modelled by the rational function

$$D(x) = \frac{x-1}{x},$$

where x tells us how many times a smoker is more likely to die of lung cancer than a non-smoker.

a. Find $D(10)$ and interpret it in the context of the problem.

b. Find the x-value corresponding to the incidence rate of 0.5.

c. Under what condition would the incidence rate equal to 0?

RT6 Applications of Rational Equations

In previous sections of this chapter, we studied operations on rational expressions, simplifying complex fractions, and solving rational equations. These skills are needed when working with real-world problems that lead to a rational equation. The common types of such problems are motion or work problems. In this section, we first discuss how to solve a rational formula for a given variable, and then present several examples of application problems involving rational equations.

Formulas Containing Rational Expressions

Solving application problems often involves working with formulas. We might need to form a formula, evaluate it, or solve it for a desired variable. The basic strategies used to solve a formula for a variable were shown in *Section L2* and *F4*. Recall the guidelines that we used to isolate the desired variable:

- **Reverse operations** to clear unwanted factors or addends;
  Example: To solve $\frac{A+B}{2} = C$ for A, we multiply by 2 and then subtract B.
- **Multiply by the LCD to keep** the desired variable **in the numerator**;
  Example: To solve $\frac{A}{1+r} = P$ for r, first, we multiply by $(1+r)$.
- **Take the reciprocal** of both sides of the equation **to keep** the desired variable **in the numerator** (this applies to proportions only);
  Example: To solve $\frac{1}{C} = \frac{A+B}{AB}$ for C, we can take the reciprocal of both sides to obtain $C = \frac{AB}{A+B}$.
- **Factor** to **keep** the desired variable **in one place**.
  Example: To solve $P + Prt = A$ for P, we first factor P out.

Below we show how to solve formulas containing rational expressions, using a combination of the above strategies.

Example 1 ► **Solving Rational Formulas for a Given Variable**

Solve each formula for the indicated variable.

a. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, for p **b.** $L = \frac{dR}{D-d}$, for D **c.** $L = \frac{dR}{D-d}$, for d

Solution ► **a.** ***Solution I:*** First, we isolate the term containing p, by ‘moving’ $\frac{1}{q}$ to the other side of the equation. So,

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{f} - \frac{1}{q} = \frac{1}{p}$$

rewrite from the right to the left

$$\frac{1}{p} = \frac{q-f}{fq}$$

and perform the subtraction to leave this side as a single fraction

Then, to bring p to the numerator, we can take the reciprocal of both sides of the equation, obtaining

$$\boldsymbol{p = \frac{fq}{q - f}}$$

Caution! This method can be applied only to a proportion (an equation with a **single fraction on each side**).

Solution II: The same result can be achieved by multiplying the original equation by the $LCD = fpq$, as shown below

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$pq = fq + fp$$

$$pq - fp = fq$$

factor p out

$$p(q - f) = fq$$

$$\boldsymbol{p = \frac{fq}{q - f}}$$

b. To solve $L = \frac{dR}{D-d}$ for D, we may start with multiplying the equation by the denominator to bring the variable D to the numerator. So,

This can be done in one step by interchanging L with $D - d$. The movement of the expressions resembles that of a teeter-totter.

$$L = \frac{dR}{D - d}$$

$$L(D - d) = dR$$

$$D - d = \frac{dR}{L}$$

$$\boldsymbol{D = \frac{dR}{L} + d = \frac{dR + dL}{L}}$$

Both forms are correct answers.

c. When solving $L = \frac{dR}{D-d}$ for d, we first observe that the variable d appears in both the numerator and denominator. Similarly as in the previous example, we bring the d from the denominator to the numerator by multiplying the formula by the denominator $D - d$. Thus,

$$L = \frac{dR}{D - d}$$

$$L(D - d) = dR.$$

Then, to keep the d in one place, we need to expand the bracket, collect terms with d, and finally factor the d out. So, we have

$$LD - Ld = dR$$

$$LD = dR + Ld$$

$$LD = d(R + L)$$

$$\frac{LD}{R + L} = d$$

The final formula can be written equivalently starting with d,

$$\boldsymbol{d = \frac{LD}{R + L}}.$$

Example 2 ► **Forming and Evaluating a Rational Formula**

Suppose a trip consists of two parts of the same distance d.

a. Given the speed v_1 for the first part of the trip and v_2 for the second part of the trip, find a formula for the average speed v for the whole trip. (*Make sure to leave this formula in the simplified form.*)

b. Find the average speed v for the whole trip, if the speed for the first part of the trip was 75 km/h and the speed for the second part of the trip was 105 km/h.

c. How does the v-value from (**b**) compare to the average of v_1 and v_2?

Solution ►

a. The total distance, D, for the whole trip is $d + d = 2d$. The total time, T, for the whole trip is the sum of the times for the two parts of the trip, t_1 and t_2. From the relation $rate \cdot time = distance$, we have

$$t_1 = \frac{d}{v_1} \quad \text{and} \quad t_2 = \frac{d}{v_2}.$$

Therefore,

$$t = \frac{d}{v_1} + \frac{d}{v_2},$$

which after substituting to the formula for the average speed, $V = \frac{D}{T}$, gives us

$$V = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}}.$$

Since the formula involves a complex fraction, it should be simplified. We can do this by multiplying the numerator and denominator by the $LCD = v_1 v_2$. So, we have

$$V = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}} \cdot \frac{v_1 v_2}{v_1 v_2}$$

$$V = \frac{2d v_1 v_2}{\frac{d \cancel{v_1} v_2}{\cancel{v_1}} + \frac{d v_1 \cancel{v_2}}{\cancel{v_2}}}$$

$$V = \frac{2dv_1v_2}{dv_2 + dv_1}$$

factor the d

$$V = \frac{2\cancel{d}v_1v_2}{\cancel{d}(v_2 + v_1)}$$

$$\boldsymbol{V = \frac{2v_1v_2}{v_2 + v_1}}$$

Note 1: The average speed in this formula does not depend on the distance travelled.

Note 2: The average speed for the total trip is <u>not</u> the average (arithmetic mean) of the speeds for each part of the trip. In fact, this formula represents the **harmonic mean** of the two speeds.

b. Since $v_1 = 75$ km/h and $v_2 = 105$ km/h, using the formula developed in *Example 2a*, we calculate

$$v = \frac{2 \cdot 75 \cdot 105}{75 + 105} = \frac{15750}{180} = \mathbf{87.5\ km/h}$$

c. The average speed for the whole trip, $v = 87.5$ km/h, is lower than the average of the speeds for each part of the trip, which is $\frac{75+105}{2} = 90$ km/h.

Applied Problems

Many types of application problems were already introduced in *Sections L3* and *E2*. Some of these types, for example motion problems, may involve solving rational equations. Below we show examples of proportion and motion problems as well as introduce another type of problems, work problems.

Proportion Problems

When forming a proportion,

$$\frac{category\ I\ before}{category\ II\ before} = \frac{category\ I\ after}{category\ II\ after},$$

it is essential that the same type of data is placed in the same row or the same column.

Recall: To solve a proportion

$$\frac{\boldsymbol{a}}{\boldsymbol{b}} = \frac{\boldsymbol{c}}{\boldsymbol{d}},$$

for example, for a, it is enough to multiply the equation by b. This gives us

$$\boldsymbol{a} = \frac{\boldsymbol{bc}}{\boldsymbol{d}}.$$

Similarly, to solve

$$\frac{a}{b} = \frac{\textcircled{c}}{d}$$

for b, we can use the cross-multiplication method, which eventually (we encourage the reader to check this) leads us to

$$a = \frac{ad}{c}.$$

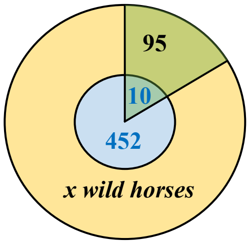
Notice that in both cases the desired variable equals the **product** of the blue variables lying **across** each other, **divided by the remaining** purple variable. This is often referred to as the ‘cross multiply and divide’ approach to solving a proportion.

In statistics, proportions are often used to estimate the population by analysing its sample in situations where the exact count of the population is too costly or not possible to obtain.

Example 3 ► **Estimating Numbers of Wild Animals**

To estimate the number of wild horses in a particular area in Nevada, a forest ranger catches 452 wild horses, tags them, and releases them. In a week, he catches 95 horses out of which 10 are found to be tagged. Assuming that the horses mix freely when they are released, estimate the number of wild horses in this region. *Round your answer to the nearest hundreds.*

Solution ►



Suppose there are x wild horses in region. 452 of them were tagged, so the ratio of the tagged horses to the whole population of the wild horses there is

$$\frac{452}{x}$$

The ratio of the tagged horses found in the sample of 95 horses caught in the later time is

$$\frac{10}{95}$$

So, we form the proportion:

$$\underbrace{\frac{452}{x}}_{\text{population}} = \underbrace{\frac{10}{95}}_{\text{sample}} \begin{array}{l} \leftarrow \text{tagged horses} \\ \leftarrow \text{all horses} \end{array}$$

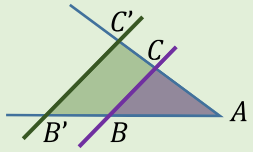
After solving for x, we have

$$x = \frac{452 \cdot 95}{10} = 4294 \approx 4300$$

So, we can say that approximatly **4300** wild horses live in this region.

In geometry, proportions are the defining properties of similar figures. One frequently used theorem that involves proportions is the theorem about similar triangles, attributed to the Greek mathematician Thales.

Thales' Theorem ► Two triangles are **similar** iff the ratios of the corresponding sides are the same.



$$\triangle ABC \sim \triangle AB'C' \Leftrightarrow \frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'}$$

Example 4 ► **Using Similar Triangles in an Application Problem**

A cross-section of a small storage room is in the shape of a right triangle with a height of 2 meters and a base of 1.2 meters, as shown in *Figure 6.1a*. Find the size of the largest cubic box fitting in this room when placed with its base on the floor.

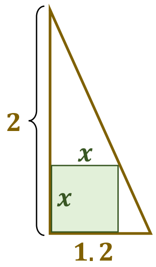


Figure 6.1a

Solution ► Suppose that the height of the box is x meters. Since the height of the storage room is 2 meters, the expression $2 - x$ represents the height of the wall above the box, as shown in *Figure 6.1b.*

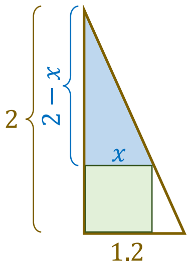


Figure 6.1b

Since the blue and brown triangles are similar, we can use the Thales' Theorem to form the proportion

$$\frac{2-x}{2} = \frac{x}{1.2}.$$

Employing cross-multiplication, we obtain

$$2.4 - 1.2x = 2x$$
$$2.4 = 3.2x$$
$$x = \frac{2.4}{3.2} = \mathbf{0.75}$$

So, the dimensions of the largest cubic box fitting in this storage room are 75 cm by 75 cm by 75 cm.

Motion Problems

Motion problems in which we compare times usually involve solving rational equations. This is because when solving the motion formula $rate\ \boldsymbol{R} \cdot time\ \boldsymbol{T} = distance\ \boldsymbol{D}$ for time, we create a fraction

$$time\ \boldsymbol{T} = \frac{distance\ \boldsymbol{D}}{rate\ \boldsymbol{R}}$$

Example 5 ► Solving a Motion Problem Where Times are the Same

Two bikers participate in a Cross-Mountain Crusher. One biker is 2 km/h faster than the other. The faster biker travels 35 km in the same amount of time that it takes the slower biker to cover only 30 km. Find the average speed of each biker.

Solution ► Let r represent the average speed of the slower biker. Then $r + 2$ represents the average speed of the faster biker. The slower biker travels 30 km, while the faster biker travels 35 km. Now, we can complete the table

	R $\cdot$	T	$= D$
slower biker	r	$\frac{30}{r}$	30
faster biker	$r+2$	$\frac{35}{r+2}$	35

To complete the ***Time*** column, we divide the ***Distance*** by the ***Rate***.

Since the time of travel is the same for both bikers, we form and then solve the equation:

$$\frac{30}{r} = \frac{35}{r+2}$$

$$6(r+2) = 7r$$

$$6r + 12 = 7r$$

$$r = 12$$

Thus, the average speed of the slower biker is $r = \mathbf{12}$ km/h and the average speed of the faster biker is $r + 2 = \mathbf{14}$ km/h.

Example 6 ► Solving a Motion Problem Where the Total Time is Given

Judy and Nathan drive from Abbotsford to Kelowna, a distance of 322 km. Judy's average driving rate is 5 km/h faster than Nathan's. Judy got tired after driving the first 154 kilometers, so Nathan drove the remaining part of the trip. If the total driving time was 3 hours, what was the average rate of each driver?

Solution ► Let r represent Nathan's average rate. Then $r + 5$ represents Judy's average rate. Since Judy drove 154 km, Nathan drove $322 - 154 = 168$ km. Now, we can complete the table:

	R $\cdot$	T	$= D$
Judy	$r+5$	$\frac{154}{r+5}$	154
Nathan	r	$\frac{168}{r}$	168
total		3	322

Note: In motion problems we may add *times* or *distances* but we usually **do not add rates!**

The equation to solve comes from the ***Time*** column.

$$\frac{154}{r+5}+\frac{168}{r}=3$$

$$154r+168(r+5)=3r(r+5)$$

$$154r+168r+840=3r^2+15r$$

$$0=3r^2-307r-840$$

$$(3r+8)(r-105)=0$$

$$r=-\frac{8}{3} \quad or \quad r=105$$

Since a rate cannot be negative, we discard the solution $r=-\frac{8}{3}$. Therefore, Nathan's average rate was $r=\mathbf{105}$ km/h and Judy's average rate was $r+5=\mathbf{110}$ km/h.

Work Problems

Notice the similarity to the formula $R \cdot T = D$ used in motion problems.

When solving work problems, refer to the formula

$$\boldsymbol{Rate}\ of\ work \cdot \boldsymbol{Time} = amount\ of\ \boldsymbol{Job}\ completed$$

and organize data in a table like this:

	R ·	***T***	= ***J***
worker I			
worker II			
together			

Note: In work problems we usually *add rates* but **do not add times!**

Example 7 ► **Solving a Work Problem Involving Addition of Rates**

Adam can trim the shrubs at Centralia College in 8 hr. Bruce can do the same job in 6 hr. To the nearest minute, how long would it take them to complete the same trimming job if they work together?

Solution ► Let t be the time needed to trim the shrubs when Adam and Bruce work together. Since trimming the shrubs at Centralia College is considered to be the whole one job to complete, then the rate R in which this work is done equals

$$\boldsymbol{R}=\frac{\boldsymbol{Job}}{\boldsymbol{Time}}=\frac{\mathbf{1}}{\boldsymbol{Time}}.$$

To organize the information, we can complete the table below.

	$R\ \cdot$	T	$= J$
Adam	$\frac{1}{8}$	8	1
Bruce	$\frac{1}{6}$	6	1
together	$\frac{1}{t}$	t	1

The job column is often equal to **1**, although sometimes other values might need to be used.

To complete the ***Rate*** column, we divide the ***Job*** by the ***Time***.

Since the rate of work when both Adam and Bruce trim the shrubs is the sum of rates of individual workers, we form and solve the equation

$$\frac{1}{8}+\frac{1}{6}=\frac{1}{t}$$

$$3t+4t=24$$

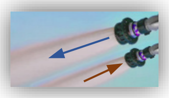
$$7t=24$$

$$t=\frac{24}{7}\approx 3.43$$

So, if Adam and Bruce work together, the amount of time needed to complete the job would be approximately 3.43 hours ≈ **3 hours 26 minutes**.

Note: The time needed for both workers is **shorter** than either of the individual times.

Example 8 ► Solving a Work Problem Involving Subtraction of Rates



The inlet pipe can fill a swimming pool in 4 hours, while the outlet pipe can empty the pool in 5 hours. If both pipes were left open, how long would it take to fill the pool?

Solution ► Suppose t is the time needed to fill the pool when both pipes are left open. If filling the pool is the whole one job to complete, then emptying the pool corresponds to -1 job. This is because when emptying the pool, we reverse the filling job.

To organize the information given in the problem, we complete the following table.

	$R\ \cdot$	T	$= J$
inlet pipe	$\frac{1}{4}$	4	1
outlet pipe	$-\frac{1}{5}$	5	-1
both pipes	$\frac{1}{t}$	t	1

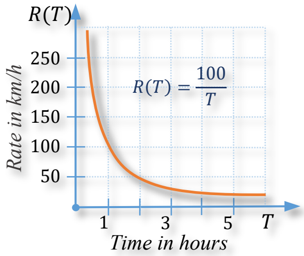
The equation to solve comes from the ***Rate*** column.

$$\frac{1}{4}-\frac{1}{5}=\frac{1}{t}$$
$$5t-4t=20$$
$$t=20$$

So, it will take **20 hours** to fill the pool when both pipes are left open.

Inverse and Combined Variation

When two quantities vary in such a way that their **product remains constant**, we say that they are **inversely proportional**. For example, consider rate R and time T of a moving object that covers a constant distance D. In particular, if $D=100$ km, we have



$$R=\frac{100}{T}=100\cdot\frac{1}{T}$$

This relation tells us that the rate is 100 times larger than the reciprocal of time. Observe though that when the time doubles, the rate is half as large. When the time triples, the rate is three times smaller, and so on. One can observe that the rate decreases proportionally to the increase of time. Such a **reciprocal relation** between the two quantities is called an **inverse variation**.

Definition 6.1 ► Two quantities, $\boldsymbol{x}$ and $\boldsymbol{y}$, are **inversely proportional** to each other (there is an **inverse variation** between them) iff there is a real constant $\boldsymbol{k\neq 0}$, such that

$$\boldsymbol{y=\frac{k}{x}}.$$

We say that $\boldsymbol{y}$ **varies inversely** as $\boldsymbol{x}$ with the **variation constant** $\boldsymbol{k}$.
(or equivalently: $\boldsymbol{y}$ is **inversely proportional to** $\boldsymbol{x}$ with the **proportionality constant** $\boldsymbol{k}$.)

Example 9 ► **Solving Inverse Variation Problems**

The volume V of a gas is inversely proportional to the pressure P of the gas. If a pressure of 30 kg/cm^2 corresponds to a volume of 240 cm^3, find the following:

a. The equation that relates V and P,
b. The pressure needed to produce a volume of 150 cm^3.

Solution ► **a.** To find the inverse variation equation that relates V and P, we need to find the variation constant k first. This can be done by substituting $V=240$ and $P=30$ into the equation $V=\frac{k}{P}$. So, we obtain

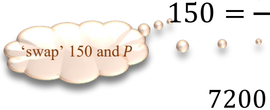
$$240=\frac{k}{30}$$
$$k=7200.$$

Therefore, our equation is $\boldsymbol{V=\frac{7200}{P}}$.

b. The required pressure can be found by substituting $V = 150$ into the inverse variation equation,

$$150 = \frac{7200}{P}.$$

This gives us



$$P = \frac{7200}{150} = 48.$$

So, the pressure of the gas that assumes the volume of 150 cm^3 is **48 kg/cm^2**.

Extension: We say that $\boldsymbol{y}$ **varies inversely** as the $\boldsymbol{n}$**-th power** of $\boldsymbol{x}$ iff $\boldsymbol{y} = \frac{\boldsymbol{k}}{\boldsymbol{x^n}}$, for some nonzero constant $\boldsymbol{k}$.

Example 10 ► **Solving an Inverse Variation Problem Involving the Square of a Variable**

The intensity of light varies inversely as the square of the distance from the light source. If 4 meters from the source the intensity of light is 9 candelas, what is the intensity of this light 3 meters from the source?

Solution ► Let I represents the intensity of the light and d the distance from the source of this light. Since I varies inversely as d^2, we set the equation

$$I = \frac{k}{d^2}$$

After substituting the data given in the problem, we find the value of k:

$$9 = \frac{k}{4^2}$$

$$k = 9 \cdot 16 = 144$$

So, the inverse variation equation is $I = \frac{144}{d^2}$. Hence, the light intensity at 3 meters from the source is $I = \frac{144}{3^2} = \mathbf{16}$ **candelas**.

Recall from *Section L2* that two variables, say $\boldsymbol{x}$ and $\boldsymbol{y}$, vary **directly** with a proportionality constant $\boldsymbol{k \neq 0}$ if $\boldsymbol{y = kx}$. Also, we say that one variable, say $\boldsymbol{z}$, varies **jointly** as other variables, say $\boldsymbol{x}$ and $\boldsymbol{y}$, with a proportionality constant $\boldsymbol{k \neq 0}$ if $\boldsymbol{z = kxy}$.

Definition 6.2 ► A combination of the **direct** or **joint** variation with the **inverse** variation is called a **combined variation.**

Example:
$\boldsymbol{w}$ may vary **jointly** as $\boldsymbol{x}$ and $\boldsymbol{y}$ and **inversely** as the square of $\boldsymbol{z}$. This means that there is a real constant $\boldsymbol{k \neq 0}$, such that

$$\boldsymbol{w = \frac{kxy}{z^2}}.$$

Example 11 ► **Solving Combined Variation Problems**

The resistance of a cable varies directly as its length and inversely as the square of its diameter. A 20-meter cable with a diameter of 1.2 cm has a resistance of 0.2 ohms. A 50-meter cable with a diameter of 0.6 cm is made out of the same material. What would be its resistance?

Solution ► Let R, l, and d represent respectively the resistance, length, and diameter of a cable. Since R varies directly as l and inversely as d^2, we set the combined variation equation

$$R = \frac{kl}{d^2}.$$

Substituting the data given in the problem, we have

$$0.2 = \frac{k \cdot 20}{1.2^2},$$

which gives us

$$k = \frac{0.2 \cdot 1.44}{20} = 0.0144$$

So, the combined variation equation is $R = \frac{0.0144l}{d^2}$. Therefore, the resistance of a 50-meter cable with the diameter of 0.6 cm is $R = \frac{0.0144 \cdot 50}{0.6^2} =$ **2 ohms**.

RT.6 Exercises

1. Using the formula $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, find q if $r = 6$ and $p = 10$.
2. The gravitational force between two masses is given by the formula $F = \frac{GMm}{d^2}$. Find M if $F = 20$, $G = 6.67 \cdot 10^{-11}$, $m = 1$, and $d = 4 \cdot 10^{-6}$. *Round your answer to one decimal place.*
3. What is the first step in solving the formula $ka + kb = a - b$ for k?
4. What is the first step in solving the formula $A = \frac{pq}{q-p}$ for p?

Solve each formula for the specified variable.

5. $m = \frac{F}{a}$ for a
6. $I = \frac{E}{R}$ for R
7. $\frac{W_1}{W_2} = \frac{d_1}{d_2}$ for d_1
8. $F = \frac{GMm}{d^2}$ for m
9. $s = \frac{(v_1+v_2)t}{2}$ for t
10. $s = \frac{(v_1+v_2)t}{2}$ for v_1

11. $\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}$ for R

12. $\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}$ for r_1

13. $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$ for q

14. $\frac{t}{a}+\frac{t}{b}=1$ for a

15. $\frac{PV}{T}=\frac{pv}{t}$ for v

16. $\frac{PV}{T}=\frac{pv}{t}$ for T

17. $A=\frac{h(a+b)}{2}$ for b

18. $a=\frac{V-v}{t}$ for V

19. $R=\frac{gs}{g+s}$ for s

20. $I=\frac{2V}{V+2r}$ for V

21. $I=\frac{nE}{E+nr}$ for n

22. $\frac{E}{e}=\frac{R+r}{r}$ for e

23. $\frac{E}{e}=\frac{R+r}{r}$ for r

24. $S=\frac{H}{m(t_1-t_2)}$ for t_1

25. $V=\frac{\pi h^2(3R-h)}{3}$ for R

26. $P=\frac{A}{1+r}$ for r

27. $\frac{V^2}{R^2}=\frac{2g}{R+h}$ for h

28. $v=\frac{d_2-d_1}{t_2-t_1}$ for t_2

Solve each problem.

29. The ratio of the weight of an object on Earth to the weight of an object on the moon is 200 to 33. What would be the weight of a 75-kg astronaut on the moon?

30. A 30-meter long ribbon is cut into two sections. How long are the two sections if the ratio of their lengths is 5 to 7?

31. Assume that burning 7700 calories causes a decrease of 1 kilogram in body mass. If walking 7 kilometers in 2 hours burns 700 calories, how many kilometers would a person need to walk at the same rate to lose 1 kg?

32. On a map of Canada, the linear distance between Vancouver and Calgary is 1.8 cm. The flight distance between the two cities is about 675 kilometers. On this same map, what would be the linear distance between Calgary and Montreal if the flight distance between the two cities is approximately 3000 kilometers?

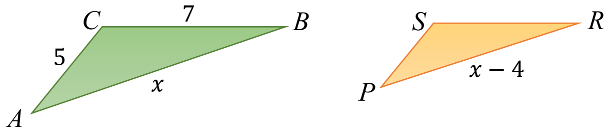
33. To estimate the population of Cape Mountain Zebra in South Africa, biologists caught, tagged, and then released 68 Cape Mountain Zebras. In a month, they caught a random sample of 84 of this type of zebras. It turned out that 5 of them were tagged. Assuming that zebras mixed freely, approximately how many Cape Mountain Zebras lived in South Africa?

34. To estimate the number of white bass fish in a particular lake, biologists caught, tagged, and then released 300 of this fish. In two weeks, they returned and collected a random sample of 196 white bass fish. This sample contained 12 previously tagged fish. Approximately how many white bass fish does the lake have?

35. Eighteen white-tailed eagles are tagged and released into the wilderness. In a few weeks, a sample of 43 white-tailed eagles was examined, and 5 of them were tagged. Estimate the white-tailed eagle population in this wilderness area.

36. A meter stick casts a 64 cm long shadow. At the same time, a 15-year old cottonwood tree casts an 18-meter long shadow. To the nearest meter, how tall is the tree?

37. The ratio of corresponding sides of similar triangles is 5 to 3. The two shorter sides of the larger triangle are 5 and 7 units long, correspondingly. Find the length of each side of the smaller triangle if its longest side is 4 units shorter than the corresponding side of the larger triangle.



38. The width of a rectangle is the same as the length of a similar rectangle. If the dimensions of the smaller rectangle are 7 cm by 12 cm, what are the dimensions of the larger rectangle?

39. Justin runs twice around a park. He averages 20 kilometers per hour during the first round and only 16 kilometers per hour during the second round. What is his average speed for the whole run? *Round your answer to one decimal place.*

40. Robert runs twice around a stadium. He averages 18 km/h during the first round. What should his average speed be during the second round to have an overall average of 20 km/h for the whole run?

41. Jim's boat moves at 20 km/h in still water. Suppose it takes the same amount of time for Jim to travel by his boat either 15 km downriver or 10 km upriver. Find the rate of the current.

42. The average speed of a plane flying west was 880 km/h. On the return trip, the same plane averaged only 620 km/h. If the total flying time in both directions was 6 hours, what was the one-way distance?

43. A plane flies 3800 kilometers with the wind, while only 3400 kilometers against the same wind. If the airplane speed in still air is 900 km/h, find the speed of the wind.

44. Walking on a moving sidewalk, Sarah could travel 40 meters forward in the same time it would take her to travel 15 meters in the opposite direction. If the rate of the moving sidewalk was 35 m/min, what was Sarah's rate of walking?

45. Arthur travelled by car from Madrid to Paris. He usually averages 100 km/h on such trips. This time, due to heavier traffic and few stops, he averaged only 85 km/h, and he reached his destination 2 hours 15 minutes later than expected. How far did Arthur travel?

46. Tony averaged 100 km/h on the first part of his trip to Lillooet, BC. The second part of his trip was 20 kilometers longer than the first, and his average speed was only 80 km/h. If the second part of the trip took him 30 minutes longer than the first part, what was the overall distance travelled by Tony?

47. Page is a college student who lives in a near-campus apartment. When she rides her bike to campus, she gets there 24 min faster than when she walks. If her average walking rate is 4 km/h and her average biking rate is 20 km/h, how far does she live from the campus?

48. Sonia can respond to all the daily e-mails in 2 hours. Betty needs 3 hours to do the same job. If they both work on responding to e-mails, what portion of this daily job can be done in 1 hour? How much more time would they need to complete the job?

49. Brenda can paint a deck in x hours, while Tony can do the same job in y hours. Write a rational expression that represents the portion of the deck that can be painted by both of them in 4 hours.

50. Aaron and Ben plan to paint a house. Aaron needs 24 hours to paint the house by himself. Ben needs 18 to do the same job. To the nearest minute, how long would it take them to paint the house if they work together?

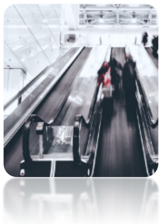
51. When working together, Adam and Brian can paint a house in 6 hours. Brian could paint this house on his own in 10 hours. How long would it take Adam to paint the house working alone?

52. An experienced floor installer can install a parquet floor twice as fast as an apprentice. Working together, it takes the two workers 2 days to install the floor in a particular house. How long would it take the apprentice to do the same job on his own?

53. A pool can be filled in 8 hr and drained in 12 hr. On one occasion, when filling the pool, the drain was accidentally left open. How long did it take to fill this pool?

54. One inlet pipe can fill a hot tub in 15 minutes. Another inlet pipe can fill the tub in 10 minutes. An outlet pipe can drain the hot tub in 18 minutes. How long would it take to fill the hot tub if all three pipes are left open?

55. Two different width escalators can empty a 1470-people auditorium in 12 min. If the wider escalator can move twice as many people as the narrower one, how many people per hour can the narrower escalator move?



56. At what times between 3:00 and 4:00 are the minute and hour hands perfectly lined up?

57. If Miranda drives to work at an average speed of 60 km/h, she is 1 min late. When she drives at an average speed of 75 km/h, she is 3 min early. How far is Miranda's workplace from her home?

58. The current in an electrical circuit at a constant potential varies inversely as the resistance of the circuit. Suppose that the current I is 9 amperes when the resistance R is 10 ohms. Find the current when the resistance is 6 ohms.

59. Assuming the same rate of work for all workers, the number of workers needed for a job varies inversely as the time required to complete the job. If it takes 3 hours for 8 workers to build a deck, how long would it take two workers to build the same deck?

60. The length of a guitar string is inversely proportional to the frequency of the string vibrations. Suppose a 60-cm long string vibrates at a frequency of 500 Hz (*1 hertz = one cycle per second*). What is the frequency of the same string when it is shortened to 50 centimeters?

61. A musical tone's pitch is inversely proportional to its wavelength. If a wavelength of 2.2 meters corresponds to a pitch of 420 vibrations per second, find the wavelength of a tone with a pitch of 660 vibrations per second.

62. The intensity, I, of a television signal is inversely proportional to the square of the distance, d, from a transmitter. If 2 km away from the transmitter the intensity is 25 W/m^2 (watts per square meter), how far from the transmitter is a TV set that receives a signal with the intensity of 2.56 W/m^2?

63. The weight W of an object is inversely proportional to the square of the distance D from the center of Earth. To the nearest kilometer, how high above the surface of Earth must a 60-kg astronaut be to weigh half as much? Assume the radius of Earth to be 6400 km.

64. The number of long-distance phone calls between two cities during a specified period in time varies jointly as the populations of the cities, P_1 and P_2, and inversely as the distance between them. Suppose 80,000 calls are made between two cities that are 400 km apart and have populations of 70,000 and 100,000. How many calls are made between Vancouver and Abbotsford that are 70 km apart and have populations of 630,000 and 140,000, respectively?

65. The force that keeps a car from skidding on a curve is inversely proportional to the radius of the curve and jointly proportional to the weight of the car and the square of its speed. Knowing that a force of 880 N (Newtons) keeps an 800-kg car moving at 50 km/h from skidding on a curve of radius 160 m, estimate the force that would keep the same car moving at 80 km/h from skidding on a curve of radius 200 meters.

66. Suppose that the renovation time is inversely proportional to the number of workers hired for the job. Will the renovation time decrease more when hiring additional 2 workers in a 4-worker company or a 6-worker company? Justify your answer.

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Radicals and Radical Functions

So far we have discussed polynomial and rational expressions and functions. In this chapter, we study algebraic expressions that contain radicals. For example, $3+\sqrt{2}$, $\sqrt[3]{x}-1$, or $\frac{1}{\sqrt{5x-1}}$. Such expressions are called ***radical expressions***. Familiarity with radical expressions is essential when solving a wide variety of problems. For instance, in algebra, some polynomial or rational equations have radical solutions that need to be simplified. In geometry, due to the frequent use of the Pythagorean equation, $a^2+b^2=c^2$, the exact distances are often radical expressions. In sciences, many formulas involve radicals.

We begin the study of radical expressions with defining radicals of various degrees and discussing their properties. Then, we show how to simplify radicals and radical expressions, and introduce operations on radical expressions. Finally, we study the methods of solving radical equations. In addition, similarly as in earlier chapters where we looked at the related polynomial and rational functions, we will also define and look at properties of radical functions.

RD1 Radical Expressions, Functions, and Graphs

Roots and Radicals

The operation of taking a **square root** of a number is the **reverse operation of squaring** a number. For example, a square root of 25 is 5 because raising 5 to the second power gives us 25.

Note: Observe that raising -5 to the second power also gives us 25. So, the square root of 25 could have two answers, 5 or -5. To avoid this duality, we choose the **nonnegative value**, called the **principal square root**, for the value of a square root of a number.

The operation of taking a square root is denoted by the symbol $\sqrt{}$. So, we have

$$\sqrt{25}=5,\quad \sqrt{0}=0,\quad \sqrt{1}=1,\quad \sqrt{9}=3, \text{ etc.}$$

What about $\sqrt{-4}=?$ Is there a number such that when it is squared, it gives us -4 ?

Since the square of any real number is nonnegative, the square root of a negative number is not a real number. So, when working in the set of real numbers, we can conclude that

$$\sqrt{\boldsymbol{positive}}=\boldsymbol{positive},\quad \sqrt{0}=0,\quad \text{and}\quad \sqrt{\boldsymbol{negative}}=\boldsymbol{DNE}$$

does not exist

The operation of taking a **cube root** of a number is the **reverse operation of cubing** a number. For example, a cube root of 8 is 2 because raising 2 to the third power gives us 8.

This operation is denoted by the symbol $\sqrt[3]{}$. So, we have

$$\sqrt[3]{8}=2,\quad \sqrt[3]{0}=0,\quad \sqrt[3]{1}=1,\quad \sqrt[3]{27}=3, \text{ etc.}$$

Note: Observe that $\sqrt[3]{-8}$ exists and is equal to -2. This is because $(-2)^3 = -8$. Generally, a cube root can be applied to any real number and the **sign** of the resulting value **is the same** as the sign of the original number.

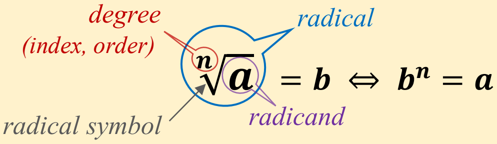
Thus, we have

$$\sqrt[3]{\boldsymbol{positive}} = \boldsymbol{positive}, \quad \sqrt[3]{0} = 0, \quad \text{and} \quad \sqrt[3]{\boldsymbol{negative}} = \boldsymbol{negative}$$

The square or cube roots are special cases of n-th degree radicals.

Definition 1.1 ► The ***n*-th degree radical** of a number a is a number b such that $\boldsymbol{b^n = a}$.

Notation:



$$\sqrt[n]{a} = b \Leftrightarrow b^n = a$$

For example, $\sqrt[4]{16} = 2$ because $2^4 = 16$,
$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$,
$\sqrt[3]{0.027} = 0.3$ because $(0.3)^3 = 0.027$.

Notice: A square root is a second degree radical, customarily denoted by $\sqrt{}$ rather than $\sqrt[2]{}$.

Example 1 ► **Evaluating Radicals**

Evaluate each radical, if possible.

a. $\sqrt{0.64}$ **b.** $\sqrt[3]{125}$ **c.** $\sqrt[4]{-16}$ **d.** $\sqrt[5]{-\frac{1}{32}}$

Solution ► **a.** Since $0.64 = (0.8)^2$, then $\sqrt{0.64} = 0.8$.

take half of the decimal places

Advice: To become fluent in evaluating square roots, it is helpful to be familiar with the following perfect square numbers:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, …, 400, …, 625, …

b. $\sqrt[3]{125} = 5$ as $5^3 = 125$

Advice: To become fluent in evaluating cube roots, it is helpful to be familiar with the following cubic numbers:
1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …

c. $\sqrt[4]{-16}$ is not a real number as there is no real number which raised to the 4-th power becomes negative.

d. $\sqrt[5]{-\frac{1}{32}} = -\frac{1}{2}$ as $\left(-\frac{1}{2}\right)^5 = -\frac{1^5}{2^5} = -\frac{1}{32}$

Note: Observe that $\frac{\sqrt[5]{-1}}{\sqrt[5]{32}} = \frac{-1}{2}$, so $\sqrt[5]{-\frac{1}{32}} = \frac{\sqrt[5]{-1}}{\sqrt[5]{32}}$.

Generally, to take a radical of a quotient, $\sqrt[n]{\frac{a}{b}}$, it is the same as to take the quotient of radicals, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Example 2 ► Evaluating Radical Expressions

Evaluate each radical expression.

a. $-\sqrt{121}$ **b.** $-\sqrt[3]{-64}$ **c.** $\sqrt[4]{(-3)^4}$ **d.** $\sqrt[3]{(-6)^3}$

Solution ►

a. $-\sqrt{121} = -11$

b. $-\sqrt[3]{-64} = -(-4) = 4$

c. $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$

the result is positive

Note: If n is **even**, then $\sqrt[n]{\boldsymbol{a^n}} = \begin{cases} a, & if\ a \geq 0 \\ -a, & if\ a < 0 \end{cases} = |\boldsymbol{a}|$.

For example, $\sqrt{7^2} = 7$ and $\sqrt{(-7)^2} = 7$.

d. $\sqrt[3]{(-6)^3} = \sqrt[3]{-216} = -6$

the result has the same sign

Note: If n is **odd**, then $\sqrt[n]{\boldsymbol{a^n}} = \boldsymbol{a}$. For example, $\sqrt[3]{5^3} = 5$ but $\sqrt[3]{(-5)^3} = -5$.

Summary of Properties of n-th Degree Radicals

- If n is **EVEN**, then

$$\sqrt[n]{\boldsymbol{positive}} = \boldsymbol{positive}, \quad \sqrt[n]{\boldsymbol{negative}} = \boldsymbol{DNE}, \quad \text{and} \quad \sqrt[n]{\boldsymbol{a^n}} = |\boldsymbol{a}|$$

- If n is **ODD**, then

$$\sqrt[n]{\boldsymbol{positive}} = \boldsymbol{positive}, \quad \sqrt[n]{\boldsymbol{negative}} = \boldsymbol{negative}, \quad \text{and} \quad \sqrt[n]{\boldsymbol{a^n}} = \boldsymbol{a}$$

- For any natural $n \geq 0$, $\sqrt[n]{\mathbf{0}} = \mathbf{0}$ and $\sqrt[n]{\mathbf{1}} = \mathbf{1}$.

Example 3 ► **Simplifying Radical Expressions Using Absolute Value Where Appropriate**

Simplify each radical, assuming that all variables represent any real number.

a. $\sqrt{9x^2y^4}$ **b.** $\sqrt[3]{-27y^3}$ **c.** $\sqrt[4]{a^{20}}$ **d.** $-\sqrt[4]{(k-1)^4}$

Solution ►

a. $\sqrt{9x^2y^4} = \sqrt{(3xy^2)^2} = |3xy^2| = \boldsymbol{3|x|y^2}$

An **even** degree radical is **nonnegative**, so we must use the **absolute value** operator.

Recall: As discussed in *Section L6*, the absolute value operator has the following properties:

$$|\boldsymbol{xy}| = |\boldsymbol{x}||\boldsymbol{y}|$$

$$\left|\frac{\boldsymbol{x}}{\boldsymbol{y}}\right| = \frac{|\boldsymbol{x}|}{|\boldsymbol{y}|}$$

Note: $|y^2| = y^2$ as y^2 is already nonnegative.

b. $\sqrt[3]{-27y^3} = \sqrt[3]{(-3y)^3} = \boldsymbol{-3y}$

An **odd** degree radical assumes the sign of the radicand, so we **do not apply the absolute value** operator.

c. $\sqrt[4]{a^{20}} = \sqrt[4]{(a^5)^4} = |a^5| = \boldsymbol{|a|^5}$

Note: To simplify an expression with an absolute value, we keep the absolute value operator as close as possible to the variable(s).

d. $-\sqrt[4]{(k-1)^4} = \boldsymbol{-|k-1|}$

Radical Functions

Since each nonnegative real number x has exactly one principal square root, we can define the **square root** function, $\boldsymbol{f(x)=\sqrt{x}}$. The **domain** D_f of this function is the set of nonnegative real numbers, $[\mathbf{0},\infty)$, and so is its **range** (as indicated in *Figure 1*).

To graph the square root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect square numbers. However, sometimes we want to use additional x-values that are not perfect squares. Since a square root of such a number, for example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$, etc., is an irrational number, we approximate these values using a calculator.

x	y
$\mathbf{0}$	0
$\mathbf{\frac{1}{4}}$	$\frac{1}{2}$
$\mathbf{1}$	1
$\mathbf{4}$	2
$\mathbf{6}$	$\sqrt{6}\approx 2.4$

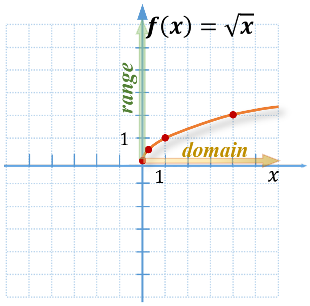


Figure 1

For example, to approximate $\sqrt{6}$, we use the sequence of keying: √ 6 *ENTER* or 6 ^ (1 / 2) *ENTER*. This is because a square root operator works the same way as the exponent of $\frac{1}{2}$.

Note: When graphing an even degree radical function, it is essential that we find its domain first. The end-point of the domain indicates the starting point of the graph, often called the vertex.
For example, since the domain of $f(x)=\sqrt{x}$ is $[0,\infty)$, the graph starts from the point $\big(0,f(0)\big)=(0,0)$, as in *Figure 1*.

Since the cube root can be evaluated for any real number, the **domain** D_f of the related **cube root** function, $\boldsymbol{f(x)=\sqrt[3]{x}}$, is the set of **all real numbers**, $\mathbb{R}$. The **range** can be observed in the graph (see *Figure 2*) or by inspecting the expression $\sqrt[3]{x}$. It is also $\mathbb{R}$.

To graph the cube root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect cube numbers. As before, sometimes we might need to estimate additional x-values. For example, to approximate $\sqrt[3]{6}$, we use the sequence of keying:
∛ 6 *ENTER* or
6 ^ (1 / 3) *ENTER*.

x	y
$\mathbf{-8}$	-2
$\mathbf{-6}$	$-\sqrt[3]{6}\approx -1.8$
$\mathbf{-1}$	-1
$\mathbf{-\frac{1}{8}}$	$-\frac{1}{2}$
$\mathbf{0}$	0
$\mathbf{\frac{1}{8}}$	$\frac{1}{2}$
$\mathbf{1}$	1
$\mathbf{6}$	$\sqrt[3]{6}\approx 1.8$
$\mathbf{8}$	2

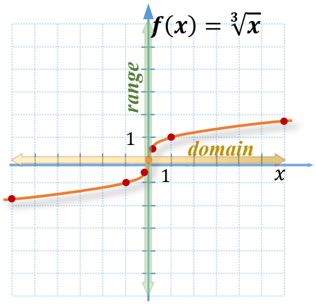


Figure 2

Example 4 ► **Finding a Calculator Approximations of Roots**

Use a calculator to approximate the given root up to three decimal places.

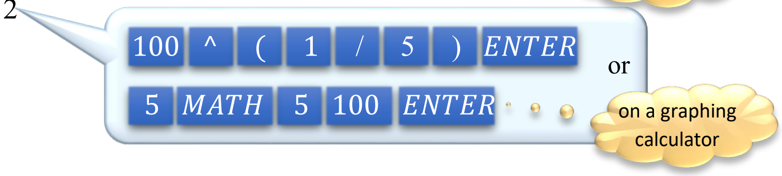
a. $\sqrt{3}$ **b.** $\sqrt[3]{5}$ **c.** $\sqrt[5]{100}$

Solution ►

a. $\sqrt{3} \approx 1.732$

b. $\sqrt[3]{5} \approx 1.710$

c. $\sqrt[5]{100} \approx 2.512$



Example 5 ► **Finding the Best Integer Approximation of a Square Root**

Without the use of a calculator, determine the best integer approximation of the given root.

a. $\sqrt{68}$ **b.** $\sqrt{140}$

Solution ►

a. Observe that 68 lies between the following two consecutive perfect square numbers, 64 and 81. Also, 68 lies closer to 64 than to 81. Therefore, $\sqrt{68} \approx \sqrt{64} = 8$.

b. 140 lies between the following two consecutive perfect square numbers, 121 and 144. In addition, 140 is closer to 144 than to 121. Therefore, $\sqrt{140} \approx \sqrt{144} = 12$.

Example 6 ► **Finding the Domain of a Radical Function**

Find the domain of each of the following functions.

a. $f(x) = \sqrt{2x+3}$ **b.** $g(x) = 2 - \sqrt{1-x}$

Solution ►

The **domain** of an **even degree** radical is the solution set of the inequality ***radicand*** $\boldsymbol{\geq 0}$

The **domain** of an **odd degree** radical is $\mathbb{R}$.

a. When finding domain D_f of function $f(x) = \sqrt{2x+3}$, we need to protect the radicand $2x+3$ from becoming negative. So, an x-value belongs to the domain D_f if it satisfies the condition

$$2x + 3 \geq 0.$$

This happens for $x \geq -\frac{3}{2}$. Therefore, $\boldsymbol{D_f} = \left[-\frac{3}{2}, \infty\right)$.

b. To find the domain D_g of function $g(x) = 2 - \sqrt{1-x}$, we solve the condition

$$1 - x \geq 0$$
$$1 \geq x$$

Thus, $\boldsymbol{D_g} = (-\infty, \mathbf{1}]$.

Example 7 ► Graphing Radical Functions

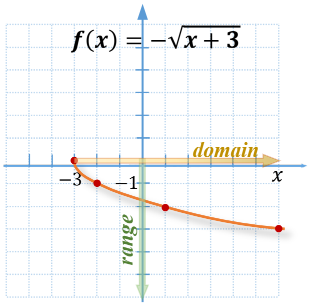
For each function, find its domain, graph it, and find its range. Then, observe what transformation(s) of a basic root function result(s) in the obtained graph.

a. $f(x) = -\sqrt{x+3}$ **b.** $g(x) = \sqrt[3]{x} - 2$

Solution ► **a.** The **domain** D_f is the solution set of the inequality $x + 3 \geq 0$, which is equivalent to $x \geq -3$. Hance, $\boldsymbol{D_f = [-3, \infty)}$.

start from the endpoint of the domain

x	y
-3	0
-2	-1
1	-2
6	-3



The projection of the graph onto the y-axis indicates the **range** of this function, which is $(-\infty, \mathbf{0}]$.

The graph of $f(x) = -\sqrt{x+3}$ has the same shape as the graph of the basic square root function $f(x) = \sqrt{x}$, except that it is flipped over the x-axis and moved to the left by three units. These transformations are illustrated in *Figure 3*.

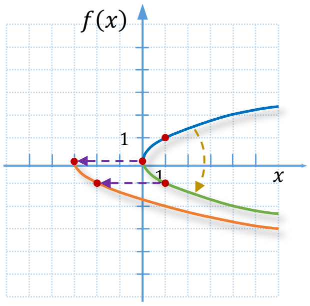
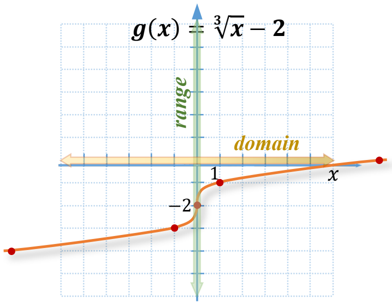


Figure 3

b. The **domain** and **range** of any odd degree radical are both the set of all real numbers. So, $\boldsymbol{D_g} = \mathbb{R}$ and $\boldsymbol{range_g} = \mathbb{R}$.

x	y
-8	-4
-1	-3
0	-2
1	-1
8	0



The graph of $g(x) = \sqrt[3]{x} - 2$ has the same shape as the graph of the basic cube root function $f(x) = \sqrt[3]{x}$, except that it is moved down by two units. This transformation is illustrated in *Figure 4*.

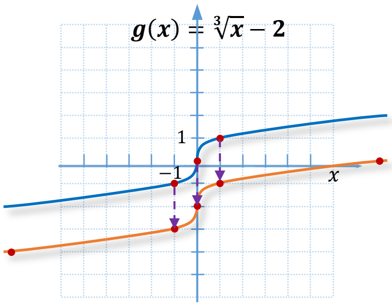


Figure 4

Radicals in Application Problems

Some application problems require evaluation of formulas that involve radicals. For example, the formula $c = \sqrt{a^2 + b^2}$ allows for finding the hypotenuse in a right angle triangle (see *Section RD3*), **Heron's** formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ allows for finding the area of any triangle given the lengths of its sides (see *Section T5*), the formula $T = 2\pi\sqrt{\frac{d^3}{Gm}}$ allows for finding the time needed for a planet to make a complete orbit around the Sun, and so on.

Example 8 ► **Using a Radical Formula in an Application Problem**

The time T, in seconds, needed for a pendulum to complete a full swing can be calculated using the formula

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where L denotes the length of the pendulum in feet, and g is the acceleration due to gravity, which is about 32 ft/sec^2. To the nearest hundredths of a second, find the time of a complete swing of an 18-inch long pendulum.

Solution ► Since $L = 18 \text{ in} = \frac{18}{12} \text{ ft} = \frac{3}{2} \text{ ft}$ and $g = 32$ ft/sec^2, then

$$T = 2\pi\sqrt{\frac{\frac{3}{2}}{32}} = 2\pi\sqrt{\frac{3}{2 \cdot 32}} = 2\pi\sqrt{\frac{3}{64}} = 2\pi \cdot \frac{\sqrt{3}}{8} = \frac{\pi\sqrt{3}}{4} \approx 1.36$$

So, the approximate time of a complete swing of an 18-in pendulum is **1.36 seconds**.

RD.1 Exercises

Evaluate each radical, if possible.

1. $\sqrt{49}$

2. $-\sqrt{81}$

3. $\sqrt{-400}$

4. $\sqrt{0.09}$

5. $\sqrt{0.0016}$

6. $\sqrt{\frac{64}{225}}$

7. $\sqrt[3]{64}$

8. $\sqrt[3]{-125}$

9. $\sqrt[3]{0.008}$

10. $-\sqrt[3]{-1000}$

11. $\sqrt[3]{\frac{1}{0.000027}}$

12. $\sqrt[4]{16}$

13. $\sqrt[5]{0.00032}$

14. $\sqrt[7]{-1}$

15. $\sqrt[8]{-256}$

16. $-\sqrt[6]{\frac{1}{64}}$

17. Decide whether the expression $-\sqrt{-a}$ is positive, negative, 0, or not a real number, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

18. Assuming that n is odd, decide whether the expression $-\sqrt[n]{a}$ is positive, negative, or 0, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

Simplify each radical. Assume that letters can represent <u>*any real number*</u>.

19. $\sqrt{15^2}$ **20.** $\sqrt{(-15)^2}$ **21.** $\sqrt{x^2}$ **22.** $\sqrt{(-x)^2}$

23. $\sqrt{81x^2}$ **24.** $\sqrt{(-12y)^2}$ **25.** $\sqrt{(a+3)^2}$ **26.** $\sqrt{(2-x)^2y^4}$

27. $\sqrt{x^2-4x+4}$ **28.** $\sqrt{9y^2+30y+25}$ **29.** $\sqrt[3]{(-5)^3}$ **30.** $\sqrt[3]{x^3}$

31. $\sqrt[3]{-125a^3}$ **32.** $-\sqrt[3]{0.008(x-1)^3}$ **33.** $\sqrt[4]{(5x)^4}$ **34.** $\sqrt[8]{(-10)^8}$

35. $\sqrt[5]{(y-3)^5}$ **36.** $\sqrt[2017]{(a+b)^{2017}}$ **37.** $\sqrt[2018]{(2a-b)^{2018}}$ **38.** $\sqrt[6]{x^{18}}$

39. $\sqrt[4]{(a+1)^{12}}$ **40.** $\sqrt[5]{(-a)^{20}}$ **41.** $\sqrt[7]{(-k)^{35}}$ **42.** $\sqrt[4]{x^4(-y)^8}$

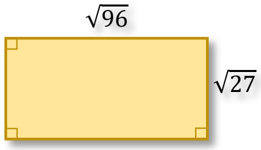
Find a decimal approximation for each radical. Round the answer to three decimal places.

43. $\sqrt{350}$ **44.** $-\sqrt{0.859}$ **45.** $\sqrt[3]{5}$ **46.** $\sqrt[5]{3}$

Without the use of a calculator, give the best integer approximation of each square root.

47. $\sqrt{67}$ **48.** $\sqrt{95}$ **49.** $\sqrt{115}$ **50.** $\sqrt{87}$

Questions in Exercises 60 and 61 refer to the accompanying rectangle. Answer these questions without the use of a calculator.



51. Give the best integer estimation of the area of the rectangle.

52. Give the best integer estimation of the perimeter of the rectangle.

Solve each problem. Do not use any calculator.

53. The length of a rectangular garden is $\sqrt{189}$ m and the width is $\sqrt{50}$ m. Give the best integral estimate for the dimentions of the garden and then estimate the perimeter of the garden to the nearest meter.

54. The sides of a triangle parking lot are $\sqrt{65}$ m, $\sqrt{78}$ m, and $\sqrt{220}$ m. Give the best integral estimate for each side and then estimate the perimeter of this parking lot to the nearest meter.

Graph *each function and give its* ***domain*** *and* ***range****. Then, discuss the* ***transformations*** *of a basic root function needed to obtain the graph of the given function.*

55. $f(x) = \sqrt{x+1}$ **56.** $g(x) = \sqrt{x}+1$ **57.** $h(x) = -\sqrt{x}$

58. $f(x) = \sqrt{x-3}$ **59.** $g(x) = \sqrt{x} - 3$ **60.** $h(x) = 2 - \sqrt{x}$

61. $f(x) = \sqrt[3]{x-2}$ **62.** $g(x) = \sqrt[3]{x} + 2$ **63.** $h(x) = -\sqrt[3]{x} + 2$

***Graph** each function and give its **domain** and **range**.*

64. $f(x) = 2 + \sqrt{x-1}$ **65.** $g(x) = 2\sqrt{x}$ **66.** $h(x) = -\sqrt{x+3}$

67. $f(x) = \sqrt{3x+9}$ **68.** $g(x) = \sqrt{3x-6}$ **69.** $h(x) = -\sqrt{2x-4}$

70. $f(x) = \sqrt{12-3x}$ **71.** $g(x) = \sqrt{8-4x}$ **72.** $h(x) = -2\sqrt{-x}$

Graph the three given functions on the same grid and discuss the relationship between them.

73. $f(x) = 2x + 1;\ g(x) = \sqrt{2x+1};\ \ h(x) = \sqrt[3]{2x+1}$

74. $f(x) = -x + 2;\ g(x) = \sqrt{-x+2};\ \ h(x) = \sqrt[3]{-x+2}$

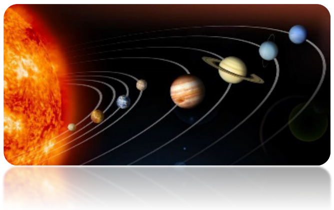
75. $f(x) = \frac{1}{2}x + 1;\ g(x) = \sqrt{\frac{1}{2}x + 1};\ \ h(x) = \sqrt[3]{\frac{1}{2}x + 1}$

Solve each problem.

76. The distance D, in kilometers, from the point of sight to the horizon is given by the formula $D = 4\sqrt{H}$, where H denotes the height of the point of sight above the sea level, in meters. To the nearest tenth of a kilometer, how far away is the horizon for a 180 cm tall man standing on a 40-m high cliff?

77. Let T represents the threshold body weight, in kilograms, above which the risk of death of a person increases significantly. Suppose the formula $h = 40\sqrt[3]{T}$ can be used to calculate the height h, in centimeters, of a middle age man with the threshold body weight T. To the nearest centimeter, find the height corresponding to a threshold weight of a 100 kg man at his forties.

78. The orbital period (time needed for a planet to make a complete rotation around the Sun) is given by the formula $T = 2\pi\sqrt{\frac{r^3}{GM}}$, where r is the average distance of the planet from the Sun, G is the universal gravitational constant, and M is the mass of the Sun. To the nearest day, find the orbital period of Mercury, knowing that its average distance from the Sun is $5.791 \cdot 10^7$ km, the mass of the Sun is $1.989 \cdot 10^{30}$ kg, and $G = 6.67408 \cdot 10^{-11}$ $m^3/(kg \cdot s^2)$. (Attention: *Watch the units!*)



79. Suppose that the time t, in seconds, needed for an object to fall a certain distance can be found by using the formula $t = \sqrt{\frac{2d}{g}}$, where d is the distance in meters, and g is the acceleration due to gravity. An astronaut standing on a platform above the moon's surface drops an object, which hits the ground 2 seconds after it was dropped. Assume that the acceleration due to gravity on the moon is 1.625 m/s^2. How high above the surface was the object at the time it was dropped?

*Half of the perimeter (**semiperimeter**) of a triangle with sides a, b, and c is* $\boldsymbol{s}=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$. *The area of such a triangle is given by the **Heron's Formula**:* $\boldsymbol{A}=\sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}$.
*In problems **89-90**, find the area of a triangular piece of land with the given sides.*

80. $a=3\text{ m}, b=4\text{ m}, c=5\text{ m}$

81. $a=80\text{ m}, b=80\text{ m}, c=140\text{ m}$

RD2 Rational Exponents

In *Sections P2* and *RT1*, we reviewed the properties of powers with natural and integral exponents. All of these properties hold for real exponents as well. In this section, we give meaning to expressions with rational exponents, such as $a^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, or $(2x)^{0.54}$, and use the rational exponent notation as an alternative way to write and simplify radical expressions.

Rational Exponents

Observe that $\sqrt{9} = 3 = 3^{2 \cdot \frac{1}{2}} = 9^{\frac{1}{2}}$. Similarly, $\sqrt[3]{8} = 2 = 2^{3 \cdot \frac{1}{3}} = 8^{\frac{1}{3}}$. This suggests the following generalization:

For any real number a and a natural number $n > 1$, we have

$$\sqrt[n]{\boldsymbol{a}} = \boldsymbol{a}^{\frac{1}{n}}.$$

Notice: The **denominator** of the rational exponent is the **index** of the radical.

Caution! If $a < 0$ and n is an even natural number, then $a^{\frac{1}{n}}$ is <u>not</u> a real number.

Example 1 ► Converting Radical Notation to Rational Exponent Notation

Convert each radical to a power with a rational exponent and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\sqrt[6]{16}$ **b.** $\sqrt[3]{27x^3}$ **c.** $\sqrt{\frac{4}{b^6}}$

Solution ► **a.** $\sqrt[6]{16} = 16^{\frac{1}{6}} = (2^4)^{\frac{1}{6}} = 2^{\frac{4}{6}} = \boldsymbol{2^{\frac{2}{3}}}$

Observation: Expressing numbers as **powers of prime numbers** often allows for further simplification.

b. $\sqrt[3]{27x^3} = (27x^3)^{\frac{1}{3}} = 27^{\frac{1}{3}} \cdot (x^{\cancel{3}})^{\cancel{\frac{1}{3}}} = (3^{\cancel{3}})^{\cancel{\frac{1}{3}}} \cdot x = \boldsymbol{3x}$

distribution
of exponents

change into a power
of a prime number

Note: The above example can also be done as follows:

$$\sqrt[3]{27x^3} = \sqrt[3]{3^3x^3} = (3^3x^3)^{\frac{1}{3}} = \boldsymbol{3x}$$

c. $\sqrt{\frac{9}{b^6}} = \left(\frac{9}{b^6}\right)^{\frac{1}{2}} = \frac{(3^{\cancel{2}})^{\cancel{\frac{1}{2}}}}{(b^{\overset{3}{\cancel{6}}})^{\cancel{\frac{1}{2}}}} = \boldsymbol{\frac{3}{b^3}}$, as $b > 0$.

Observation: $\sqrt{a^4} = a^{\frac{4}{2}} = a^2.$

Generally, for any real number $a \neq 0$, natural number $n > 1$, and integral number m, we have

$$\sqrt[n]{\boldsymbol{a^m}} = (a^m)^{\frac{1}{n}} = \boldsymbol{a^{\frac{m}{n}}}$$

Rational exponents are introduced in such a way that they automatically agree with the rules of exponents, as listed in *Section RT1*.

Furthermore, the rules of exponents hold not only for rational but also for **real exponents**.

Observe that following the rules of exponents and the commutativity of multiplication, we have

$$\sqrt[n]{\boldsymbol{a^m}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{\boldsymbol{a}}\right)^m,$$

provided that $\sqrt[n]{a}$ exists.

Example 2 ► **Converting Rational Exponent Notation to the Radical Notation**

Convert each power with a rational exponent to a radical and simplify, if possible.

a. $5^{\frac{3}{4}}$ **b.** $(-27)^{\frac{1}{3}}$ **c.** $3x^{-\frac{2}{5}}$

Solution ►

a. $5^{\frac{3}{4}} = \sqrt[4]{5^3} = \boldsymbol{\sqrt[4]{125}}$

b. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = \boldsymbol{-3}$

c. $3x^{-\frac{2}{5}} = \frac{3}{x^{\frac{2}{5}}} = \boldsymbol{\frac{3}{\sqrt[5]{x^2}}}$

Notice that $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$, so $(-27)^{\frac{1}{3}} = -27^{\frac{1}{3}}$.

However, $(-9)^{\frac{1}{2}} \neq -9^{\frac{1}{2}}$, as $(-9)^{\frac{1}{2}}$ is not a real number while $-9^{\frac{1}{2}} = -\sqrt{9} = -3$.

Caution: A **negative exponent** indicates a **reciprocal** not a negative number!

Also, the exponent refers to x only, so 3 remains in the numerator.

Observation: If $a^{\frac{m}{n}}$ is a real number, then

$$\boldsymbol{a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}},$$

provided that $a \neq 0$.

Caution! Make sure to distinguish between a negative exponent and a negative result. A negative exponent leads to a reciprocal of the base. The result can be either positive or negative, depending on the sign of the base. For example,

$$8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{2}, \text{ but } (-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{-2} = -\frac{1}{2} \text{ and } -8^{-\frac{1}{3}} = -\frac{1}{8^{\frac{1}{3}}} = -\frac{1}{2}.$$

Example 3 ► Applying Rules of Exponents When Working with Rational Exponents

Simplify each expression. Write your answer with only positive exponents. Assume that all variables represent positive real numbers.

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}}$

b. $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}}$

c. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$

Solution ►

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}} = 2a^{\frac{3}{4}+\left(-\frac{2}{3}\right)} = 2a^{\frac{9}{12}-\frac{8}{12}} = \boldsymbol{2a^{\frac{1}{12}}}$

b. $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}} = 4^{\frac{1}{3}-\frac{5}{3}} = 4^{-\frac{4}{3}} = \boldsymbol{\frac{1}{4^{\frac{4}{3}}}}$

c. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}} = x^{\frac{\cancel{3}}{\underset{2}{\cancel{8}}} \cdot \frac{\cancel{4}}{\cancel{3}}} \cdot y^{\frac{5}{\cancel{2}} \cdot \frac{\overset{2}{\cancel{4}}}{3}} = \boldsymbol{x^{\frac{1}{2}} y^{\frac{10}{3}}}$

Example 4 ► Evaluating Powers with Rational Exponents

Evaluate each power.

a. $64^{-\frac{1}{3}}$

b. $\left(-\frac{8}{125}\right)^{\frac{2}{3}}$

Solution ►

a. $64^{-\frac{1}{3}} = (2^6)^{-\frac{1}{3}} = 2^{-2} = \frac{1}{2^2} = \boldsymbol{\frac{1}{4}}$

It is helpful to change the base into a **power of a prime number**, if possible.

b. $\left(-\frac{8}{125}\right)^{\frac{2}{3}} = \left(\left(-\frac{2}{5}\right)^{\cancel{3}}\right)^{\frac{2}{\cancel{3}}} = \left(-\frac{2}{5}\right)^2 = \boldsymbol{\frac{4}{25}}$

Observe that if m in $\sqrt[n]{\boldsymbol{a}^{m}}$ is a multiple of n, that is if $m = kn$ for some integer k, then

$$\sqrt[n]{\boldsymbol{a}^{\boldsymbol{kn}}} = a^{\frac{k\cancel{n}}{\cancel{n}}} = \boldsymbol{a}^{\boldsymbol{k}}$$

Example 5 ► Simplifying Radical Expressions by Converting to Rational Exponents

Simplify. Assume that all variables represent positive real numbers. Leave your answer in simplified single radical form.

a. $\sqrt[5]{3^{20}}$ **b.** $\sqrt{x} \cdot \sqrt[4]{x^3}$ **c.** $\sqrt[3]{2\sqrt{2}}$

Solution ► **a.** $\sqrt[5]{3^{20}} = (3^{20})^{\frac{1}{5}} = 3^4 = \mathbf{81}$

divide at the exponential level

b. $\sqrt{x} \cdot \sqrt[4]{x^3} = x^{\frac{1}{2}} \cdot x^{\frac{3}{4}} = x^{\frac{1\cdot 2}{2\cdot 2}+\frac{3}{4}} = x^{\frac{5}{4}} = x^1 \cdot x^{\frac{1}{4}} = \boldsymbol{x\sqrt[4]{x}}$

add exponents as $\frac{5}{4}=1+\frac{1}{4}$

c. $\sqrt[3]{2\sqrt{2}} = \left(2 \cdot 2^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{1+\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{\frac{\cancel{3}}{2}}\right)^{\frac{1}{\cancel{3}}} = 2^{\frac{1}{2}} = \boldsymbol{\sqrt{2}}$

This bracket is essential!

Another solution:

$$\sqrt[3]{2\sqrt{2}} = 2^{\frac{1}{3}} \cdot \left(2^{\frac{1}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}} = 2^{\frac{1}{2}} = \boldsymbol{\sqrt{2}}$$

RD.2 Exercises

Match each expression from Column I with the equivalent expression from Column II.

1.

Column I	Column II
a. $9^{\frac{1}{2}}$	**A.** $\frac{1}{3}$
b. $9^{-\frac{1}{2}}$	**B.** 3
c. $-9^{\frac{3}{2}}$	**C.** -27
d. $-9^{-\frac{1}{2}}$	**D.** not a real number
e. $(-9)^{\frac{1}{2}}$	**E.** $\frac{1}{27}$
f. $9^{-\frac{3}{2}}$	**F.** $-\frac{1}{3}$

2.

Column I	Column II
a. $(-32)^{\frac{2}{5}}$	**A.** 2
b. $-27^{\frac{2}{3}}$	**B.** $\frac{1}{4}$
c. $32^{\frac{1}{5}}$	**C.** -8
d. $32^{-\frac{2}{5}}$	**D.** -9
e. $-4^{\frac{3}{2}}$	**E.** not a real number
f. $(-4)^{\frac{3}{2}}$	**F.** 4

Write the base as a ***power of a prime number*** *to evaluate each expression, if possible.*

3. $32^{\frac{1}{5}}$

4. $27^{\frac{4}{3}}$

5. $-49^{\frac{3}{2}}$

6. $16^{\frac{3}{4}}$

7. $-100^{-\frac{1}{2}}$

8. $125^{-\frac{1}{3}}$

9. $\left(\frac{64}{81}\right)^{\frac{3}{4}}$

10. $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

11. $(-36)^{\frac{1}{2}}$
12. $(-64)^{\frac{1}{3}}$
13. $\left(-\frac{1}{8}\right)^{-\frac{1}{3}}$
14. $(-625)^{-\frac{1}{4}}$

*Rewrite **with** rational exponents and simplify, if possible. Assume that all variables represent positive real numbers.*

15. $\sqrt{5}$
16. $\sqrt[3]{6}$
17. $\sqrt{x^6}$
18. $\sqrt[5]{y^2}$
19. $\sqrt[3]{64x^6}$
20. $\sqrt[3]{16x^2y^3}$
21. $\sqrt{\frac{25}{x^5}}$
22. $\sqrt[4]{\frac{16}{a^6}}$

*Rewrite **without** rational exponents, and simplify, if possible. Assume that all variables represent positive real numbers.*

23. $4^{\frac{5}{2}}$
24. $8^{\frac{3}{4}}$
25. $x^{\frac{3}{5}}$
26. $a^{\frac{7}{3}}$
27. $(-3)^{\frac{2}{3}}$
28. $(-2)^{\frac{3}{5}}$
29. $2x^{-\frac{1}{2}}$
30. $x^{\frac{1}{3}}y^{-\frac{1}{2}}$

*Use the **laws of exponents** to simplify. Write the answers with positive exponents. Assume that all variables represent positive real numbers.*

31. $3^{\frac{3}{4}} \cdot 3^{\frac{1}{8}}$
32. $x^{\frac{2}{3}} \cdot x^{-\frac{1}{4}}$
33. $\frac{2^{\frac{5}{8}}}{2^{-\frac{1}{8}}}$
34. $\frac{a^{\frac{1}{3}}}{a^{\frac{2}{3}}}$
35. $\left(5^{\frac{15}{8}}\right)^{\frac{2}{3}}$
36. $\left(y^{\frac{2}{3}}\right)^{-\frac{3}{7}}$
37. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$
38. $\left(a^{-\frac{2}{3}} \cdot b^{\frac{5}{8}}\right)^{-4}$
39. $\left(\frac{y^{-\frac{3}{2}}}{x^{-\frac{5}{3}}}\right)^{\frac{1}{3}}$
40. $\left(\frac{a^{-\frac{2}{3}}}{b^{-\frac{5}{6}}}\right)^{\frac{3}{4}}$
41. $x^{\frac{2}{3}} \cdot 5x^{-\frac{2}{5}}$
42. $x^{\frac{2}{5}} \cdot \left(4x^{-\frac{4}{5}}\right)^{-\frac{1}{4}}$

*Use rational exponents to **simplify**. Write the answer **in radical notation** if appropriate. Assume that all variables represent positive real numbers.*

43. $\sqrt[6]{x^2}$
44. $\left(\sqrt[3]{ab}\right)^{15}$
45. $\sqrt[6]{y^{-18}}$
46. $\sqrt{x^4y^{-6}}$
47. $\sqrt[6]{81}$
48. $\sqrt[14]{128}$
49. $\sqrt[3]{8y^6}$
50. $\sqrt[4]{81p^6}$
51. $\sqrt[3]{(4x^3y)^2}$
52. $\sqrt[5]{64(x+1)^{10}}$
53. $\sqrt[4]{16x^4y^2}$
54. $\sqrt[5]{32a^{10}d^{15}}$

*Use rational exponents to rewrite in a **single radical expression** in a simplified form. Assume that all variables represent positive real numbers.*

55. $\sqrt[3]{5} \cdot \sqrt{5}$
56. $\sqrt[3]{2} \cdot \sqrt[4]{3}$
57. $\sqrt{a} \cdot \sqrt[3]{3a}$
58. $\sqrt[3]{x} \cdot \sqrt[5]{2x}$
59. $\sqrt[6]{x^5} \cdot \sqrt[3]{x^2}$
60. $\sqrt[3]{xz} \cdot \sqrt{z}$
61. $\frac{\sqrt{x^5}}{\sqrt{x^8}}$
62. $\frac{\sqrt[3]{a^5}}{\sqrt{a^3}}$

63. $\frac{\sqrt[3]{8x}}{\sqrt[4]{x^3}}$

64. $\sqrt[3]{\sqrt{a}}$

65. $\sqrt[4]{\sqrt[3]{xy}}$

66. $\sqrt{\sqrt[3]{(3x)^2}}$

67. $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$

68. $\sqrt[3]{3\sqrt{3}}$

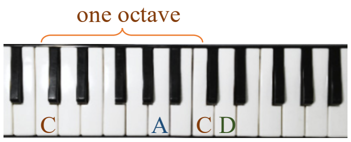
69. $\sqrt[4]{x\sqrt{x}}$

70. $\sqrt[3]{2\sqrt{x}}$

71. Consider two expressions: $\sqrt[n]{x^n + y^n}$ and $x + y$. Observe that for $x = 1$ and $y = 0$ both expressions are equal: $\sqrt[n]{x^n + y^n} = \sqrt[n]{1^n + 0^n} = 1 = 1 + 9 = x + y$. Does this mean that $\sqrt[n]{x^n + y^n} = x + y$? Justify your answer.

Solve each problem.

72. When counting both the black and white keys on a piano, an octave contains 12 keys. The frequencies of consecutive keys increase by a factor of $2^{\frac{1}{12}}$. For example, the frequency of the tone D that is two keys above middle C is



$$2^{\frac{1}{12}} \cdot 2^{\frac{1}{12}} = \left(2^{\frac{1}{12}}\right)^2 = 2^{\frac{1}{6}} \approx 1.12$$

times the frequency of the middle C.

a. If tone G, which is five keys below the middle C, has a frequency of about 196 cycles per second, estimate the frequency of the middle C to the nearest tenths of a cycle.

b. Find the relation between frequencies of two tones that are one octave apart.

73. An animal's heart rate is related to the animal's weight. Suppose that the average heart rate R, in beats per minute, for an animal that weighs k kilograms can be estimated by using the function $R(w) = 600w^{-\frac{1}{2}}$. What is the expected average heart rate of a horse that weighs 400 kilograms?

74. Suppose that the duration of a storm T, in hours, can be determined by using the function $T(D) = 0.03D^{\frac{3}{2}}$, where D denotes the diameter of a storm in kilometers. To the nearest minute, what is the duration of a storm with a diameter of 20 kilometers?

RD3 Simplifying Radical Expressions and the Distance Formula

In the previous section, we simplified some radical expressions by replacing radical signs with rational exponents, applying the rules of exponents, and then converting the resulting expressions back into radical notation. In this section, we broaden the above method of simplifying radicals by examining products and quotients of radicals with the same indexes, as well as explore the possibilities of decreasing the index of a radical.

In the second part of this section, we will apply the skills of simplifying radicals in problems involving the Pythagorean Theorem. In particular, we will develop the distance formula and apply it to calculate distances between two given points in a plane.

Multiplication, Division, and Simplification of Radicals

Suppose we wish to multiply radicals with the same indexes. This can be done by converting each radical to a rational exponent and then using properties of exponents as follows:

PRODUCT RULE

$$\sqrt[n]{\boldsymbol{a}} \cdot \sqrt[n]{\boldsymbol{b}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{\boldsymbol{ab}}$$

This shows that the **product of same index radicals** is the **radical of the product** of their radicands.

Similarly, the **quotient of same index radicals** is the **radical of the quotient** of their radicands, as we have

QUOTIENT RULE

$$\frac{\sqrt[n]{\boldsymbol{a}}}{\sqrt[n]{\boldsymbol{b}}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{\boldsymbol{a}}{\boldsymbol{b}}}$$

So, $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = \mathbf{4}$. Similarly, $\frac{\sqrt[3]{16}}{\sqrt[3]{2}} = \sqrt[3]{\frac{16}{2}} = \sqrt[3]{8} = \mathbf{2}$.

Attention! There is no such rule for addition or subtraction of terms. For instance,

$$\sqrt{\boldsymbol{a} + \boldsymbol{b}} \neq \sqrt{\boldsymbol{a}} \pm \sqrt{\boldsymbol{b}},$$

and generally

$$\sqrt[n]{\boldsymbol{a} \pm \boldsymbol{b}} \neq \sqrt[n]{\boldsymbol{a}} \pm \sqrt[n]{\boldsymbol{b}}.$$

Here is a counterexample: $\sqrt[n]{2} = \sqrt[n]{1+1} \neq \sqrt[n]{1} + \sqrt[n]{1} = 1 + 1 = 2$

Example 1 ► Multiplying and Dividing Radicals of the Same Indexes

Perform the indicated operations and simplify, if possible. Assume that all variables are positive.

a. $\sqrt{10} \cdot \sqrt{15}$

b. $\sqrt{2x^3}\sqrt{6xy}$

c. $\frac{\sqrt{10x}}{\sqrt{5}}$

d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}}$

Solution ►

a. $\sqrt{10}\cdot\sqrt{15} = \sqrt{10\cdot 15} = \sqrt{2\cdot 5\cdot 5\cdot 3} = \sqrt{5\cdot 5\cdot 2\cdot 3} = \sqrt{25}\cdot\sqrt{6} = \mathbf{5\sqrt{6}}$

product rule　prime factorization　commutativity of multiplication　product rule

b. $\sqrt{2x^3}\,\sqrt{6xy} = \sqrt{2\cdot 2\cdot 3x^4y} = \sqrt{4x^4}\cdot\sqrt{3y} = \mathbf{2x^2\sqrt{3y}}$

use commutativity of multiplication to isolate perfect square factors

Here the multiplication sign is assumed, even if it is not indicated.

c. $\frac{\sqrt{10x}}{\sqrt{5}} = \sqrt{\frac{\overset{2}{\cancel{10}}x}{\cancel{5}}} = \mathbf{\sqrt{2x}}$

quotient rule

d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}} = \sqrt[4]{\frac{\overset{16}{\cancel{32}}x^{\cancel{3}\,2}}{\cancel{2x}}} = \sqrt[4]{16x^2} = \sqrt[4]{16}\cdot\sqrt[4]{x^2} = \mathbf{2\sqrt{x}}$

Recall that
$\sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}.$

Caution! Remember to indicate the index of the radical for indexes higher than two.

The product and quotient rules are essential when simplifying radicals.

To **simplify a radical** means to:

1. Make sure that all **power factors of the radicand have exponents smaller than the index of the radical**.

   For example, $\sqrt[3]{2^4x^8y} = \sqrt[3]{2^3x^6}\cdot\sqrt[3]{2x^2y} = 2x^2\,\sqrt[3]{2x^2y}$.

2. Leave the radicand with **no fractions**.

   For example, $\sqrt{\frac{2x}{25}} = \frac{\sqrt{2x}}{\sqrt{25}} = \frac{\sqrt{2x}}{5}$.

3. **Rationalize** any **denominator**. (Make sure that denominators are **free from radicals,** see Section RD4.)

   For example, $\sqrt{\frac{4}{x}} = \frac{\sqrt{4}}{\sqrt{x}} = \frac{2\cdot\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}} = \frac{2\sqrt{x}}{x}$, provided that $x > 0$.

4. **Reduce the power** of the radicand **with the index** of the radical, if possible.

   For example, $\sqrt[4]{x^2} = x^{\frac{\cancel{2}}{\cancel{4}\,2}} = x^{\frac{1}{2}} = \sqrt{x}$.

Example 2 ► **Simplifying Radicals**

Simplify each radical. Assume that all variables are positive.

a. $\sqrt[5]{96x^7y^{15}}$　**b.** $\sqrt[4]{\frac{a^{12}}{16b^4}}$　**c.** $\sqrt{\frac{25x^2}{8x^3}}$　**d.** $\sqrt[6]{27a^{15}}$

Solution ► **a.** $\sqrt[5]{96x^7y^{15}} = \sqrt[5]{2^5 \cdot 3x^7y^{15}} = \mathbf{2xy^3\sqrt[5]{3x^2}}$

$\sqrt[5]{y^{15}} = y^3$

$\sqrt[5]{x^7} = x\sqrt[5]{x^2}$

Generally, to simplify $\sqrt[d]{x^a}$, we perform the division

$$a \div d = quotient\ \boldsymbol{q} + remainder\ \boldsymbol{r},$$

and then pull the $\boldsymbol{q}$-th power of x out of the radical, leaving the $\boldsymbol{r}$-th power of x under the radical. So, we obtain

$$\sqrt[d]{\boldsymbol{x^a}} = \boldsymbol{x^q\sqrt[d]{x^r}}$$

b. $\sqrt[4]{\frac{a^{12}}{16b^4}} = \frac{\sqrt[4]{a^{12}}}{\sqrt[4]{2^4b^4}} = \boldsymbol{\frac{a^3}{2b}}$

c. $\sqrt{\frac{25\cancel{x^2}}{8\cancel{x^3}^{1}}} = \sqrt{\frac{25}{2^3x}} = \frac{\sqrt{25}}{\sqrt{2^3x}} = \frac{5}{2\sqrt{2x}} \cdot \frac{\sqrt{2x}}{\sqrt{2x}} = \frac{5\sqrt{2x}}{2{\cdot}2x} = \boldsymbol{\frac{5\sqrt{2x}}{4x}}$

d. $\sqrt[6]{27a^{15}} = \sqrt[6]{3^3a^{15}} = a^2\sqrt[6]{3^3a^3} = a^2 \cdot \sqrt[\overset{2}{\cancel{6}}]{(3a)^{\cancel{3}}} = \boldsymbol{a^2\sqrt{3a}}$

Example 3 ► Simplifying Expressions Involving Multiplication, Division, or Composition of Radicals with Different Indexes

Simplify each expression. Leave your answer in simplified single radical form. Assume that all variables are positive.

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^2y}$ **b.** $\frac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}}$ **c.** $\sqrt[3]{x^2\sqrt{2x}}$

Solution ► **a.** $\sqrt{xy^5} \cdot \sqrt[3]{x^2y} = x^{\frac{1}{2}}y^{\frac{5}{2}} \cdot x^{\frac{2}{3}}y^{\frac{1}{3}} = x^{\frac{1\cdot3}{2\cdot3}+\frac{2\cdot2}{3\cdot2}}y^{\frac{5\cdot3}{2\cdot3}+\frac{1\cdot2}{3\cdot2}} = x^{\frac{7}{6}}y^{\frac{17}{6}} = (x^7y^{17})^{\frac{1}{6}}$

$= \sqrt[6]{x^7\,y^{17}} = \boldsymbol{xy^2\sqrt[6]{xy^5}}$

If radicals are of different indexes, convert them to exponential form.

b. $\frac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}} = \frac{a^{\frac{\cancel{2}}{\cancel{4}^{2}}}b^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}} = a^{\frac{1\cdot3}{2\cdot3}-\frac{1\cdot2}{3\cdot2}}b^{\frac{3\cdot3}{4\cdot3}-\frac{1\cdot4}{3\cdot4}} = a^{\frac{1\cdot2}{6\cdot2}}\,b^{\frac{5}{12}} = (a^2\,b^5)^{\frac{1}{12}} = \boldsymbol{\sqrt[12]{a^2\,b^5}}$

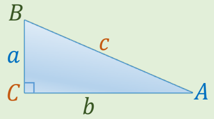
Bring the exponents to the LCD in order to leave the answer as a single radical.

c. $\sqrt[3]{x^2\,\sqrt{2x}} = x^{\frac{2}{3}} \cdot \left((2x)^{\frac{1}{2}}\right)^{\frac{1}{3}} = x^{\frac{2}{3}} \cdot 2^{\frac{1}{6}} \cdot x^{\frac{1}{6}} = x^{\frac{2\cdot2}{3\cdot2}+\frac{1}{6}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1}{6}}x^{\frac{5}{6}} = (2x^5)^{\frac{1}{6}} = \boldsymbol{\sqrt[6]{2x^5}}$

Pythagorean Theorem and Distance Formula

One of the most famous theorems in mathematics is the Pythagorean Theorem.

Pythagorean Theorem ►



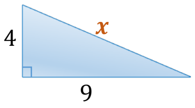
Suppose angle C in a triangle ABC is a 90° angle.
Then the **sum of the squares** of the lengths of the two **legs**, $\boldsymbol{a}$ and $\boldsymbol{b}$, equals to the **square** of the length of the **hypotenuse** $\boldsymbol{c}$:

$$\boldsymbol{a^2 + b^2 = c^2}$$

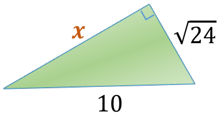
Example 4 ► Using The Pythagorean Equation

For the first two triangles, find the exact length x of the unknown side. For triangle (**c**), express length x in terms of the unknown n.

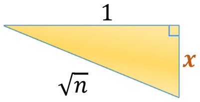
a.



b.



c.



Solution ►

Caution: Generally, $\sqrt{x^2} = |x|$ However, the length of a side of a triangle is positive. So, we can write $\sqrt{x^2} = x$

a. The length of the hypotenuse of the given right triangle is equal to x. So, the Pythagorean equation takes the form

$$x^2 = 4^2 + 9^2.$$

To solve it for x, we take a square root of each side of the equation. This gives us

$$\begin{aligned} \sqrt{x^2} &= \sqrt{4^2 + 9^2} \\ x &= \sqrt{16 + 81} \\ x &= \boldsymbol{\sqrt{97}} \end{aligned}$$

b. Since 10 is the length of the hypotenuse, we form the Pythagorean equation

$$10^2 = x^2 + {\sqrt{24}}^2.$$

To solve it for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we have

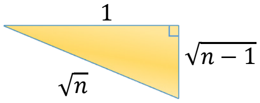
Customarily, we simplify each root, if possible.

$$\begin{aligned} 10^2 - {\sqrt{24}}^2 &= x^2 \\ 100 - 24 &= x^2 \\ x^2 &= 76 \\ x &= \sqrt{76} = \sqrt{4 \cdot 19} = \boldsymbol{2\sqrt{19}} \end{aligned}$$

c. The length of the hypotenuse is $\sqrt{n}$, so we form the Pythagorean equation as below.

$$\left(\sqrt{n}\right)^2 = 1^2 + x^2$$

To solve this equation for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we obtain



$$n = 1 + x^2$$
$$n - 1 = x^2$$
$$x = \sqrt{\boldsymbol{n - 1}}$$

Note: Since the hypotenuse of length $\sqrt{n}$ must be longer than the leg of length 1, $n > 1$. This means that $n - 1 > 0$, and therefore $\sqrt{n-1}$ is a positive real number.

The Pythagorean Theorem allows us to find the distance between any two given points in a plane.
Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a coordinate plane. Then $|x_2 - x_1|$ represents the horizontal distance between A and B and $|y_2 - y_1|$ represents the vertical distance between A and B, as shown in *Figure 1*. Notice that by applying the absolute value operator to each difference of the coordinates we guarantee that the resulting horizontal and vertical distance is indeed a nonnegative number.

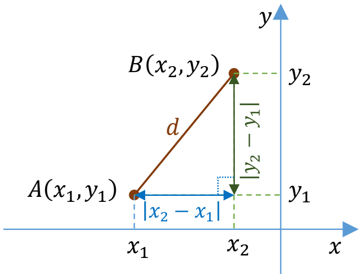


Figure 1

Applying the Pythagorean Theorem to the right triangle shown in *Figure 1*, we form the equation

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2,$$

where d is the distance between A and B.

Notice that $|x_2 - x_1|^2 = (x_2 - x_1)^2$ as a perfect square automatically makes the expression nonnegative. Similarly, $|y_2 - y_1|^2 = (y_2 - y_1)^2$. So, the Pythagorean equation takes the form

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

After solving this equation for d, we obtain the **distance formula**:

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Note: Observe that due to squaring the difference of the corresponding coordinates, **the distance between two points is the same regardless of which point is chosen as first**, (x_1, y_1), and second, (x_2, y_2).

Example 5 ► **Finding the Distance Between Two Points**

Find the exact distance between the points $(-2,4)$ and $(5,3)$.

Solution ► Let $(-2,4) = (x_1, y_1)$ and $(5,3) = (x_2, y_2)$. To find the distance d between the two points, we follow the distance formula:

$$d=\sqrt{\left(5-(-2)\right)^2+(3-4)^2}=\sqrt{7^2+(-1)^2}=\sqrt{49+1}=\sqrt{50}=\mathbf{5\sqrt{2}}$$

So, the points $(-2,4)$ and $(5,3)$ are $5\sqrt{2}$ units apart.

RD.3 Exercises

Multiply and simplify, if possible. Assume that all variables are positive.

1. $\sqrt{5}\cdot\sqrt{5}$

2. $\sqrt{18}\cdot\sqrt{2}$

3. $\sqrt{6}\cdot\sqrt{3}$

4. $\sqrt{15}\cdot\sqrt{6}$

5. $\sqrt{45}\cdot\sqrt{60}$

6. $\sqrt{24}\cdot\sqrt{75}$

7. $\sqrt{3x^3}\cdot\sqrt{6x^5}$

8. $\sqrt{5y^7}\cdot\sqrt{15a^3}$

9. $\sqrt{12x^3y}\,\sqrt{8x^4y^2}$

10. $\sqrt{30a^3b^4}\,\sqrt{18a^2b^5}$

11. $\sqrt[3]{4x^2}\,\sqrt[3]{2x^4}$

12. $\sqrt[4]{20a^3}\,\sqrt[4]{4a^5}$

Divide and simplify, if possible. Assume that all variables are positive.

13. $\frac{\sqrt{90}}{\sqrt{5}}$

14. $\frac{\sqrt{48}}{\sqrt{6}}$

15. $\frac{\sqrt{42a}}{\sqrt{7a}}$

16. $\frac{\sqrt{30x^3}}{\sqrt{10x}}$

17. $\frac{\sqrt{52ab^3}}{\sqrt{13a}}$

18. $\frac{\sqrt{56xy^3}}{\sqrt{8x}}$

19. $\frac{\sqrt{128x^2y}}{2\sqrt{2}}$

20. $\frac{\sqrt{48a^3b}}{2\sqrt{3}}$

21. $\frac{\sqrt[4]{80}}{\sqrt[4]{5}}$

22. $\frac{\sqrt[3]{108}}{\sqrt[3]{4}}$

23. $\frac{\sqrt[3]{96a^5b^2}}{\sqrt[3]{12a^2b}}$

24. $\frac{\sqrt[4]{48x^9y^{13}}}{\sqrt[4]{3xy^5}}$

Simplify each expression. Assume that all variables are positive.

25. $\sqrt{144x^4y^9}$

26. $-\sqrt{81m^8n^5}$

27. $\sqrt[3]{-125a^6b^9c^{12}}$

28. $\sqrt{50x^3y^4}$

29. $\sqrt[4]{\frac{1}{16}m^8n^{20}}$

30. $-\sqrt[3]{-\frac{1}{27}x^2y^7}$

31. $\sqrt{7a^7b^6}$

32. $\sqrt{75p^3q^4}$

33. $\sqrt[5]{64x^{12}y^{15}}$

34. $\sqrt[5]{p^{14}q^7r^{23}}$

35. $-\sqrt[4]{162a^{15}b^{10}}$

36. $-\sqrt[4]{32x^5y^{10}}$

37. $\sqrt{\frac{16}{49}}$

38. $\sqrt[3]{\frac{27}{125}}$

39. $\sqrt{\frac{121}{y^2}}$

40. $\sqrt{\frac{64}{x^4}}$

41. $\sqrt[3]{\frac{81a^5}{64}}$

42. $\sqrt{\frac{36x^5}{y^6}}$

43. $\sqrt[4]{\frac{16x^{12}}{y^4z^{16}}}$

44. $\sqrt[5]{\frac{32y^8}{x^{10}}}$

45. $\sqrt[4]{36}$

46. $\sqrt[6]{27}$

47. $-\sqrt[10]{x^{25}}$

48. $\sqrt[12]{x^{44}}$

49. $-\sqrt{\frac{1}{x^3y}}$

50. $\sqrt[3]{\frac{64x^{15}}{y^4z^5}}$

51. $\sqrt[6]{\frac{x^{13}}{y^6z^{12}}}$

52. $\sqrt[6]{\frac{p^9q^{24}}{r^{18}}}$

53. To simplify the radical $\sqrt{x^3+x^2}$, a student wrote $\sqrt{x^3+x^2} = x\sqrt{x} + x = x\left(\sqrt{x}+1\right)$. Is this correct? Justify your answer.

Perform operations. Leave the answer in simplified ***single radical*** *form. Assume that all variables are positive.*

54. $\sqrt{3}\cdot\sqrt[3]{4}$

55. $\sqrt{x}\cdot\sqrt[5]{x}$

56. $\sqrt[3]{x^2}\cdot\sqrt[4]{x}$

57. $\sqrt[3]{4}\cdot\sqrt[5]{8}$

58. $\dfrac{\sqrt[3]{a^2}}{\sqrt{a}}$

59. $\dfrac{\sqrt{x}}{\sqrt[4]{x}}$

60. $\dfrac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$

61. $\dfrac{\sqrt[5]{16a^2}}{\sqrt[3]{2a^2}}$

62. $\sqrt[3]{2\sqrt{x}}$

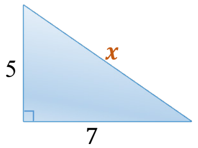
63. $\sqrt{x\sqrt[3]{2x^2}}$

64. $\sqrt[4]{3\sqrt[3]{9}}$

65. $\sqrt[3]{x^2\sqrt[4]{x^3}}$

For each right triangle, find length x. Simplify the answer if possible. In problems ***73*** *and* ***74,*** *expect the length x to be an expression in terms of n.*

66.



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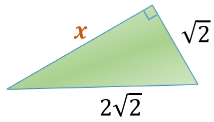

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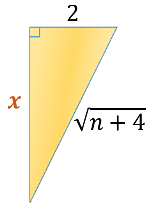

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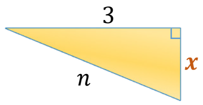

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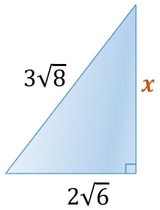

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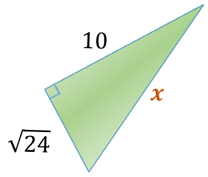


Find the exact distance between each pair of points.

72. $(8,13)$ and $(2,5)$

73. $(-8,3)$ and $(-4,1)$

74. $(-6,5)$ and $(3,-4)$

75. $\left(\frac{5}{7},\frac{1}{14}\right)$ and $\left(\frac{1}{7},\frac{11}{14}\right)$

76. $\left(0,\sqrt{6}\right)$ and $\left(\sqrt{7},0\right)$

77. $\left(\sqrt{2},\sqrt{6}\right)$ and $\left(2\sqrt{2},-4\sqrt{6}\right)$

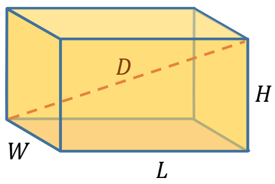
78. $\left(-\sqrt{5},6\sqrt{3}\right)$ and $\left(\sqrt{5},\sqrt{3}\right)$

79. $(0,0)$ and (p,q)

80. $(x+h,y+h)$ and (x,y) (*assume that* $h>0$)

Solve each problem.

81. To find the diagonal of a box, we can use the formula $D=\sqrt{W^2+L^2+H^2}$, where W, L, and H are, respectively, the width, length, and height of the box. Find the diagonal D of a storage container that is 6.1 meters long, 2.4 meters wide, and 2.6 meters high. *Round your answer to the nearest centimeter.*

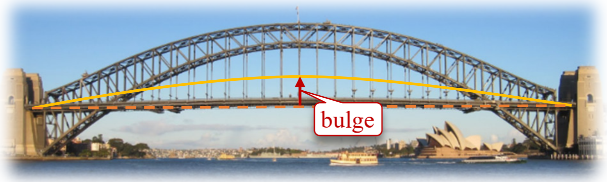


82. The screen of a 32-inch television is 27.9-inch wide. To the nearest tenth of an inch, what is the measure of its height? (Note: *TVs are measured diagonally, so a 32-inch television means that its screen measures diagonally 32 inches.*)

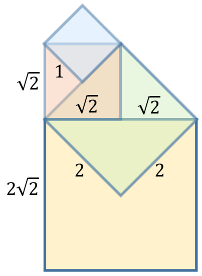
83. Suppose $A=(0,-3)$ and P is a point on the x-axis of a Cartesian coordinate system. Find all possible coordinates of P if $AP=5$.

84. Suppose $B = (1, 0)$ and P is a point on the y-axis of a Cartesian coordinate system. Find all possible coordinates of P if $BP = 2$.

85. Due to high temperatures, a 3-km bridge may expand up to 0.6 meters in length. If the maximum bulge occurs at the middle of the bridge, find the height of such a bulge. *The answer may be surprising. To avoid such situations, engineers design bridges with expansion spaces.*



RD4 Operations on Radical Expressions; Rationalization of Denominators



Unlike operations on fractions or decimals, sums and differences of many radicals cannot be simplified. For instance, we cannot combine $\sqrt{2}$ and $\sqrt{3}$, nor simplify expressions such as $\sqrt[3]{2}-1$. These types of radical expressions can only be approximated with the aid of a calculator.

However, some radical expressions can be combined (added or subtracted) and simplified. For example, the sum of $2\sqrt{2}$ and $\sqrt{2}$ is $3\sqrt{2}$, similarly as $2x+x=3x$.

In this section, first, we discuss the addition and subtraction of radical expressions. Then, we show how to work with radical expressions involving a combination of the four basic operations. Finally, we examine how to rationalize denominators of radical expressions.

Addition and Subtraction of Radical Expressions

Recall that to perform addition or subtraction of two variable terms we need these terms to be **like**. This is because the addition and subtraction of terms are performed by factoring out the variable "like" part of the terms as a common factor. For example,

$$x^2+3x^2=(1+3)x^2=4x^2$$

The same strategy works for addition and subtraction of the same types of radicals or **radical terms** (terms containing radicals).

Definition 4.1 ► Radical terms containing radicals with the <u>same index</u> and the <u>same radicands</u> are referred to as **like radicals** or **like radical terms**.

For example,

$\sqrt{5x}$ and $2\sqrt{5x}$ are **like** (the indexes and the radicands are the same)

while

$5\sqrt{2}$ and $2\sqrt{5}$ are **<u>not</u> like** (the radicands are different)

and

$\sqrt{x}$ and $\sqrt[3]{x}$ are **<u>not</u> like radicals** (the indexes are different).

To **add** or **subtract like radical expressions** we **factor out the common radical** and any other common factor, if applicable. For example,

$$4\sqrt{2}+3\sqrt{2}=(4+3)\sqrt{2}=7\sqrt{2},$$

and

$$4xy\sqrt{2}-3x\sqrt{2}=(4y-3)x\sqrt{2}.$$

Caution! Unlike radical expressions cannot be combined. For example, we are unable to perform the addition $\sqrt{6}+\sqrt{3}$. Such a sum can only be approximated using a calculator.

Notice that unlike radicals may become like if we simplify them first. For example, $\sqrt{200}$ and $\sqrt{50}$ are not like, but $\sqrt{200}=10\sqrt{2}$ and $\sqrt{50}=5\sqrt{2}$. Since $10\sqrt{2}$ and $5\sqrt{2}$ are like radical terms, they can be combined. So, we can perform, for example, the addition:

$$\sqrt{200}+\sqrt{50}=10\sqrt{2}+5\sqrt{2}=15\sqrt{2}$$

Example 1 ► Adding and Subtracting Radical Expressions

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

a. $5\sqrt{3}-8\sqrt{3}$

b. $3\sqrt[5]{2}-7x\sqrt[5]{2}+6\sqrt[5]{2}$

c. $7\sqrt{45}+\sqrt{80}-\sqrt{12}$

d. $3\sqrt[3]{y^5}-5y\sqrt[3]{y^2}+\sqrt[5]{32y^7}$

e. $\sqrt{\frac{x}{16}}+2\sqrt{\frac{x^3}{9}}$

f. $\sqrt{25x^2-25}-\sqrt{9x^2-9}$

Solution ►

a. To subtract like radicals, we combine their coefficients via factoring.

$$5\sqrt{3}-8\sqrt{3}=(5-8)\sqrt{3}=\mathbf{-3\sqrt{3}}$$

b. $3\sqrt[5]{2}-7x\sqrt[5]{2}+6\sqrt[5]{2}=(3-7x+6)\sqrt[5]{2}=\mathbf{(9-7x)\sqrt[5]{2}}$

The brackets are essential here.

Note: Even if not all coefficients are like, factoring the common radical is a useful strategy that allows us to combine like radical expressions.

c. The expression $7\sqrt{45}+\sqrt{80}-\sqrt{12}$ consists of unlike radical terms, so they cannot be combined in this form. However, if we simplify the radicals, some of them may become like and then become possible to combine.

$$7\sqrt{45}+\sqrt{80}-\sqrt{12}=7\sqrt{9\cdot5}+\sqrt{16\cdot5}-\sqrt{4\cdot3}=7\cdot3\sqrt{5}+4\sqrt{5}-2\sqrt{3}$$

$$=21\sqrt{5}+4\sqrt{5}-2\sqrt{3}=\mathbf{25\sqrt{5}-2\sqrt{3}}$$

d. As in the previous example, we simplify each radical expression before attempting to combine them.

$$3\sqrt[3]{y^5}-5y\sqrt[3]{y^2}+\sqrt[5]{32y^7}=3y\sqrt[3]{y^2}-5y\sqrt[3]{y^2}+2y\sqrt[5]{y^2}$$

$$=(3y-5y)\sqrt[3]{y^2}+2y\sqrt[5]{y^2}=\boldsymbol{-2y\sqrt[3]{y^2}+2y\sqrt[5]{y^2}}$$

Remember to write the index with each radical.

Note: The last two radical expressions cannot be combined because of different indexes.

e. To perform the addition $\sqrt{\frac{x}{16}}+2\sqrt{\frac{x^3}{9}}$, we may simplify each radical expression first. Then, we add the expressions by bringing them to the least common denominator and finally, factor the common radical, as shown below.

$$\sqrt{\frac{x}{16}}+2\sqrt{\frac{x^3}{9}}=\frac{\sqrt{x}}{\sqrt{16}}+2\frac{\sqrt{x^3}}{\sqrt{9}}=\frac{\sqrt{x}}{4}+2\frac{\sqrt{x^3}}{3}=\frac{3\sqrt{x}+2\cdot4\cdot x\sqrt{x}}{12}=\left(\frac{\mathbf{3+8x}}{\mathbf{12}}\right)\boldsymbol{\sqrt{x}}$$

f. In an attempt to simplify radicals in the expression $\sqrt{25x^2-25}-\sqrt{9x^2-9}$, we factor each radicand first. So, we obtain

$$\sqrt{25x^2-25}-\sqrt{9x^2-9}=\sqrt{25(x^2-1)}-\sqrt{9(x^2-1)}=5\sqrt{x^2-1}-3\sqrt{x^2-1}$$

$$=\mathbf{2\sqrt{x^2-1}}$$

Caution! **The root of a sum <u>does not</u> equal the sum of the roots.** For example,

$$\sqrt{5}=\sqrt{\mathbf{1+4}}\neq\sqrt{\mathbf{1}}+\sqrt{\mathbf{4}}=1+2=3$$

So, radicals such as $\sqrt{25x^2-25}$ or $\sqrt{9x^2-9}$ can be simplified only via factoring a perfect square out of their radicals while $\sqrt{x^2-1}$ cannot be simplified any further.

Multiplication of Radical Expressions with More than One Term

Similarly as in the case of multiplication of polynomials, multiplication of radical expressions where at least one factor consists of more than one term is performed by applying the distributive property.

Example 2 ► **Multiplying Radical Expressions with More than One Term**

Multiply and then simplify each product. Assume that all variables represent positive real numbers.

a. $5\sqrt{2}\left(3\sqrt{2x}-\sqrt{6}\right)$

b. $\sqrt[3]{x}\left(\sqrt[3]{3x^2}-\sqrt[3]{81x^2}\right)$

c. $\left(2\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-3\sqrt{2}\right)$

d. $\left(x\sqrt{x}-\sqrt{y}\right)\left(x\sqrt{x}+\sqrt{y}\right)$

e. $\left(3\sqrt{2}+2\sqrt[3]{x}\right)\left(3\sqrt{2}-2\sqrt[3]{x}\right)$

f. $\left(\sqrt{5y}+y\sqrt{y}\right)^2$

Solution ► **a.**

$$5\sqrt{2}\left(3\sqrt{2x}-\sqrt{6}\right)=15\sqrt{4x}-5\sqrt{2\cdot2\cdot3}=15\cdot2\sqrt{x}-5\cdot2\sqrt{3}=\mathbf{30\sqrt{x}-10\sqrt{3}}$$

$5\sqrt{2}\cdot3\sqrt{2x}=5\cdot3\sqrt{2\cdot2x}$

These are unlike terms. So, they cannot be combined.

b.

simplification

$$\sqrt[3]{x}\left(\sqrt[3]{3x^2}-\sqrt[3]{81x^2}\right)=\sqrt[3]{3x^2\cdot x}-\sqrt[3]{81x^2\cdot x}=x\sqrt[3]{3}-3x\sqrt[3]{3}=\mathbf{-2x\sqrt[3]{3}}$$

distribution

combining like terms

c. To multiply two binomial expressions involving radicals we may use the **FOIL** method. Recall that the acronym **FOIL** refers to multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials.

$$(2\sqrt{3}+\sqrt{2})(\sqrt{3}-3\sqrt{2}) = \overset{\textbf{F}}{2\cdot 3} - \overset{\textbf{O}}{6\sqrt{3\cdot 2}} + \overset{\textbf{I}}{\sqrt{2\cdot 3}} - \overset{\textbf{L}}{3\cdot 2} = \cancel{6} - 6\sqrt{6} + \sqrt{6} - \cancel{6}$$

$$= \boldsymbol{-5\sqrt{6}}$$

d. To multiply two conjugate binomial expressions we follow the difference of squares formula, $(a-b)(a+b) = a^2 - b^2$. So, we obtain

$(\sqrt{x})^2 = x$

$$(x\sqrt{x}-\sqrt{y})(x\sqrt{x}+\sqrt{y}) = (x\sqrt{x})^2 - (\sqrt{y})^2 = x^2\cdot x - y = \boldsymbol{x^3 - y}$$

square each factor

e. Similarly as in the previous example, we follow the difference of squares formula.

$$(3\sqrt{2}+2\sqrt[3]{x})(3\sqrt{2}-2\sqrt[3]{x}) = (3\sqrt{2})^2 - (2\sqrt[3]{x})^2 = 9\cdot 2 - 4\sqrt[3]{x^2} = \boldsymbol{18 - 4\sqrt[3]{x^2}}$$

f. To multiply two identical binomial expressions we follow the perfect square formula, $(a+b)(a+b) = a^2 + 2ab + b^2$. So, we obtain

$$(\sqrt{5y}+y\sqrt{y})^2 = (\sqrt{5y})^2 + 2(\sqrt{5y})(y\sqrt{y}) + (y\sqrt{y})^2 = 5y + 2y\sqrt{5y^2} + y^2y$$

$$= \boldsymbol{5y + 2\sqrt{5}\,y^2 + y^3}$$

Rationalization of Denominators

As mentioned in *Section RD3*, the process of simplifying radicals involves rationalization of any emerging denominators. Similarly, a radical expression is not in its simplest form unless all its denominators are rational. This agreement originated before the days of calculators when computation was a tedious process performed by hand. Nevertheless, even in present time, the agreement of keeping denominators rational does not lose its validity, as we often work with variable radical expressions. For example, the expressions $\frac{2}{\sqrt{2}}$ and $\sqrt{2}$ are equivalent, as

$$\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Similarly, $\frac{x}{\sqrt{x}}$ is equivalent to $\sqrt{x}$, as

$$\frac{x}{\sqrt{x}} = \frac{x}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = \frac{x\sqrt{x}}{x} = \sqrt{x}$$

While one can argue that evaluating $\frac{2}{\sqrt{2}}$ is as easy as evaluating $\sqrt{2}$ when using a calculator, the expression $\sqrt{x}$ is definitely easier to use than $\frac{x}{\sqrt{x}}$ in any further algebraic manipulations.

Definition 4.2 ► The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalization** of the denominator.

Rationalization of denominators is carried out by multiplying the given fraction by a factor of 1, as shown in the next two examples.

Example 3 ► **Rationalizing Monomial Denominators**

Simplify, if possible. Leave the answer with a rational denominator. Assume that all variables represent positive real numbers.

a. $\frac{-1}{3\sqrt{5}}$ **b.** $\frac{5}{\sqrt[3]{32x}}$ **c.** $\sqrt[4]{\frac{81x^5}{y}}$

Solution ► **a.** Notice that $\sqrt{5}$ can be converted to a rational number by multiplying it by another $\sqrt{5}$. Since the denominator of a fraction cannot be changed without changing the numerator in the same way, we multiply both, the numerator and denominator of $\frac{-1}{3\sqrt{5}}$ by $\sqrt{5}$. So, we obtain

$$\frac{-1}{3\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{-\sqrt{5}}{3 \cdot 5} = -\frac{\mathbf{\sqrt{5}}}{\mathbf{15}}$$

b. First, we may want to simplify the radical in the denominator. So, we have

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{\sqrt[3]{8 \cdot 4x}} = \frac{5}{2\sqrt[3]{4x}}$$

Then, notice that since $\sqrt[3]{4x} = \sqrt[3]{2^2x}$, it is enough to multiply it by $\sqrt[3]{2x^2}$ to nihilate the radical. This is because $\sqrt[3]{2^2x} \cdot \sqrt[3]{2x^2} = \sqrt[3]{2^3x^3} = 2x$. So, we proceed

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{2\sqrt[3]{4x}} \cdot \frac{\sqrt[3]{2x^2}}{\sqrt[3]{2x^2}} = \frac{5\sqrt[3]{2x^2}}{2 \cdot 2x} = \frac{\mathbf{5\sqrt[3]{2x^2}}}{\mathbf{4x}}$$

Caution: A common mistake in the rationalization of $\sqrt[3]{4x}$ is the attempt to multiply it by a copy of $\sqrt[3]{4x}$. However, $\sqrt[3]{4x} \cdot \sqrt[3]{4x} = \sqrt[3]{16x^2} = 2\sqrt[3]{3x^2}$ is still not rational. This is because we work with a cubic root, not a square root. So, to rationalize $\sqrt[3]{4x}$ we must look for 'filling' the radicand to a perfect cube. This is achieved by multiplying $4x$ by $2x^2$ to get $8x^3$.

c. To simplify $\sqrt[4]{\frac{81x^5}{y}}$, first, we apply the quotient rule for radicals, then simplify the radical in the numerator, and finally, rationalize the denominator. So, we have

$$\sqrt[4]{\frac{81x^5}{y}} = \frac{\sqrt[4]{81x^5}}{\sqrt[4]{y}} = \frac{3x\sqrt[4]{x}}{\sqrt[4]{y}} \cdot \frac{\sqrt[4]{y^3}}{\sqrt[4]{y^3}} = \frac{\mathbf{3x\sqrt[4]{xy^3}}}{\mathbf{y}}$$

To rationalize a binomial containing square roots, such as $2-\sqrt{x}$ or $\sqrt{2}-\sqrt{3}$, we need to find a way to square each term separately. This can be achieved through multiplying by a conjugate binomial, in order to benefit from the difference of squares formula. In particular, we can rationalize denominators in expressions below as follows:

$$\frac{1}{2-\sqrt{x}}=\frac{1}{(2-\sqrt{x})}\cdot\frac{(2+\sqrt{x})}{(2+\sqrt{x})}=\frac{\mathbf{2+\sqrt{x}}}{\mathbf{4-x}}$$

Apply the difference of squares formula: $(a-b)(a+b)=a^2-b^2$

or

$$\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}}=\frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})}\cdot\frac{(\sqrt{2}-\sqrt{3})}{(\sqrt{2}-\sqrt{3})}=\frac{2-\sqrt{6}}{2-3}=\frac{2-\sqrt{6}}{-1}=\mathbf{\sqrt{6}-2}$$

Example 4 ► **Rationalizing Binomial Denominators**

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\frac{1-\sqrt{3}}{1+\sqrt{3}}$

b. $\frac{\sqrt{xy}}{2\sqrt{x}-\sqrt{y}}$

Solution ►

factor

a. $\frac{1-\sqrt{3}}{1+\sqrt{3}}\cdot\frac{(1-\sqrt{3})}{(1-\sqrt{3})}=\frac{1-2\sqrt{3}+3}{1-3}=\frac{4-2\sqrt{3}}{-2}=\frac{\cancel{-2}(-2+\sqrt{3})}{\cancel{-2}}=\mathbf{\sqrt{3}-2}$

b. $\frac{\sqrt{xy}}{2\sqrt{x}-\sqrt{y}}\cdot\frac{(2\sqrt{x}+\sqrt{y})}{(2\sqrt{x}+\sqrt{y})}=\frac{2x\sqrt{y}+y\sqrt{x}}{4x-y}$

Some of the challenges in algebraic manipulations involve simplifying quotients with radical expressions, such as $\frac{4-2\sqrt{3}}{-2}$, which appeared in the solution to *Example 4a*. The key concept that allows us to simplify such expressions is **factoring**, as only common factors can be reduced.

Example 5 ► **Writing Quotients with Radicals in Lowest Terms**

Write each quotient in lowest terms.

a. $\frac{15-6\sqrt{5}}{6}$

b. $\frac{3x+\sqrt{8x^2}}{6x}$

Solution ► **a.** To reduce this quotient to the lowest terms we may factor the numerator first,

$$\frac{15-6\sqrt{5}}{6}=\frac{\cancel{3}(5-2\sqrt{5})}{\cancel{6}_{2}}=\frac{\mathbf{5-2\sqrt{5}}}{\mathbf{2}},$$

or alternatively, rewrite the quotient into two fractions and then simplify,

$$\frac{15-6\sqrt{5}}{6}=\frac{15}{6}-\frac{6\sqrt{5}}{6}=\frac{\mathbf{5}}{\mathbf{2}}-\sqrt{\mathbf{5}}.$$

Caution: Here are the common errors to avoid:

$\cancel{\frac{15-\cancel{6}\sqrt{5}}{\cancel{6}}=15-\sqrt{5}}$ - only common factors can be reduced!

$\cancel{\frac{15-6\sqrt{5}}{6}=\frac{\overset{3}{\cancel{9}}\sqrt{5}}{\underset{2}{\cancel{6}}}}=\frac{3\sqrt{5}}{2}$ - subtraction is performed after multiplication!

b. To reduce this quotient to the lowest terms, we simplify the radical and factor the numerator first. So,

$$\frac{3x+\sqrt{8x^2}}{6x}=\frac{3x+2x\sqrt{2}}{6x}=\frac{\cancel{x}(3+2\sqrt{2})}{6\cancel{x}}=\frac{\mathbf{3+2\sqrt{2}}}{\mathbf{6}}$$

This expression cannot be simplified any further.

RD.4 Exercises

1. A student claims that $24-4\sqrt{x}=20\sqrt{x}$ because for $x=1$ both sides of the equation equal to 20. Is this a valid justification? Explain.

2. Generally, $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$. For example, if $a=b=1$, we have $\sqrt{1+1}=\sqrt{2}\neq 2=1+1=\sqrt{1}+\sqrt{1}$. Can you think of a situation when $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$?

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

3. $2\sqrt{3}+5\sqrt{3}$

4. $6\sqrt[3]{x}-4\sqrt[3]{x}$

5. $9y\sqrt{3x}+4y\sqrt{3x}$

6. $12a\sqrt{5b}-4a\sqrt{5b}$

7. $5\sqrt{32}-3\sqrt{8}+2\sqrt{3}$

8. $-2\sqrt{48}+4\sqrt{75}-\sqrt{5}$

9. $\sqrt[3]{16}+3\sqrt[3]{54}$

10. $\sqrt[4]{32}-3\sqrt[4]{2}$

11. $\sqrt{5a}+2\sqrt{45a^3}$

12. $\sqrt[3]{24x}-\sqrt[3]{3x^4}$

13. $4\sqrt{x^3}-2\sqrt{9x}$

14. $7\sqrt{27x^3}+\sqrt{3x}$

15. $6\sqrt{18x}-\sqrt{32x}+2\sqrt{50x}$

16. $2\sqrt{128a}-\sqrt{98a}+2\sqrt{72a}$

17. $\sqrt[3]{6x^4} + \sqrt[3]{48x} - \sqrt[3]{6x}$

18. $9\sqrt{27y^2} - 14\sqrt{108y^2} + 2\sqrt{48y^2}$

19. $3\sqrt{98n^2} - 5\sqrt{32n^2} - 3\sqrt{18n^2}$

20. $-4y\sqrt{xy^3} + 7x\sqrt{x^3y}$

21. $6a\sqrt{ab^5} - 9b\sqrt{a^3b}$

22. $\sqrt[3]{-125p^9} + p\sqrt[3]{-8p^6}$

23. $3\sqrt[4]{x^5y} + 2x\sqrt[4]{xy}$

24. $\sqrt{125a^5} - 2\sqrt[3]{125a^4}$

25. $x\sqrt[3]{16x} + \sqrt{2} - \sqrt[3]{2x^4}$

26. $\sqrt{9a-9} + \sqrt{a-1}$

27. $\sqrt{4x+12} - \sqrt{x+3}$

28. $\sqrt{x^3-x^2} - \sqrt{4x-4}$

29. $\sqrt{25x-25} - \sqrt{x^3-x^2}$

30. $\frac{4\sqrt{3}}{3} - \frac{2\sqrt{3}}{9}$

31. $\frac{\sqrt{27}}{2} - \frac{3\sqrt{3}}{4}$

32. $\sqrt{\frac{49}{x^4}} + \sqrt{\frac{81}{x^8}}$

33. $2a\sqrt[4]{\frac{a}{16}} - 5a\sqrt[4]{\frac{a}{81}}$

34. $-4\sqrt[3]{\frac{4}{y^9}} + 3\sqrt[3]{\frac{9}{y^{12}}}$

35. A student simplifies the below expression as follows:

$$\begin{aligned}\sqrt{8} + \sqrt[3]{16} &\overset{?}{=} \sqrt{4\cdot 2} + \sqrt[3]{8\cdot 2}\\ &\overset{?}{=} \sqrt{4}\cdot\sqrt{2} + \sqrt[3]{8}\cdot\sqrt[3]{2}\\ &\overset{?}{=} 2\sqrt{2} + 2\sqrt[3]{2}\\ &\overset{?}{=} 4\sqrt{4}\\ &\overset{?}{=} 8\end{aligned}$$

Check each equation for correctness and disscuss any errors that you can find. What would you do differently and why?

36. Match each expression from ***Column I*** with the equivalent expression in ***Column II***. Assume that A and B represent positive real numbers.

Column I	***Column II***
A. $(A+\sqrt{B})(A-\sqrt{B})$	**a.** $A-B$
B. $(\sqrt{A}+B)(\sqrt{A}-B)$	**b.** $A+2B\sqrt{A}+B^2$
C. $(\sqrt{A}+\sqrt{B})(\sqrt{A}-\sqrt{B})$	**c.** $A-B^2$
D. $(\sqrt{A}+\sqrt{B})^2$	**d.** $A-2\sqrt{AB}+B$
E. $(\sqrt{A}-\sqrt{B})^2$	**e.** A^2-B
F. $(\sqrt{A}+B)^2$	**f.** $A+2\sqrt{AB}+B$

Multiply, and then simplify each product. Assume that all variables represent positive real numbers.

37. $\sqrt{5}\left(3-2\sqrt{5}\right)$

38. $\sqrt{3}\left(3\sqrt{3}-\sqrt{2}\right)$

39. $\sqrt{2}\left(5\sqrt{2}-\sqrt{10}\right)$

40. $\sqrt{3}\left(-4\sqrt{3}+\sqrt{6}\right)$

41. $\sqrt[3]{2}\left(\sqrt[3]{4}-2\sqrt[3]{32}\right)$

42. $\sqrt[3]{3}\left(\sqrt[3]{9}+2\sqrt[3]{21}\right)$

43. $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$

44. $\left(\sqrt{5}+\sqrt{7}\right)\left(\sqrt{5}-\sqrt{7}\right)$

45. $\left(2\sqrt{3}+5\right)\left(2\sqrt{3}-5\right)$

46. $\left(6+3\sqrt{2}\right)\left(6-3\sqrt{2}\right)$

47. $\left(5-\sqrt{5}\right)^2$

48. $\left(\sqrt{2}+3\right)^2$

49. $\left(\sqrt{a}+5\sqrt{b}\right)\left(\sqrt{a}-5\sqrt{b}\right)$

50. $\left(2\sqrt{x}-3\sqrt{y}\right)\left(2\sqrt{x}+3\sqrt{y}\right)$

51. $\left(\sqrt{3}+\sqrt{6}\right)^2$

52. $\left(\sqrt{5}-\sqrt{10}\right)^2$

53. $\left(2\sqrt{5}+3\sqrt{2}\right)^2$

54. $\left(2\sqrt{3}-5\sqrt{2}\right)^2$

55. $\left(4\sqrt{3}-5\right)\left(\sqrt{3}-2\right)$

56. $\left(4\sqrt{5}+3\sqrt{3}\right)\left(3\sqrt{5}-2\sqrt{3}\right)$

57. $\left(\sqrt[3]{2y}-5\right)\left(\sqrt[3]{2y}+1\right)$

58. $\left(\sqrt{x+5}-3\right)\left(\sqrt{x+5}+3\right)$

59. $\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)$

60. $\left(\sqrt{x+2}+\sqrt{x-2}\right)^2$

Given $f(x)$ and $g(x)$, find $(f+g)(x)$ and $(fg)(x)$.

61. $f(x)=5x\sqrt{20x}$ and $g(x)=3\sqrt{5x^3}$

62. $f(x)=2x\sqrt[4]{64x}$ and $g(x)=-3\sqrt[4]{4x^5}$

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

63. $\frac{\sqrt{5}}{2\sqrt{2}}$

64. $\frac{3}{5\sqrt{3}}$

65. $\frac{12}{\sqrt{6}}$

66. $-\frac{15}{\sqrt{24}}$

67. $-\frac{10}{\sqrt{20}}$

68. $\sqrt{\frac{3x}{20}}$

69. $\sqrt{\frac{5y}{32}}$

70. $\frac{\sqrt[3]{7a}}{\sqrt[3]{3b}}$

71. $\frac{\sqrt[3]{2y^4}}{\sqrt[3]{6x^4}}$

72. $\frac{\sqrt[3]{3n^4}}{\sqrt[3]{5m^2}}$

73. $\frac{pq}{\sqrt[4]{p^3q}}$

74. $\frac{2x}{\sqrt[5]{18x^8}}$

75. $\frac{17}{6+\sqrt{2}}$

76. $\frac{4}{3-\sqrt{5}}$

77. $\frac{2\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

78. $\frac{6\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$

79. $\frac{3}{3\sqrt{5}+2\sqrt{3}}$

80. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+5\sqrt{2}}$

81. $\frac{m-4}{\sqrt{m}+2}$

82. $\frac{4}{\sqrt{x}-2\sqrt{y}}$

83. $\frac{\sqrt{3}+2\sqrt{x}}{\sqrt{3}-2\sqrt{x}}$

84. $\frac{\sqrt{x}-2}{3\sqrt{x}+\sqrt{y}}$

85. $\frac{2\sqrt{a}}{\sqrt{a}-\sqrt{b}}$

86. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Write each quotient in lowest terms. Assume that all variables represent positive real numbers.

87. $\frac{10-20\sqrt{5}}{10}$

88. $\frac{12+6\sqrt{3}}{6}$

89. $\frac{12-9\sqrt{72}}{18}$

90. $\frac{2x+\sqrt{8x^2}}{2x}$

91. $\frac{6p-\sqrt{24p^3}}{3p}$

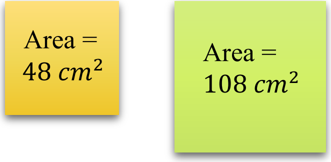
92. $\frac{9x+\sqrt{18}}{15}$

93. When solving one of the trigonometry problems, a student come up with the answer $\frac{\sqrt{3}-1}{1+\sqrt{3}}$. The textbook answer to this problem was $2-\sqrt{3}$. Was the student's answer equivalent to the textbook answer?

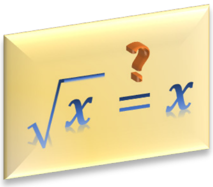
Solve each problem.

94. The base of the second tallest of the Pyramids of Giza is a square with an area of 46,225 m^2. What is its perimeter?

95. The areas of two types of square wall tiles sold at the local Home Depot store are 48 cm^2 and 108 cm^2, respectively. What is the difference in the length of sides of the two tiles? *Give the exact answer in a simplified radical form and its approximation to the nearest tenth.*



RD5 Radical Equations



In this section, we discuss techniques for solving radical equations. These are equations containing at least one radical expression with a variable, such as $\sqrt{3x-2}=x$, or a variable expression raised to a fractional exponent, such as $(2x)^{\frac{1}{3}}+1=5$.
At the end of this section, we revisit working with formulas involving radicals as well as application problems that can be solved with the use of radical equations.

Radical Equations

Definition 5.1 ► A **radical equation** is an equation in which a variable appears in one or more radicands. This includes radicands 'hidden' under fractional exponents.
For example, since $(x-1)^{\frac{1}{2}}=\sqrt{x-1}$, then the base $x-1$ is, in fact, the 'hidden' radicand.

Some examples of radical equations are

$$x=\sqrt{2x},\quad \sqrt{x}+\sqrt{x-2}=5,\quad (x-4)^{\frac{3}{2}}=8,\quad \sqrt[3]{3+x}=5$$

Note that $x=\sqrt{2}$ is not a radical equation since there is no variable under the radical sign.

The process of solving radical equations involves clearing radicals by raising both sides of an equation to an appropriate power. This method is based on the following property of equality.

Power Rule: For any **odd** natural number $\boldsymbol{n}$, the equation $\boldsymbol{a}=\boldsymbol{b}$ is equivalent to the equation $\boldsymbol{a^n}=\boldsymbol{b^n}$.

For any **even** natural number $\boldsymbol{n}$, if an equation $\boldsymbol{a}=\boldsymbol{b}$ is true, then $\boldsymbol{a^n}=\boldsymbol{b^n}$ is true.

When rephrased, the power rule for odd powers states that the solution sets to both equations, $a=b$ and $a^n=b^n$, are exactly the same.

However, the power rule for even powers states that the solutions to the original equation $a=b$ are among the solutions to the 'power' equation $a^n=b^n$.

Unfortunately, the reverse implication does not hold for even numbers n. We cannot conclude that $a=b$ from the fact that $a^n=b^n$ is true. For instance, $3^2=(-3)^2$ is true but $3\neq-3$. This means that not all solutions of the equation $a^n=b^n$ are in fact true solutions to the original equation $a=b$. Solutions that do not satisfy the original equation are called **extraneous solutions** or **extraneous roots**. Such solutions must be rejected.

For example, to solve $\sqrt{2-x}=x$, we may square both sides of the equation to obtain the quadratic equation

$$2-x=x^2.$$

Then, we solve it via factoring and the zero-product property:

$$x^2+x-2=0$$
$$(x+2)(x-1)=0$$

So, the possible solutions are $x = -2$ and $x = 1$.

Notice that $x = 1$ satisfies the original equation, as $\sqrt{2-1} = 1$ is true. However, $x = -2$ does not satisfy the original equation as its left side equals to $\sqrt{2-(-2)} = \sqrt{4} = 2$, while the right side equals to -2. Thus, $x = -2$ is the extraneous root and as such, it does not belong to the solution set of the original equation. So, the solution set of the original equation is $\{1\}$.

Caution: When the power rule for **even powers** is used to solve an equation, **every solution** of the 'power' equation **must be checked in the original equation**.

Example 1 ► Solving Equations with One Radical

Solve each equation.

a. $\sqrt{3x+4} = 4$

b. $\sqrt{2x-5} + 4 = 0$

c. $2\sqrt{x+1} = x - 7$

d. $\sqrt[3]{x-8} + 2 = 0$

Solution ►

a. Since the radical in $\sqrt{3x+4} = 4$ is isolated on one side of the equation, squaring both sides of the equation allows for clearing (reversing) the square root. Then, by solving the resulting polynomial equation, one can find the possible solution(s) to the original equation.

$(\sqrt{a})^2 = \left(a^{\frac{1}{2}}\right)^2 = a$

$$\left(\sqrt{3x+4}\right)^2 = (4)^2$$
$$3x + 4 = 16$$
$$3x = 12$$
$$x = 4$$

To check if 4 is a true solution, it is enough to check whether or not $x = 4$ satisfies the original equation.

$$\sqrt{3 \cdot 4 + 4} \overset{?}{=} 4$$
$$\sqrt{16} \overset{?}{=} 4$$
$$4 = 4 \quad \checkmark$$

true

Since $x = 4$ satisfies the original equation, the solution set is $\{\mathbf{4}\}$.

b. To solve $\sqrt{2x-5} + 4 = 0$, it is useful to isolate the radical on one side of the equation. So, consider the equation

$$\sqrt{2x-5} = -4$$

Notice that the left side of the above equation is nonnegative for any x-value while the right side is constantly negative. Thus, such an equation cannot be satisfied by any x-value. Therefore, this equation has **no solution**.

c. Squaring both sides of the equation gives us

$$\left(2\sqrt{x+1}\right)^2 = (x-7)^2$$

$$4(x+1) = x^2 - 14x + 49$$

the bracket is essential here

apply the **perfect square** formula $(a-b)^2 = a^2 - 2ab + b^2$

$$4x + 4 = x^2 - 14x + 49$$

$$x^2 - 18x + 45 = 0$$

$$(x-3)(x-15) = 0$$

So, the possible solutions are $x = 3$ or $x = 15$. We check each of them by substituting them into the original equation.

If $x = 3$, then

$$2\sqrt{3+1} \overset{?}{=} 3 - 7$$

$$2\sqrt{4} \overset{?}{=} -4$$

$$4 \neq -4$$ ✗ false

So $x = 3$ is an **extraneous** root.

If $x = 15$, then

$$2\sqrt{15+1} \overset{?}{=} 15 - 7$$

$$2\sqrt{16} \overset{?}{=} 8$$

$$8 = 8$$ ✓ true

Since only 15 satisfies the original equation, the solution set is $\{15\}$.

d. To solve $\sqrt[3]{x-8} + 2 = 0$, we first isolate the radical by subtracting 2 from both sides of the equation.

$$\sqrt[3]{x-8} = -2$$

Then, to clear the cube root, we raise both sides of the equation to the third power.

$$\left(\sqrt[3]{x-8}\right)^3 = (-2)^3$$

So, we obtain

$$x - 8 = -8$$

$$x = 0$$

Since we applied the power rule for odd powers, the obtained solution is the true solution. So the solution set is $\{0\}$.

Observation: When using the power rule for odd powers checking the obtained solutions against the original equation is not necessary. This is because there is no risk of obtaining extraneous roots when applying the power rule for odd powers.

To solve radical equations with more than one radical term, we might need to apply the power rule repeatedly until all radicals are cleared. In an efficient solution, each application of the power rule should cause clearing of at least one radical term. For that reason, it is a good idea to isolate a single radical term on one side of the equation before each application of the power rule. For example, to solve the equation

$$\sqrt{x-3}+\sqrt{x+5}=4,$$

we isolate one of the radicals before squaring both sides of the equation. So, we have

$$\left(\sqrt{x-3}\right)^2=\left(4-\sqrt{x+5}\right)^2$$

$$\cancel{x}-3=\underbrace{16}_{a^2}-\underbrace{8\sqrt{x+5}}_{2ab}+\underbrace{\cancel{x}+5}_{b^2}$$

Remember that the **perfect square** formula consists of **three terms**.

Then, we isolate the remaining radical term and simplify, if possible. This gives us

$$8\sqrt{x+5}=24$$

$$\sqrt{x+5}=3$$

Squaring both sides of the last equation gives us

$$x+5=9$$

$$x=4$$

The reader is encouraged to check that $x=\mathbf{4}$ is the true solution to the original equation.

A general strategy for solving radical equations, including those with two radical terms, is as follows.

Summary of Solving a Radical Equation

- **Isolate one of the radical terms**. Make sure that one radical term is alone on one side of the equation.
- **Apply an appropriate power rule.** Raise each side of the equation to a power that is the same as the index of the isolated radical.
- **Solve the resulting equation.** If it still contains a radical, repeat the first two steps.
- **Check** all proposed solutions in the original equation.
- **State the solution set** to the original equation.

Example 2 ► Solving Equations Containing Two Radical Terms

Solve each equation.

a. $\sqrt{3x+1}-\sqrt{x+4}=1$ **b.** $\sqrt[3]{4x-5}=2\sqrt[3]{x+1}$

Solution ► **a.** We start solving the equation $\sqrt{3x+1}-\sqrt{x+4}=1$ by isolating one radical on one side of the equation. This can be done by adding $\sqrt{x+4}$ to both sides of the equation. So, we have

$$\sqrt{3x+1}=1+\sqrt{x+4}$$

which after squaring give us

$$
\begin{aligned}
\left(\sqrt{3x+1}\right)^2 &= \left(1+\sqrt{x+4}\right)^2 \\
3x+1 &= 1+2\sqrt{x+4}+x+4 \\
2x-4 &= 2\sqrt{x+4} \\
x-2 &= \sqrt{x+4}.
\end{aligned}
$$

To clear the remaining radical, we square both sides of the above equation again.

$$
\begin{aligned}
(x-2)^2 &= \left(\sqrt{x+4}\right)^2 \\
x^2-4x+4 &= x+4 \\
x^2-5x &= 0.
\end{aligned}
$$

The resulting polynomial equation can be solved by factoring and applying the zero-product property. Thus,

$$x(x-5)=0.$$

So, the possible roots are $x=0$ or $x=5$.

We check each of them by substituting to the original equation.

If $x=0$, then

$$
\begin{aligned}
\sqrt{3\cdot 0+1}-\sqrt{0+4} &\overset{?}{=} 1 \\
\sqrt{1}-\sqrt{4} &\overset{?}{=} 1 \\
1-2 &\overset{?}{=} 1 \\
-1 &\neq 1 \quad ✗
\end{aligned}
$$

false

If $x=5$, then

$$
\begin{aligned}
\sqrt{3\cdot 5+1}-\sqrt{5+4} &\overset{?}{=} 1 \\
\sqrt{16}-\sqrt{9} &\overset{?}{=} 1 \\
4-3 &\overset{?}{=} 1 \\
1 &= 1 \quad ✓
\end{aligned}
$$

true

Since $x=0$ is the **extraneous** root, it does not belong to the solution set.

Only 5 satisfies the original equation. So, the solution set is $\{\mathbf{5}\}$.

b. To solve the equation $\sqrt[3]{4x-5}=2\sqrt[3]{x+1}$, we would like to clear the cubic roots. This can be done by cubing both of its sides, as shown below.

$$
\begin{aligned}
\left(\sqrt[3]{4x-5}\right)^3 &= \left(2\sqrt[3]{x+1}\right)^3 \\
4x-5 &= 2^3(x+1) \\
4x-5 &= 8x+8 \\
-13 &= 4x \\
x &= -\frac{\mathbf{13}}{\mathbf{4}}
\end{aligned}
$$

the bracket is essential here

Since we applied the power rule for cubes, the obtained root is the true solution of the original equation.

Formulas Containing Radicals

Many formulas involve radicals. For example, the period T, in seconds, of a pendulum of length L, in feet, is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Sometimes, we might need to solve a radical formula for a specified variable. In addition to all the strategies for solving formulas for a variable, discussed in *Sections L2*, *F4*, and *RT6*, we may need to apply the power rule to clear the radical(s) in the formula.

Example 3 ► **Solving Radical Formulas for a Specified Variable**

Solve each formula for the indicated variable.

a. $N = \frac{1}{2\pi}\sqrt{\frac{\boldsymbol{a}}{r}}$ for $\boldsymbol{a}$

b. $r = \sqrt[3]{\frac{A}{\boldsymbol{P}}} - 1$ for $\boldsymbol{P}$

Solution ►

a. Since $\boldsymbol{a}$ appears in the radicand, to solve $N = \frac{1}{2\pi}\sqrt{\frac{\boldsymbol{a}}{r}}$ for $\boldsymbol{a}$, we may want to clear the radical by squaring both sides of the equation. So, we have

$$N^2 = \left(\frac{1}{2\pi}\sqrt{\frac{\boldsymbol{a}}{r}}\right)^2$$

$$N^2 = \frac{1}{(2\pi)^2}\cdot\frac{\boldsymbol{a}}{r}$$

$$\boldsymbol{4\pi^2N^2r} = \boldsymbol{a}$$

Note: We could also first multiply by 2π and then square both sides of the equation.

b. First, observe the position of $\boldsymbol{P}$ in the equation $r = \sqrt[3]{\frac{A}{\boldsymbol{P}}} - 1$. It appears in the denominator of the radical. Therefore, to solve for $\boldsymbol{P}$, we may plan to isolate the cube root first, cube both sides of the equation to clear the radical, and finally bring $\boldsymbol{P}$ to the numerator. So, we have

$$r = \sqrt[3]{\frac{A}{\boldsymbol{P}}} - 1$$

$$(r+1)^3 = \left(\sqrt[3]{\frac{A}{\boldsymbol{P}}}\right)^3$$

$$(r+1)^3 = \frac{A}{P}$$

$$\boldsymbol{P = \frac{A}{(r+1)^3}}$$

Radicals in Applications

Many application problems in sciences, engineering, or finances translate into radical equations.

Example 4 ► **Finding the Velocity of a Skydiver**

After d meters of a free fall from an airplane, a skydiver's velocity v, in kilometers per hour, can be estimated according to the formula $v = 15.9\sqrt{d}$. Approximately how far, in meters, does a skydiver need to fall to attain the velocity of 100 km/h?

Solution ► We may substitute $v = 100$ into the equation $v = 15.9\sqrt{d}$ and solve it for d, as below.

$$100 = 15.9\sqrt{d}$$

$$6.3 \approx \sqrt{d}$$

$$\boldsymbol{40 \approx d}$$

Thus, a skydiver falls at 100 kph approximately after 40 meters of free falling.

RD.5 Exercises

True or false.

1. $\sqrt{2}x = x^2 - \sqrt{5}$ is a radical equation.

2. When raising each side of a radical equation to a power, the resulting equation is equivalent to the original equation.

3. $\sqrt{3x+9} = x$ cannot have negative solutions.

4. -9 is a solution to the equation $\sqrt{x} = -3$.

Solve each equation.

5. $\sqrt{7x-3}=6$ **6.** $\sqrt{5y+2}=7$ **7.** $\sqrt{6x}+1=3$ **8.** $\sqrt{2k}-4=6$

9. $\sqrt{x+2}=-6$ **10.** $\sqrt{y-3}=-2$ **11.** $\sqrt[3]{x}=-3$ **12.** $\sqrt[3]{a}=-1$

13. $\sqrt[4]{y-3}=2$ **14.** $\sqrt[4]{n+1}=3$ **15.** $5=\frac{1}{\sqrt{a}}$ **16.** $\frac{1}{\sqrt{y}}=3$

17. $\sqrt{3r+1}-4=0$ **18.** $\sqrt{5x-4}-9=0$ **19.** $4-\sqrt{y-2}=0$

20. $9-\sqrt{4a+1}=0$ **21.** $x-7=\sqrt{x-5}$ **22.** $x+2=\sqrt{2x+7}$

23. $2\sqrt{x+1}-1=x$ **24.** $3\sqrt{x-1}-1=x$ **25.** $y-4=\sqrt{4-y}$

26. $x+3=\sqrt{9-x}$ **27.** $x=\sqrt{x^2+4x-20}$ **28.** $x=\sqrt{x^2+3x+9}$

29. Discuss the validity of the following solution:

$$\begin{aligned}\sqrt{2x+1}&=4-x\\ 2x+1&=16+x^2\\ x^2-2x+15&=0\\ (x-5)(x+3)&=0\\ \text{so}\quad x=5 &\text{ or } x=-3\end{aligned}$$

30. Discuss the validity of the following solution:

$$\begin{aligned}\sqrt{3x+1}-\sqrt{x+4}&=1\\ (3x+1)-(x+4)&=1\\ 2x-3&=1\\ 2x&=4\\ x&=2\end{aligned}$$

Solve each equation.

31. $\sqrt{5x+1}=\sqrt{2x+7}$ **32.** $\sqrt{5y-3}=\sqrt{2y+3}$ **33.** $\sqrt[3]{p+5}=\sqrt[3]{2p-4}$

34. $\sqrt[3]{x^2+5x+1}=\sqrt[3]{x^2+4x}$ **35.** $2\sqrt{x-3}=\sqrt{7x+15}$ **36.** $\sqrt{6x-11}=3\sqrt{x-7}$

37. $3\sqrt{2t+3}-\sqrt{t+10}=0$ **38.** $2\sqrt{y-1}-\sqrt{3y-1}=0$ **39.** $\sqrt{x-9}+\sqrt{x}=1$

40. $\sqrt{y-5}+\sqrt{y}=5$ **41.** $\sqrt{3n}+\sqrt{n-2}=4$ **42.** $\sqrt{x+5}-2=\sqrt{x-1}$

43. $\sqrt{14-n}=\sqrt{n+3}+3$ **44.** $\sqrt{p+15}-\sqrt{2p+7}=1$

45. $\sqrt{4a+1}-\sqrt{a-2}=3$ **46.** $4-\sqrt{a+6}=\sqrt{a-2}$

47. $\sqrt{x-5}+1=-\sqrt{x+3}$

48. $\sqrt{3x-5}+\sqrt{2x+3}+1=0$

49. $\sqrt{2m-3}+2-\sqrt{m+7}=0$

50. $\sqrt{x+2}+\sqrt{3x+4}=2$

51. $\sqrt{6x+7}-\sqrt{3x+3}=1$

52. $\sqrt{4x+7}-4=\sqrt{4x-1}$

53. $\sqrt{5y+4}-3=\sqrt{2y-2}$

54. $\sqrt{2\sqrt{x+11}}=\sqrt{4x+2}$

55. $\sqrt{1+\sqrt{24+10x}}=\sqrt{3x+5}$

56. $(2x-9)^{\frac{1}{2}}=2+(x-8)^{\frac{1}{2}}$

57. $(3k+7)^{\frac{1}{2}}=1+(k+2)^{\frac{1}{2}}$

58. $(x+1)^{\frac{1}{2}}-(x-6)^{\frac{1}{2}}=1$

59. $\sqrt{(x^2-9)^{\frac{1}{2}}}=2$

60. $\sqrt{\sqrt{x}+4}=\sqrt{x}-2$

61. $\sqrt{a^2+30a}=a+\sqrt{5a}$

62. Discuss how to evaluate the expression $\sqrt{5+3\sqrt{3}}-\sqrt{5-3\sqrt{3}}$ without the use of a calculator.

Solve each formula for the indicated variable.

63. $Z=\sqrt{\frac{L}{C}}$ for L

64. $V=\sqrt{\frac{2K}{m}}$ for K

65. $V=\sqrt{\frac{2K}{m}}$ for m

66. $r=\sqrt{\frac{Mm}{F}}$ for M

67. $r=\sqrt{\frac{Mm}{F}}$ for F

68. $Z=\sqrt{L^2+R^2}$ for R

69. $F=\frac{1}{2\pi\sqrt{LC}}$ for C

70. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for a

71. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for r

Solve each problem.

72. One of Einstein's special relativity principles states that time passes faster for bodies that travel with greater speed. The ratio of the time that passes for a body that moves with a speed v to the elapsed time that passes on Earth is called the ***aging rate*** and can be calculated by using the formula $\boldsymbol{r}=\frac{\sqrt{c^2-v^2}}{\sqrt{c^2}}$, where c is the speed of light, and v is the speed of the travelling body. For example, the aging rate of 0.5 means that one year for the person travelling at the speed v corresponds to two years spent on Earth.

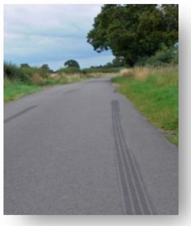
a. Find the aging rate for a person travelling at 80% of the speed of light.

b. Find the elapsed time on Earth for 20 days of travelling time at 60% of the speed of light.

73. Assume that the formula $\boldsymbol{BSA}=\sqrt{\frac{11wh}{18000}}$ can be used to calculate the ***Body Surface Area***, in square meters, of a person with the weight w, in kilograms, and the height h, in centimeters. Greg weighs 78 kg and has a BSA of 3 m^2. To the nearest centimeter, how tall is he?

74. The distance d, in kilometers, to the horizon for an object h kilometers above the Earth's surface can be approximated by using the equation $\boldsymbol{d = \sqrt{12800h + h^2}}$. Estimate the distance between a satellite that is 1000 km above the Earth's surface and the horizon.

75. The formula $\boldsymbol{S = \frac{24}{5}\sqrt{10fL}}$, where f is the drag factor of the road surface, and L is the length of a skid mark, in meters, allows for calculating the speed S, in kilometers per hour, of a car before it started skidding to a stop. To the nearest meter, calculate the length of the skid marks left by a stopping car on a road surface with a drag factor of 0.5, if the car was travelling at 50 km/h at the time of applying the brakes.



RD6 Complex Numbers

Have you wondered if there's a solution to an equation like $x^2 = -4$? We know there is no solution in the set of real numbers since the square of any real number is positive; however, a solution does exist in the set of *complex numbers*. Complex numbers allow us to work with square roots of negative numbers and solve equations like $x^2 = -4$. This is important because equations with complex solutions arise frequently in mathematics, physics, engineering, electronics, and many other fields.

In this section, we introduce the imaginary unit and use it to perform operations with complex numbers.

Imaginary and Complex Numbers

Definition 6.1 ► The **imaginary unit** i is the number whose square is -1,

$$i^2 = -1 \quad \text{and} \quad i = \sqrt{-1}$$

The imaginary unit can be used to simplify the square roots of negative numbers,

$$\sqrt{-p} = \sqrt{p}\, i,$$

where p is a positive real number.

Note: The i **multiplies** the radical and is **not** part of the radicand.

Example 1 ► **Rewriting Square Roots of Negative Numbers Using *i***

Write each expression in terms of i and simplify if possible.

a. $\sqrt{-25}$

b. $\sqrt{-7}$

c. $\sqrt{-72}$

d. $-\sqrt{-60}$

Solution ►

a. We use *Definition 6.1* to rewrite the expression and simplify:

$$\sqrt{-25} = \sqrt{25}\, i = \mathbf{5}\boldsymbol{i}$$

b. $\sqrt{-7} = \boldsymbol{\sqrt{7}\, i}$ since the radicand 7 has no perfect square factors.

c. $\sqrt{-72} = \sqrt{72}\, i = \sqrt{36 \cdot 2}\, i = \boldsymbol{6\sqrt{2}\, i}$

d. $-\sqrt{-60} = -\sqrt{60}\, i = -\sqrt{4 \cdot 15}\, i = -\boldsymbol{2\sqrt{15}\, i}$

the leading negative remains unchanged

Definition 6.2 ► A **complex number** in standard form is $a + bi$, where a and b are real numbers.

$$\underset{\textit{real part}}{\boxed{a}} + \underset{\textit{imaginary part}}{\boxed{b}}\, i$$

Observation: When the real part of a complex number is zero, $a = 0$, the number is imaginary (bi). When the imaginary part is zero, $b = 0$, the number is real (a). So there are three types of complex numbers - real numbers, imaginary numbers, and numbers that have both a real part and an imaginary part.

Addition and Subtraction of Complex Numbers

Now we are ready to perform some operations on complex numbers.

To add or subtract complex numbers, combine the real parts together and the imaginary parts together, as in the example below:

$$(1+2i)+(7-3i) = 1+7+2i-3i = \underbrace{(1+7)}_{\textit{real parts}} + \underbrace{(2-3)}_{\textit{imaginary parts}}i = 8-i$$

Caution: If the complex numbers are not already in standard form, convert them **before** performing operations.

Example 2 ► Adding and Subtracting Complex Numbers

Perform operations and simplify, if possible.

a. $(9-4i)-(2+6i)$

b. $\sqrt{-81}+\sqrt{-1}$

c. $\sqrt{-72}-3\sqrt{-2}$

d. $\left(10-2\sqrt{-3}\right)+\left(5+6\sqrt{-27}\right)$

Solution ►

a. First rewrite the subtraction to release the brackets, then combine like terms.

$$9-4i-2-6i = (9-2)+(-4i-6i) = \mathbf{7-10}\boldsymbol{i}$$

b. Use $\sqrt{-1} = i$ to rewrite in standard form **before** performing the addition

$$\sqrt{-81}+\sqrt{-1} = 9i+i = \mathbf{10}\boldsymbol{i}$$

remember to simplify any radicals with perfect square factors

c. $\sqrt{-72}-3\sqrt{-2} = \sqrt{72}\,i-3\sqrt{2}\,i = 6\sqrt{2}\,i-3\sqrt{2}\,i = \mathbf{3\sqrt{2}}\,\boldsymbol{i}$

d.
$$\begin{aligned}\left(10-2\sqrt{-3}\right)+\left(5+6\sqrt{-27}\right) &= \left(10-2\sqrt{3}\,i\right)+\left(5+6\cdot3\sqrt{3}\,i\right)\\ &= \left(10-2\sqrt{3}\,i\right)+\left(5+18\sqrt{3}\,i\right)\\ &= \mathbf{15+16\sqrt{3}}\,\boldsymbol{i}\end{aligned}$$

Multiplication of Complex Numbers

Most of our algebraic rules for real numbers hold for complex numbers. One notable exception is that the product rule for radicals, $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$, **is not true** if a and b are **both negative**. To illustrate this, recall our definition of i

$$i^2=-1 \text{ and } \sqrt{-1}=i$$

Now calculate $i^2=\sqrt{-1}\cdot\sqrt{-1}$ using $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$:

$$\sqrt{-1}\cdot\sqrt{-1}=\sqrt{-1\cdot-1}=\sqrt{1}=1$$

This contradicts our original definition that $i^2=-1$ and therefore $\sqrt{a}\cdot\sqrt{b}\neq\sqrt{ab}$ for negative a and b.

To avoid accidentally using an invalid rule, we always change $\sqrt{-1}$ to i first, then carry out our calculations. This way, the order of operations will be applied correctly.

Example 3 ► **Multiplying Complex Numbers**

Multiply.

a. $\sqrt{-8}\cdot\sqrt{-2}$

b. $2i\cdot 7i$

c. $(4+3i)(1-5i)$

d. $\left(5+\sqrt{-6}\right)\left(3-\sqrt{-2}\right)$

Solution ►

a. Rewrite in standard form **before** simplifying

$$\sqrt{-8}\cdot\sqrt{-2}=2\sqrt{2}\,i\cdot\sqrt{2}\,i=2\sqrt{4}\,i^2=2\cdot 2\cdot(-1)=\mathbf{-4}$$

product rule valid here since $2>0$

replace i^2 with its value of -1 when it appears

b. Multiply imaginary numbers like monomials then use $i^2=-1$ where appropriate

$$2i\cdot 7i=14i^2=14(-1)=\mathbf{-14}$$

c. Use distribution to multiply complex numbers in the same way as binomials

$$(4+3i)(1-5i)=4-20i+3i-15i^2=4-17i-15(-1)=\mathbf{19-17}\boldsymbol{i}$$

collect like terms

d. $\left(5+\sqrt{-6}\right)\left(3-\sqrt{-2}\right)=\left(5+\sqrt{6}\,i\right)\left(3-\sqrt{2}\,i\right)=15-5\sqrt{2}\,i+3\sqrt{6}\,i-\sqrt{12}\,i^2$

$$=15+\left(-5\sqrt{2}+3\sqrt{6}\right)i-2\sqrt{3}\,(-1)$$

$$=\mathbf{15+2\sqrt{3}+\left(-5\sqrt{2}+3\sqrt{6}\right)}\boldsymbol{i}$$

Since complex numbers behave like binomials when multiplied, patterns can help us simplify some products more efficiently than using distribution or FOIL. In the following example, we use the perfect squares formula, $(a\pm b)^2=a^2\pm 2ab+b^2$, and the difference of squares formula, $(a+b)(a-b)=a^2-b^2$, which were introduced in *Section P2*.

Example 4 ► **Multiplying Complex Numbers Using Patterns**

Multiply.

a. $(4+i)^2$

b. $(7-6i)^2$

c. $(1+2i)(1-2i)$

d. $(-3+i)(-3-i)$

Solution ► **a.** We recognize the perfect squares pattern with $a=4$ and $b=i$.

$$(4+i)^2=4^2+2\cdot 4\cdot i+i^2=16+8i+(-1)=\mathbf{15+8}\boldsymbol{i}$$

b. Here $a=7, b=6i$ in the perfect squares formula. The binomial is a difference, so the formula uses the subtraction sign for the middle term.

$$(7-6i)^2=7^2-2\cdot 7\cdot(6i)+(6i)^2=49-84i+36i^2=49-84i-36$$
$$=\mathbf{13-84}\boldsymbol{i}$$

c. This is a product of conjugates, so we use the difference of squares formula with $a=1, b=2i$.

$$(1+2i)(1-2i)=1^2-(2i)^2=1-4i^2=1+4=\mathbf{5}$$

d. Apply the difference of squares formula with $a=-3, b=i$.

$$(-3+i)(-3-i)=(-3)^2-i^2=9+1=\mathbf{10}$$

Notice in *Example 4c* and *4d* that the product of conjugate pairs resulted in a real value. In general, we have

$$(a+bi)(a-bi)=a^2-(bi)^2=a^2-b^2i^2=a^2-b^2(-1)=a^2+b^2$$

Therefore, the product of any complex conjugate pair, $a+bi$ and $a-bi$, is real and equal to a^2+b^2. This new pattern is referred to as the **product of complex conjugates formula**

$$\boxed{\boldsymbol{(a+bi)(a-bi)=a^2+b^2.}}$$

Powers of *i*

We simplify complex expressions by treating i much like a variable; however, it is important to remember that i has a **constant value** of $\sqrt{-1}$. This means we can simplify powers of i further than we would be able to simplify powers of variables. In fact, any power of i can be simplified to one of four values: i, $-1, -i$, or 1. Look for the pattern in the first several powers:

$$\begin{aligned}
&i = i\\
&i^2 = -1\\
&i^3 = i \cdot i^2 = -i\\
&i^4 = (i^2)^2 = (-1)^2 = 1\\
&i^5 = i^4 \cdot i = 1 \cdot i = i\\
&i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1\\
&i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i\\
&i^8 = (i^4)^2 = (1)^2 = 1
\end{aligned}$$

As we go to higher powers, the pattern $i, -1, -i, 1$ repeats over and over as above.

In the evaluation of the 5th to 8th powers above, the power i^4 was used repeatedly to rewrite the original power. This is because $i^4 = 1$, which is a very nice number to work with. When simplifying powers of i, it is easiest and most efficient to rewrite the power in terms of i^4 using exponent rules, as in the example below.

Example 5 ► **Simplifying Powers of *i***

Simplify.

a. i^{12}

b. i^{33}

c. i^{42}

d. i^{63}

Solution ►

a. Since $12 \div 4 = 3$, we can rewrite i^{12} as $i^{4 \cdot 3}$, which is equivalent to $(i^4)^3 = (1)^3 = \mathbf{1}$

b. This time, $33 \div 4 = 8$ with remainder 1. So we can write $33 = 4 \cdot 8 + 1$, which is used in our simplification as

$$i^{4 \cdot 8 + 1} = (i^4)^8 \cdot i = (1)^8 \cdot i = \boldsymbol{i}$$

c. Here, $42 \div 4 = 10$ with a remainder of 2:

$$i^{42} = i^{40} i^2 = (1)(-1) = \mathbf{-1}$$

d. 63 has a remainder of 3 when divided by 4:

$$i^{63} = i^{60} i^3 = (1)(-i) = \boldsymbol{-i}$$

Observation: Since any perfect 4th power of i simplifies to 1, when simplifying higher powers of i, we can divide the exponent by 4, note any remainder, r, and replace the power with i^r.

Division of Complex Numbers

Dividing complex numbers is very similar to rationalizing denominators. We get rid of any imaginary numbers in the denominator by using the product of complex conjugates formula, then simplify to standard form.

Example 6 ► **Dividing Complex Numbers**

Simplify.

a. $\frac{2+3i}{1+2i}$

b. $\frac{3}{i}$

c. $\frac{5+2i}{5-2i}$

d. $(2-i) \div (i-3)$

Solution ►

a. We identify the complex conjugate of the denominator as $1-2i$, then multiply numerator and denominator by this value:

$$\frac{2+3i}{1+2i} = \frac{2+3i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{2-4i+3i-6i^2}{1^2+2^2} = \frac{2-i+6}{5} = \frac{8-i}{5} = \boldsymbol{\frac{8}{5} - \frac{1}{5}i}$$

$(a+bi)(a-bi) = a^2+b^2$

standard form, $a+bi$

b. The denominator is $i = 0 + 1i$, so the complex conjugate is $0 - 1i = -i$

$$\frac{3}{i} \cdot \frac{-i}{-i} = \frac{-3i}{-i^2} = \frac{-3i}{-(-1)} = -3i$$

Alternatively, we could multiply numerator and denominator by i and obtain the same result (but that's only the case when the denominator is a purely imaginary number)

$$\frac{3}{i} \cdot \frac{i}{i} = \frac{3i}{i^2} = \frac{3i}{-1} = -3i$$

c. Here, the complex conjugate of the denominator, $5-2i$, is $5+2i$

$$\frac{5+2i}{5-2i} \cdot \frac{5+2i}{5+2i} = \frac{5^2+20i+(2i)^2}{5^2+2^2} = \frac{25+20i-4}{25+4} = \frac{21+20i}{29} = \frac{21}{29} + \frac{20}{29}i$$

d. Rewrite the division as a fraction and change the denominator into standard form, $(2-i) \div (i-3) = \frac{2-i}{-3+i}$, then multiply the numerator and denominator by the complex conjugate of the the denominator, $-3-i$

$$\frac{2-i}{-3+i} \cdot \frac{-3-i}{-3-i} = \frac{-6-2i+3i+i^2}{(-3)^2+1^2} = \frac{-6+i-1}{9+1} = \frac{-7+i}{10} = \boldsymbol{-\frac{7}{10} + \frac{1}{10}i}$$

RD.6 Exercises

Find the mistake.

1. $\sqrt{-3}\cdot\sqrt{-15}=\sqrt{-3\cdot -15}=\sqrt{45}=\sqrt{9\cdot 5}=3\sqrt{5}$

Match each number in Column I to its complex conjugate in Column II.

2.

Column I	Column II
a. $8+21i$	**A.** $8+21i$
b. $-8-21i$	**B.** $8-21i$
c. $8-21i$	**C.** $-8-21i$
d. $-8+21i$	**D.** $-8+21i$

3. Quinn says the solution to $x^2=-64$ is $8i$, while Finn says the solution is $-8i$. Who is correct?

Complete the indicated operation(s) and simplify.

4. $\sqrt{-81}$

5. $\sqrt{-100}$

6. $\sqrt{-72}$

7. $\sqrt{-98}$

8. $\sqrt{-5}\cdot\sqrt{-5}$

9. $\sqrt{-7}\cdot\sqrt{-7}$

10. $\sqrt{-10}\cdot\sqrt{-5}$

11. $\sqrt{-7}\cdot\sqrt{-21}$

12. $\sqrt{-75}+\sqrt{-108}$

13. $\sqrt{-32}-\sqrt{-128}$

14. $2\sqrt{-45}-7\sqrt{-80}$

15. $-3\sqrt{-40}+6\sqrt{-250}$

16. $\left(4+\sqrt{-64}\right)\left(7+2\sqrt{-16}\right)$

17. $\left(-2+9\sqrt{-1}\right)\left(5+\sqrt{-49}\right)$

18. $\left(6-\sqrt{-8}\right)\left(3+\sqrt{-50}\right)$

19. $\left(3-\sqrt{-75}\right)\left(5-\sqrt{-147}\right)$

20. $\left(1+\sqrt{-18}\right)\left(1-\sqrt{-18}\right)$

21. $\left(8+\sqrt{-48}\right)\left(8-\sqrt{-48}\right)$

22. $3i(4+7i)$

23. $2i(1-9i)$

24. $(1+4i)(5-6i)$

25. $(8-i)(2-10i)$

26. $(5-3i)^2$

27. $(6+7i)^2$

28. $(8+5i)(8-5i)$

29. $(10-9i)(10+9i)$

30. i^{45}

31. i^{56}

32. i^{103}

33. i^{201}

34. i^{90}

35. i^{79}

36. $\frac{6+\sqrt{-60}}{2}$

37. $\frac{6-\sqrt{-504}}{3}$

38. $\frac{5-3\sqrt{-525}}{10}$

39. $\frac{8+\sqrt{-624}}{40}$

40. $\frac{5}{i}$

41. $\frac{6}{5i}$

42. $\frac{7}{4+i}$

43. $\frac{3}{5+7i}$

44. $\frac{3-2i}{3+2i}$

45. $\frac{4+3i}{4-3i}$

46. $\frac{8-9i}{1-6i}$

47. $\frac{7+5i}{11+4i}$

Determine if the complex number is a solution to the equation given.

48. $5i\ ;\ x^2 + 25 = 0$

49. $-2i\ ;\ x^2 = -4$

50. $1 + 2i\ ;\ x^2 - 2x + 5 = 0$

51. $3 - 2i\ ;\ x^2 - 6x + 13 = 0$

52. $4 - 3i\ ;\ x^2 - 3x + 10 = 0$

53. $5 + i\ ;\ x^2 + 5x + 60 = 0$

Attributions

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Quadratic Equations and Functions

In this chapter, we discuss various ways of solving quadratic equations, $ax^2 + bx + c = 0$, including equations quadratic in form, such as $x^{-2} + x^{-1} - 20 = 0$, and solving formulas for a variable that appears in the first and second power, such as k in $k^2 - 3k = 2N$. Frequently used strategies of solving quadratic equations include the **completing the square** procedure and its generalization in the form of the **quadratic formula**. Completing the square allows for rewriting quadratic functions in vertex form, $f(x) = a(x - h)^2 + k$, which is very useful for graphing as it provides information about the location, shape, and direction of the parabola.

In the second part of this chapter, we examine properties and graphs of quadratic functions, including basic transformations of these graphs.

Finally, these properties are used in solving application problems, particularly problems involving **optimization**.

Q1 Methods of Solving Quadratic Equations

As defined in *Section F4*, a quadratic equation is a second-degree polynomial equation in one variable that can be written in standard form as

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $\boldsymbol{a \neq 0}$. Such equations can be solved in many different ways, as presented below.

Solving by Graphing

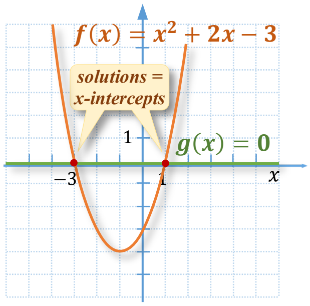


Figure 1.1

To solve a quadratic equation, for example $x^2 + 2x - 3 = 0$, we can consider its left side as a function $f(x) = x^2 + 2x - 3$ and the right side as a function $g(x) = 0$. To satisfy the original equation, both function values must be equal. After graphing both functions on the same grid, one can observe that this happens at points of intersection of the two graphs.

So the **solutions** to the original equation are the x-coordinates of the intersection points of the two graphs. In our example, these are the $\boldsymbol{x}$**-intercepts** or the **roots** of the function $f(x) = x^2 + 2x - 3$, as indicated in *Figure 1.1*.

Thus, the solutions to $x^2 + 2x - 3 = 0$ are $\boldsymbol{x = -3}$ and $\boldsymbol{x = 1}$.

Note: Notice that the graphing method, although visually appealing, is not always reliable. For example, the solutions to the equation $49x^2 - 4 = 0$ are $x = \frac{2}{7}$ and $x = -\frac{2}{7}$. Such numbers would be very hard to read from the graph.

Thus, the graphing method is advisable to use when searching for integral solutions or estimations of solutions.

To find exact solutions, we can use one of the algebraic methods presented below.

Solving by Factoring

Many quadratic equations can be solved by factoring and employing the zero-product property, as in *Section F4*.

For example, the equation $x^2 + 2x - 3 = 0$ can be solved as follows:

$$(x+3)(x-1) = 0$$

so, by zero-product property,

$$x + 3 = 0 \text{ or } x - 1 = 0,$$

which gives us the solutions

$$\boldsymbol{x = -3} \text{ or } \boldsymbol{x = 1}.$$

Solving by Using the Square Root Property

Quadratic equations of the form $\boldsymbol{ax^2 + c = 0}$ can be solved by applying the **square root property**.

Square Root Property:

For any positive real number a, if $\boldsymbol{x^2 = a}$, then $\boldsymbol{x = \pm\sqrt{a}}$.
This is because $\sqrt{x^2} = |x|$. So, after applying the square root operator to both sides of the equation $x^2 = a$, we have

$$\sqrt{x^2} = \sqrt{a}$$
$$|x| = \sqrt{a}$$
$$x = \pm\sqrt{a}$$

The $\pm\sqrt{a}$ is a shorter recording of two solutions: $\sqrt{a}$ and $-\sqrt{a}$.

For example, the equation $49x^2 - 4 = 0$ can be solved as follows:

$$49x^2 - 4 = 0$$
$$49x^2 = 4$$
$$x^2 = \frac{4}{49}$$

apply square root to both sides of the equation

$$\sqrt{x^2} = \sqrt{\frac{4}{49}}$$

Here we use the **square root property**. Remember the ± sign!

$$x = \pm\sqrt{\frac{4}{49}}$$
$$\boldsymbol{x = \pm\frac{2}{7}}$$

Note: Using the square root property is a common solving strategy for quadratic equations where **one side is a perfect square** of an unknown quantity and the **other side is a constant** number.

Example 1 ► **Solve by the Square Root Property**

Solve each equation using the square root property.

a. $(x-3)^2 = 49$ **b.** $2(3x-6)^2 - 54 = 0$

Solution ► **a.** Applying the square root property, we have

$$\begin{aligned} \sqrt{(x-3)^2} &= \sqrt{49} \\ x-3 &= \pm 7 \\ x &= 3 \pm 7 \end{aligned}$$

so

$$\boldsymbol{x = 10} \text{ or } \boldsymbol{x = -4}$$

b. To solve $2(3x-6)^2 - 54 = 0$, we isolate the perfect square first and then apply the square root property. So,

$$\begin{aligned} 2(3x-6)^2 - 54 &= 0 \\ (3x-6)^2 &= \frac{54}{2} \\ \sqrt{(3x-6)^2} &= \sqrt{27} \\ 3x-6 &= \pm 3\sqrt{3} \\ 3x &= 6 \pm 3\sqrt{3} \\ x &= \frac{6 \pm 3\sqrt{3}}{3} \\ x &= \frac{\cancel{3}(2 \pm \sqrt{3})}{\cancel{3}} \\ x &= 2 \pm \sqrt{3} \end{aligned}$$

Thus, the solution set is $\{\mathbf{2} - \sqrt{\mathbf{3}},\ \mathbf{2} + \sqrt{\mathbf{3}}\}$.

Caution: To simplify expressions such as $\frac{6+3\sqrt{3}}{3}$, we **factor the numerator** first. The common errors to avoid are

incorrect order of operations $\frac{6+3\sqrt{3}}{3} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$

or

incorrect canceling $\frac{6+\cancel{3}\sqrt{3}}{\cancel{3}} = 6 + \sqrt{3}$

or

incorrect canceling $\frac{\overset{2}{\cancel{6}}+3\sqrt{3}}{\cancel{3}} = 2 + 3\sqrt{3}$

Solving by Completing the Square

So far, we have seen how to solve quadratic equations, $ax^2+bx+c=0$, if the expression ax^2+bx+c is factorable or if the coefficient b is equal to zero. To solve other quadratic equations, we may try to rewrite the variable terms in the form of a perfect square, so that the resulting equation can be solved by the square root property.

For example, to solve $x^2+6x-3=0$, we observe that the variable terms x^2+6x could be written in **perfect square** form if we add 9, as illustrated in *Figure 1.2*. This is because

$$x^2+6x+9=(x+3)^2$$

observe that 3 comes from taking half of 6

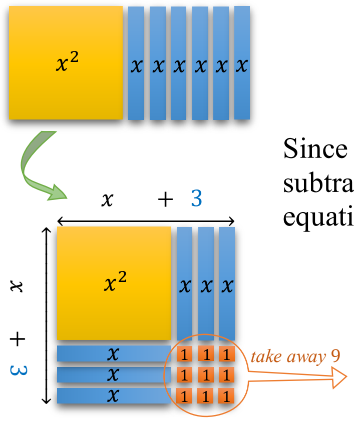


Figure 1.2

Since the original equation can only be changed to an equivalent form, if we add 9, we must subtract 9 as well. *(Alternatively, we could add 9 to both sides of the equation.)* So, the equation can be transformed as follows:

Completing the Square Procedure

$$x^2+6x-3=0$$

$$\underbrace{x^2+6x+9}_{perfect\ square}-9-3=0$$

$$(x+3)^2=12$$

square root property

$$\sqrt{(x+3)^2}=\sqrt{12}$$

$$x+3=\pm2\sqrt{3}$$

$$x=\mathbf{-3\pm2\sqrt{3}}$$

Generally, to **complete the square** for the first two terms of the equation

$$\boldsymbol{x^2+bx+c=0},$$

we take half of the x-coefficient, which is $\frac{b}{2}$, and square it. Then, we add and subtract that number, $\left(\frac{b}{2}\right)^2$. *(Alternatively, we could add $\left(\frac{b}{2}\right)^2$ to both sides of the equation.)*
This way, we produce an equivalent equation

$$\boldsymbol{x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c=0},$$

and consequently,

$$\boldsymbol{\left(x+\frac{b}{2}\right)^2-\frac{b^2}{4}+c=0}.$$

We can write this equation directly, by following the rule:
Write the sum of x and **half of the middle coefficient**, **square the binomial**, and **subtract the perfect square of the constant** appearing in the bracket.

To **complete the square** for the first two terms of a quadratic equation with a leading coefficient of $a \neq 1$,

$$\boldsymbol{ax^2 + bx + c = 0},$$

we

- divide the equation by $\boldsymbol{a}$ *(alternatively, we could factor a out of the first two terms)* so that the leading coefficient is 1, and then
- complete the square as in the previous case, where $\boldsymbol{a = 1}$.

So, after division by $\boldsymbol{a}$, we obtain

$$\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}.$$

Since half of $\boldsymbol{\frac{b}{a}}$ is $\boldsymbol{\frac{b}{2a}}$, then we complete the square as follows:

$$\boldsymbol{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0}.$$

Remember to **subtract the perfect square of the constant** appearing in the bracket!

Example 2 ► **Solve by Completing the Square**

Solve each equation using the completing the square method.

a. $x^2 + 5x - 1 = 0$ **b.** $3x^2 - 12x - 5 = 0$

Solution ►

a. First, we complete the square for $x^2 + 5x$ by adding and subtracting $\left(\frac{5}{2}\right)^2$ and then we apply the square root property. So, we have

$$\underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2}_{perfect\ square} - \left(\frac{5}{2}\right)^2 - 1 = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 1 \cdot \frac{4}{4} = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{29}{4} = 0$$

apply square root to both sides of the equation

$$\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$$

remember to use the $\pm$ sign!

$$x + \frac{5}{2} = \pm\sqrt{\frac{29}{4}}$$

$$x+\frac{5}{2}=\pm\frac{\sqrt{29}}{2}$$

$$x=\frac{-5\pm\sqrt{29}}{2}$$

Thus, the solution set is $\left\{\frac{-5-\sqrt{29}}{2},\frac{-5+\sqrt{29}}{2}\right\}$.

Note: Unless specified otherwise, we are expected to state the **exact solutions** rather than their calculator approximations. Sometimes, however, especially when solving application problems, we may need to use a calculator to approximate the solutions. The reader is encouraged to check that the two decimal **approximations** of the above solutions are

$$\frac{-5-\sqrt{29}}{2}\approx\mathbf{-5.19}\quad\text{and}\quad\frac{-5+\sqrt{29}}{2}\approx\mathbf{0.19}$$

b. In order to apply the strategy as in the previous example, we divide the equation by the leading coefficient, 3. So, we obtain

$$3x^2-12x-5=0$$

$$x^2-4x-\frac{5}{3}=0$$

Then, to complete the square for x^2-4x, we may add and subtract 4. This allows us to rewrite the equation equivalently, with the variable part in perfect square form.

$$(x-2)^2-4-\frac{5}{3}=0$$

$$(x-2)^2=4\cdot\frac{3}{3}+\frac{5}{3}$$

$$(x-2)^2=\frac{17}{3}$$

$$x-2=\pm\sqrt{\frac{17}{3}}$$

$$x=\mathbf{2}\pm\frac{\sqrt{17}}{\sqrt{3}}$$

Note: The final answer could be written as a single fraction as shown below:

$$x=\frac{2\sqrt{3}\pm\sqrt{17}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{6\pm\sqrt{51}}{3}$$

Solving with Quadratic Formula

Applying the completing the square procedure to the quadratic equation

$$ax^2+bx+c=0,$$

with real coefficients $a \neq 0$, b, and c, allows us to develop a general formula for finding the solution(s) to any such equation.

Quadratic Formula ► The solution(s) to the equation $\boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c} = 0$, where $\boldsymbol{a} \neq 0$, $\boldsymbol{b}$, $\boldsymbol{c}$ are real coefficients, are given by the formula

$$\boxed{\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}}$$

Here $\boldsymbol{x_{1,2}}$ denotes the two solutions, $\boldsymbol{x_1} = \frac{\boldsymbol{-b + \sqrt{b^2 - 4ac}}}{\boldsymbol{2a}}$, and $\boldsymbol{x_2} = \frac{\boldsymbol{-b - \sqrt{b^2 - 4ac}}}{\boldsymbol{2a}}$.

Proof: ► First, since $a \neq 0$, we can divide the equation $ax^2 + bx + c = 0$ by a. So, the equation to solve is

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then, we complete the square for $x^2 + \frac{b}{a}x$ by adding and subtracting the perfect square of half of the middle coefficient, $\left(\frac{b}{2a}\right)^2$. So, we obtain

$$\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2}_{perfect\ square} - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and finally,

QUADRATIC FORMULA

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}},$$

which concludes the proof.

Example 3 ► **Solving Quadratic Equations with the Use of the Quadratic Formula**

Using the Quadratic Formula, solve each equation, if possible. Then visualize the solutions graphically.

a. $2x^2 + 3x - 20 = 0$ **b.** $3x^2 - 4 = 2x$ **c.** $x^2 - \sqrt{2}x + 3 = 0$

Solution ▶ **a.** To apply the quadratic formula, first, we identify the values of a, b, and c. Since the equation is in standard form, $a = 2$, $b = 3$, and $c = -20$. The solutions are equal to

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2(-20)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{9 + 160}}{4}$$

$$= \frac{-3 \pm 13}{4} = \begin{cases} \dfrac{-3 + 13}{4} = \dfrac{10}{4} = \boldsymbol{\dfrac{5}{2}} \\[2ex] \dfrac{-3 - 13}{4} = \dfrac{-16}{4} = \boldsymbol{-4} \end{cases}$$

Thus, the solution set is $\left\{\boldsymbol{-4}, \boldsymbol{\frac{5}{2}}\right\}$.

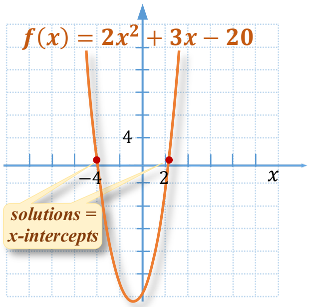


Figure 1.3

These solutions can be seen as x-intercepts of the function $f(x) = 2x^2 + 3x - 20$, as shown in *Figure 1.3*.

b. Before we identify the values of a, b, and c, we need to write the given equation $3x^2 - 4 = 2x$ in standard form. After subtracting $2x$ from both sides of the given equation, we obtain

$$3x^2 - 2x - 4 = 0$$

Since $a = 3$, $b = -2$, and $c = -4$, we evaluate the quadratic formula,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3(-4)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

$$= \frac{2 \pm \sqrt{4 \cdot 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{\cancel{2}(1 \pm \sqrt{13})}{\cancel{6}} = \boldsymbol{\frac{1 \pm \sqrt{13}}{3}}$$

simplify by factoring

So, the solution set is $\left\{\boldsymbol{\frac{1-\sqrt{13}}{3}}, \boldsymbol{\frac{1+\sqrt{13}}{3}}\right\}$.

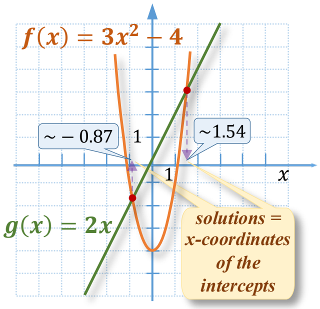


Figure 1.4

We may visualize solutions to the original equation, $3x^2 - 4 = 2x$, by graphing functions $f(x) = 3x^2 - 4$ and $g(x) = 2x$. The x-coordinates of the intersection points are the solutions to the equation $f(x) = g(x)$, and consequently to the original equation. As indicated in *Figure 1.4*, the approximations of these solutions are $\frac{1-\sqrt{13}}{3} \approx -0.87$ and $\frac{1+\sqrt{13}}{3} \approx 1.54$.

c. Substituting $a = 1$, $b = -\sqrt{2}$, and $c = 3$ into the Quadratic Formula, we obtain

$$x_{1,2} = \frac{\sqrt{2} \pm \sqrt{\left(-\sqrt{2}\right)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{\sqrt{2} \pm \sqrt{2 - 12}}{2} = \frac{\sqrt{2} \pm \sqrt{-10}}{2}$$

not a real number!

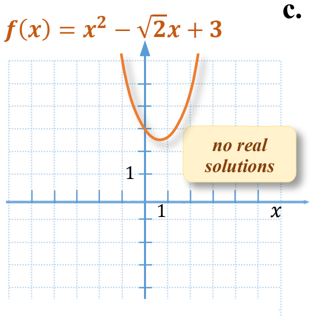


Figure 1.5

Since a square root of a negative number is not a real value, we have **no real solutions**. In a graphical representation, this means that the graph of the function $f(x) = x^2 - \sqrt{2}x + 3$ never equals 0 and therefore does not cross the x-axis. See *Figure 1.5*.

Although there are no real solutions to the equation, there are complex solutions that can be simplified as in *Section RD6*:

$$x_{1,2}=\frac{\sqrt{2}\pm\sqrt{10}\,i}{2}$$

Observation: Notice that we could find information about the solutions in *Example 3c* just by evaluating the radicand b^2-4ac. Since this radicand was negative, we concluded that there was no real solution to the given equation as a root of a negative number is not a real number. There was no need to evaluate the whole Quadratic Formula to determine the nature of the solutions.

So, the radicand in the Quadratic Formula carries important information about the number and nature of roots. Because of it, this radicand earned a special name, the discriminant.

Definition 1.1 ► The radicand $\boldsymbol{b^2-4ac}$ in the Quadratic Formula is called the **discriminant** and it is denoted by Δ.

Notice that in terms of Δ, the Quadratic Formula takes the form

$$\boxed{x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}}$$

Observing the behaviour of the expression $\sqrt{\Delta}$ allows us to classify the number and type of solutions (roots) of a quadratic equation with rational coefficients.

Characteristics of Roots (Solutions) Depending on the Discriminant

Suppose $ax^2+bx+c=0$ has **rational** coefficients $a\neq 0$, b, c, and $\Delta=b^2-4ac$.

- If $\Delta<\mathbf{0}$, then the equation has **two complex conjugate solutions,** $\frac{-b-\sqrt{|\Delta|}\,i}{2a}$ and $\frac{-b+\sqrt{|\Delta|}\,i}{2a}$, as $\sqrt{negative}$ is an imaginary number.
- If $\Delta=\mathbf{0}$, then the equation has **one rational solution,** $\frac{-b}{2a}$.
- If $\Delta>\mathbf{0}$, then the equation has **two real solutions,** $\frac{-b-\sqrt{\Delta}}{2a}$ and $\frac{-b+\sqrt{\Delta}}{2a}$.
  These solutions are
  - **irrational**, if Δ is **not** a **perfect square number**
  - **rational**, if Δ is a **perfect square number** (as $\sqrt{perfect\ square}=integer$)

  In addition, if $\Delta\geq\mathbf{0}$ is a **perfect square number**, then the equation could be solved by **factoring**.

Example 4 ► Determining the Number and Type of Solutions of a Quadratic Equation

Using the discriminant, determine the number and type of solutions of each equation without solving the equation. If the equation can be solved by factoring, show the factored form of the trinomial.

a. $2x^2 + 7x - 15 = 0$

b. $4x^2 - 12x + 9 = 0$

c. $3x^2 - x + 1 = 0$

d. $2x^2 - 7x + 2 = 0$

Solution ►

a. $\Delta = 7^2 - 4 \cdot 2 \cdot (-15) = 49 + 120 = 169$

Since 169 is a perfect square number, the equation has **two rational solutions** and it can be solved by factoring. Indeed, $2x^2 + 7x - 15 = (2x - 3)(x + 5)$.

b. $\Delta = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$

$\Delta = 0$ indicates that the equation has **one rational solution** and it can be solved by factoring. Indeed, the expression $4x^2 - 12x + 9$ is a perfect square, $(2x - 3)^2$.

c. $\Delta = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11$

Since $\Delta < 0$, the equation has **two complex conjugate solutions** and therefore it cannot be solved by factoring.

d. $\Delta = (-7)^2 - 4 \cdot 2 \cdot 2 = 49 - 16 = 33$

Since $\Delta > 0$ but is not a perfect square number, the equation has **two real solutions** but it cannot be solved by factoring.

Example 5 ► Solving Equations Equivalent to Quadratic

Solve each equation.

a. $2 + \frac{7}{x} = \frac{5}{x^2}$

b. $1 - 2x^2 = (x + 2)(x - 1)$

Solution ►

a. This is a rational equation, with the set of $\mathbb{R} \setminus \{0\}$ as its domain. To solve it, we multiply the equation by the $LCD = x^2$. This brings us to a quadratic equation

$$2x^2 + 7x = 5$$

or equivalently

$$2x^2 + 7x - 5 = 0,$$

which can be solved by following the Quadratic Formula for $a = 2$, $b = 7$, and $c = -5$. So, we have

$$x_{1,2}=\frac{-7\pm\sqrt{7^2-4\cdot2(-5)}}{2\cdot2}=\frac{-7\pm\sqrt{49+40}}{4}=\frac{\mathbf{-7\pm\sqrt{89}}}{\mathbf{4}}$$

Since both solutions are in the domain, the solution set is $\left\{\frac{\mathbf{-7-\sqrt{89}}}{\mathbf{4}},\frac{\mathbf{-7+\sqrt{89}}}{\mathbf{4}}\right\}$.

b. To solve $1-2x^2=(x+2)(x-1)$, we simplify the equation first and rewrite it in standard form. So, we have

$$1-2x^2=x^2+x-2$$
$$-3x^2-x+3=0$$
$$3x^2+x-3=0$$

Since the left side of this equation is not factorable, we may use the Quadratic Formula. So, the solutions are

$$x_{1,2}=\frac{-1\pm\sqrt{1^2-4\cdot3(-3)}}{2\cdot3}=\frac{-1\pm\sqrt{1+36}}{6}=\frac{\mathbf{-1\pm\sqrt{37}}}{\mathbf{6}}.$$

Q.1 Exercises

True or False.

1. A quadratic equation is an equation that can be written in the form $ax^2+bx+c=0$, where a, b, and c are any real numbers.

2. If the graph of $f(x)=ax^2+bx+c$ intersects the x-axis twice, the equation $ax^2+bx+c=0$ has two solutions.

3. If the equation $ax^2+bx+c=0$ has no real solution, the graph of $f(x)=ax^2+bx+c$ does not intersect the x-axis.

4. The Quadratic Formula cannot be used to solve the equation $x^2-5=0$ because the equation does not contain a linear term.

5. The solution set for the equation $x^2=16$ is $\{4\}$.

6. To complete the square for x^2+bx, we add $\left(\frac{b}{2}\right)^2$.

7. If the discriminant is positive, the equation can be solved by factoring.

For each function f,
a) *graph $f(x)$ using a table of values;*
b) *find the x-intercepts of the graph;*
c) *solve the equation $f(x)=0$ by factoring and compare these solutions to the x-intercepts of the graph.*

8. $f(x) = -x^2 - 3x + 2$ **9.** $f(x) = x^2 + 2x - 3$ **10.** $f(x) = 3x + x(x - 2)$

11. $f(x) = 2x - x(x - 3)$ **12.** $f(x) = 4x^2 - 4x - 3$ **13.** $f(x) = -\frac{1}{2}(2x^2 + 5x - 12)$

Solve each equation using the ***square root property****.*

14. $x^2 = 49$ **15.** $x^2 = 32$ **16.** $a^2 - 50 = 0$

17. $n^2 - 24 = 0$ **18.** $3x^2 - 72 = 0$ **19.** $5y^2 - 200 = 0$

20. $(x - 4)^2 = 64$ **21.** $(x + 3)^2 = 16$ **22.** $(3n - 1)^2 = 7$

23. $(5t + 2)^2 = 12$ **24.** $x^2 - 10x + 25 = 45$ **25.** $y^2 + 8y + 16 = 44$

26. $4a^2 + 12a + 9 = 32$ **27.** $25(y - 10)^2 = 36$ **28.** $16(x + 4)^2 = 81$

29. $(4x + 3)^2 = -25$ **30.** $(3n - 2)(3n + 2) = -5$ **31.** $2x - 1 = \frac{18}{2x-1}$

Solve each equation using the ***completing the square*** *procedure.*

32. $x^2 + 12x = 0$ **33.** $y^2 - 3y = 0$ **34.** $x^2 - 8x + 2 = 0$

35. $n^2 + 7n = 3n - 4$ **36.** $p^2 - 4p = 4p - 16$ **37.** $y^2 + 7y - 1 = 0$

38. $2x^2 - 8x = -4$ **39.** $3a^2 + 6a = -9$ **40.** $3y^2 - 9y + 15 = 0$

41. $5x^2 - 60x + 80 = 0$ **42.** $2t^2 + 6t - 10 = 0$ **43.** $3x^2 + 2x - 2 = 0$

44. $2x^2 - 16x + 25 = 0$ **45.** $9x^2 - 24x = -13$ **46.** $25n^2 - 20n = 1$

47. $x^2 - \frac{4}{3}x = -\frac{1}{9}$ **48.** $x^2 + \frac{5}{2}x = -1$ **49.** $x^2 - \frac{2}{5}x - 3 = 0$

In problems ***50-51****, find all values of x such that $f(x) = g(x)$ for the given functions f and g.*

50. $f(x) = x^2 - 9$ and $g(x) = 4x - 6$ **51.** $f(x) = 2x^2 - 5x$ and $g(x) = -x + 14$

52. Explain the errors in the following solutions of the equation $5x^2 - 8x + 2 = 0$:

a. $x = \frac{8 \pm \sqrt{-8^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64 - 40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \frac{1}{2} \pm 2\sqrt{6}$

b. $x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64 - 40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \begin{cases} \frac{10\sqrt{6}}{16} = \frac{5\sqrt{6}}{8} \\ \frac{6\sqrt{6}}{16} = \frac{3\sqrt{6}}{8} \end{cases}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve each equation with the aid of the ***Quadratic Formula****, if possible. Illustrate your solutions graphically, using a table of values.*

53. $x^2 + 3x + 2 = 0$ **54.** $y^2 - 2 = y$ **55.** $x^2 + x = -3$

56. $2y^2 + 3y = -2$ **57.** $x^2 - 8x + 16 = 0$ **58.** $4n^2 + 1 = 4n$

Solve each equation with the aid of the ***Quadratic Formula****. Give the* ***exact*** *and* ***approximate*** *solutions up to two decimal places.*

59. $a^2 - 4 = 2a$

60. $2 - 2x = 3x^2$

61. $0.2x^2 + x + 0.7 = 0$

62. $2t^2 - 4t + 2 = 3$

63. $y^2 + \frac{y}{3} = \frac{1}{6}$

64. $\frac{x^2}{4} - \frac{x}{2} = 1$

65. $5x^2 = 17x - 2$

66. $15y = 2y^2 + 16$

67. $6x^2 - 8x = 2x - 3$

Use the discriminant to determine the ***number and type of solutions*** *for each equation. Also, without solving, decide whether the equation can be solved by* ***factoring*** *or whether the quadratic formula should be used.*

68. $3x^2 - 5x - 2 = 0$

69. $4x^2 = 4x + 3$

70. $x^2 + 3 = -2\sqrt{3}x$

71. $4y^2 - 28y + 49 = 0$

72. $3y^2 - 10y + 15 = 0$

73. $9x^2 + 6x = -1$

In problems 74-76, find all values of constant k, so that each equation will have ***exactly one*** *rational solution.*

74. $x^2 + ky + 49 = 0$

75. $9y^2 - 30y + k = 0$

76. $kx^2 + 8x + 1 = 0$

77. Suppose that one solution of a quadratic equation with integral coefficients is irrational. Assuming that the equation has two solutions, can the other solution be a rational number? Justify your answer.

Solve each equation using any algebraic method. State the solutions in their exact form.

78. $-2x(x + 2) = -3$

79. $(x + 2)(x - 4) = 1$

80. $(x + 2)(x + 6) = 8$

81. $(2x - 3)^2 = 8(x + 1)$

82. $(3x + 1)^2 = 2(1 - 3x)$

83. $2x^2 - (x + 2)(x - 3) = 12$

84. $(x - 2)^2 + (x + 1)^2 = 0$

85. $1 + \frac{2}{x} + \frac{5}{x^2} = 0$

86. $x = \frac{2(x+3)}{x+5}$

87. $2 + \frac{1}{x} = \frac{3}{x^2}$

88. $\frac{3}{x} + \frac{x}{3} = \frac{5}{2}$

89. $\frac{1}{x} + \frac{1}{x+4} = \frac{1}{7}$

Q2 Applications of Quadratic Equations

Some polynomial, rational or even radical equations are **quadratic in form.** As such, they can be solved using techniques described in the previous section. For instance, the rational equation $\frac{1}{x^2}+\frac{1}{x}-6=0$ is quadratic in form because if we replace $\frac{1}{x}$ with a single variable, say a, then the equation becomes quadratic, $a^2+a-6=0$. In this section, we explore applications of quadratic equations in solving equations quadratic in form as well as solving formulas containing variables in the second power.

We also revisit application problems that involve solving quadratic equations. Some of the application problems that are typically solved with the use of quadratic or polynomial equations were discussed in *Sections F4* and *RT6*. However, in the previous sections, the equations used to solve such problems were all possible to solve by factoring. In this section, we include problems that require the use of methods other than factoring.

Equations Quadratic in Form

Definition 2.1 ► A nonquadratic equation is referred to as **quadratic in form** or **reducible to quadratic** if it can be written in the form

$$a\boldsymbol{u}^2+b\boldsymbol{u}+c=0,$$

where $a \neq 0$ and $\boldsymbol{u}$ represents any *algebraic expression*.

Equations quadratic in form are usually easier to solve by using strategies for solving the related quadratic equation $a\boldsymbol{u}^2+b\boldsymbol{u}+c=0$ for the expression $\boldsymbol{u}$, and then solving for the original variable, as shown in the example below.

Example 1 ► **Solving Equations Quadratic in Form**

Solve each equation.

a. $(x^2-1)^2-(x^2-1)=2$ **b.** $x-3\sqrt{x}=10$

c. $\frac{1}{(a+2)^2}+\frac{1}{a+2}-6=0$

Solution ► **a.** First, observe that the expression x^2-1 appears in the given equation in the first and second power. So, it may be useful to replace x^2-1 with a new variable, for example u. After this substitution, the equation becomes quadratic,

This can be any letter, as long as it is different than the original variable.

$$u^2-u=2,$$

and can be solved via factoring

$$u^2-u-2=0$$

$$(u-2)(u+1)=0$$

$$u=2 \text{ or } u=-1$$

Since we need to solve the original equation for x, not for u, we replace u back with x^2-1. This gives us

$$x^2 - 1 = 2 \quad \text{or} \quad x^2 - 1 = -1$$

$$x^2 = 3 \quad \text{or} \quad x^2 = 0$$

$$x = \pm\sqrt{3} \quad \text{or} \quad x = 0$$

Thus, the solution set is $\{-\boldsymbol{\sqrt{3}}, \mathbf{0}, \boldsymbol{\sqrt{3}}\}$.

b. If we replace $\sqrt{x}$ with, for example, a, then $x = a^2$, and the equation becomes

$$a^2 - 3a = 10,$$

which can be solved by factoring

$$a^2 - 3a - 10 = 0$$

$$(a+2)(a-5) = 0$$

$$a = -2 \quad \text{or} \quad a = 5$$

After replacing a back with $\sqrt{x}$, we have

$$\sqrt{x} = -2 \quad \text{or} \quad \sqrt{x} = 5.$$

The first equation, $\sqrt{x} = -2$, does not give us any solution as the square root cannot be negative. After squaring both sides of the second equation, we obtain $x = 25$. So, the solution set is $\{\mathbf{25}\}$.

c. The equation $\frac{1}{(a+2)^2} + \frac{1}{a+2} - 6 = 0$ can be solved as any other rational equation, by clearing the denominators via multiplying by the $LCD = (a+2)^2$. However, it can also be seen as a quadratic equation as soon as we replace $\frac{1}{a+2}$ with, for example, x. By doing so, we obtain

$$x^2 + x - 6 = 0,$$

which after factoring

$$(x+3)(x-2) = 0,$$

gives us

$$x = -3 \quad \text{or} \quad x = 2$$

Remember to use a different letter than the variable in the original equation.

Again, since we need to solve the original equation for a, we replace x back with $\frac{1}{a+2}$. This gives us

$$\frac{1}{a+2} = -3 \quad \text{or} \quad \frac{1}{a+2} = 2$$

take the reciprocal of both sides

$$a + 2 = \frac{1}{-3} \quad \text{or} \quad a + 2 = \frac{1}{2}$$

$$a = -\frac{7}{3} \quad \text{or} \quad a = -\frac{3}{2}$$

Since both values are in the domain of the original equation, which is $\mathbb{R} \setminus \{0\}$, then the solution set is $\left\{-\frac{7}{3}, -\frac{3}{2}\right\}$.

Solving Formulas

When solving formulas for a variable that appears in the second power, we use the same strategies as in solving quadratic equations. For example, we may use the square root property or the quadratic formula.

Example 2 ► **Solving Formulas for a Variable that Appears in the Second Power**

Solve each formula for the given variable.

a. $E = mc^2$, for c **b.** $N = \frac{k^2-3k}{2}$, for k

Solution ► **a.** To solve for c, first, we reverse the multiplication by m via the division by m. Then, we reverse the operation of squaring by taking the square root of both sides of the equation.

$$E = mc^2$$

$$\frac{E}{m} = c^2$$

Then, we reverse the operation of squaring by taking the square root of both sides of the equation. So, we have

$$\sqrt{\frac{E}{m}} = \sqrt{c^2},$$

and therefore

$$\boldsymbol{c} = \pm\sqrt{\frac{\boldsymbol{E}}{\boldsymbol{m}}}$$

Remember that $\sqrt{c^2} = |c|$, so we use the $\pm$ sign in place of $|\ |$.

b. Observe that the variable k appears in the formula $N = \frac{k^2-3k}{2}$ in two places. Once in the first and once in the second power of k. This means that we can treat this formula as a quadratic equation with respect to k and solve it with the aid of the quadratic formula. So, we have

$$N = \frac{k^2 - 3k}{2}$$

$$2N = k^2 - 3k$$

$$\boldsymbol{k}^2 - 3\boldsymbol{k} - 2N = 0$$

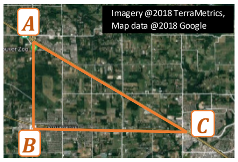
Substituting $a = 1$, $b = -3$, and $c = -2N$ to the quadratic formula, we obtain

$$\boldsymbol{k_{1,2}} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-2N)}}{2} = \frac{\mathbf{3} \pm \sqrt{\mathbf{9} + \mathbf{8}\boldsymbol{N}}}{\mathbf{2}}$$

Application Problems

Many application problems require solving quadratic equations. Sometimes this can be achieved via factoring, but often it is helpful to use the quadratic formula.

Example 3 ► Solving a Distance Problem with the Aid of the Quadratic Formula



Three towns A, B, and C are positioned as shown in the accompanying figure. The roads at B form a right angle. The towns A and C are connected by a straight road as well. The distance from A to B is 7 kilometers less than the distance from B to C. The distance from A to C is 20 km. Approximate the remaining distances between the towns up to the tenth of a kilometer.

Solution ► Since the roads between towns form a right triangle, we can employ the Pythagorean equation

$$AC^2 = AB^2 + BC^2$$

Suppose that $BC = x$. Then $AB = x - 7$, and we have

$$20^2 = (x-7)^2 + x^2$$

$$400 = x^2 - 14x + 49 + x^2$$

$$2x^2 - 14x - 351 = 0$$

Applying the quadratic formula, we obtain

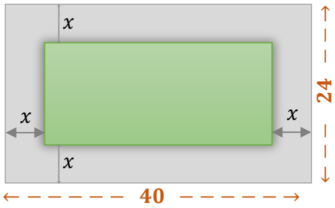
$$x_{1,2} = \frac{14 \pm \sqrt{14^2 + 4 \cdot 2 \cdot 351}}{4} = \frac{14 \pm \sqrt{196 + 2808}}{4} = \frac{14 \pm \sqrt{3004}}{4} \approx 17.2 \; or \; -\xcancel{10.2}$$

Since x represents a distance, it must be positive. So, the only solution is $x \approx 17.2$ km. Thus, the distance $\boldsymbol{BC} \approx \mathbf{17.2}$ **km** and hence $\boldsymbol{AB} \approx 17.2 - 7 = \mathbf{10.2}$ **km**.

Example 4 ► Solving a Geometry Problem with the Aid of the Quadratic Formula

A city designated a 24 m by 40 m rectangular area for a playground with a sidewalk of uniform width around it. The playground itself is using $\frac{2}{3}$ of the original rectangular area. To the nearest centimeter, what is the width of the sidewalk?

Solution ► To visualize the situation, we may draw a diagram as below.



Suppose x represents the width of the sidewalk. Then, the area of the playground (the green area) can be expressed as $(40-2x)(24-2x)$. Since the green area is $\frac{2}{3}$ of the original rectangular area, we can form the equation

$$(40-2x)(24-2x) = \frac{2}{3}(40\cdot 24)$$

To solve it, we may want to lower the coefficients by dividing both sides of the equation by 4 first. This gives us

$$\begin{aligned} \frac{\cancel{2}(20-x)\cdot\cancel{2}(12-x)}{\cancel{4}} &= \frac{2}{\cancel{3}}\cdot\frac{(\overset{10}{\cancel{40}}\cdot\overset{8}{\cancel{24}})}{\cancel{4}} \\ (20-x)(12-x) &= 160 \\ 240-32x+x^2 &= 160 \\ x^2-32x+80 &= 0, \end{aligned}$$

which can be solved using the Quadratic Formula:

$$x_{1,2} = \frac{32\pm\sqrt{(-32)^2-4\cdot 80}}{2} = \frac{32\pm\sqrt{704}}{2} \approx \frac{32\pm 8\sqrt{11}}{2}$$

$$= 16\pm 4\sqrt{11} \approx \begin{cases} \cancel{29.27} \\ 2.73 \end{cases}$$

The width x must be smaller than 12, so this value is too large to be considered.

Thus, the sidewalk is approximately $\mathbf{2.73}$ meters wide.

Example 5 ► Solving a Motion Problem That Requires the Use of the Quadratic Formula

The Columbia River flows at a rate of 2 mph for the length of a popular boating route. In order for a boat to travel 3 miles upriver and return in a total of 4 hours, how fast must the boat be able to travel in still water?

Solution ►

Suppose the rate of the boat moving in still water is r. Then, $r-2$ represents the rate of the boat moving upriver and $r+2$ represents the rate of the boat moving downriver. We can arrange these data in the table below.

	R ·	T	$= D$
upriver	$r-2$	$\frac{3}{r-2}$	3
downriver	$r+2$	$\frac{3}{r+2}$	3
total		4	

We fill the time-column by following the formula $T=\frac{D}{R}$.

By adding the times needed for traveling upriver and downriver, we form the rational equation

$$\frac{3}{r-2}+\frac{3}{r+2}=4,$$

which, after multiplying by the $LCD = r^2-4$, becomes a quadratic equation.

$$3(r+2)+3(r-2)=4(r^2-4)$$

$$\begin{aligned} 3r+6+3r-6 &= 4r^2-16 \\ 0 &= 4r^2-6r-16 \\ 0 &= 2r^2-3r-8 \end{aligned}$$

Using the Quadratic Formula, we have

$$r_{1,2}=\frac{3\pm\sqrt{(-3)^2-4\cdot 2\cdot(-8)}}{2\cdot 2}=\frac{3\pm\sqrt{9+64}}{4}=\frac{3\pm\sqrt{73}}{4}\approx\begin{cases}2.9\\ \cancel{-1.4}\end{cases}$$

Since the rate cannot be negative, the boat moves in still water at approximately **2.9 mph**.

Example 6 ► Solving a Work Problem That Requires the Use of the Quadratic Formula

Krista and Joanna work in the same office. Krista can file the daily office documents in 1 hour less time than Joanna can. Working together, they can do the job in 1 hr 45 min. To the nearest minute, how long would it take each person working alone to file these documents?

Solution ► Suppose the time needed for Joanna to complete the job is t, in hours. Then, $t-1$ represents the time needed for Krista to complete the same job. Since we keep time in hours, we need to convert 1 hr 45 min into $1\frac{3}{4}$ hr $=\frac{7}{4}$ hr. Now, we can arrange the given data in a table, as below.

	R ·	T	= Job
Joanna	$\frac{1}{t}$	t	1
Krista	$\frac{1}{t-1}$	$t-1$	1
together	$\frac{4}{7}$	$\frac{7}{4}$	1

We fill the rate-column by following the formula $R=\frac{Job}{T}$.

By adding the rates of work for each person, we form the rational equation

$$\frac{1}{t}+\frac{1}{t-1}=\frac{4}{7},$$

which, after multiplying by the $LCD = 7t(t-1)$, becomes a quadratic equation.

$$\begin{aligned} 7(t-1)+7t &= 4(t^2-t) \\ 7t-7+7t &= 4t^2-4t \\ 0 &= 4t^2-18t+7 \end{aligned}$$

Using the Quadratic Formula, we have

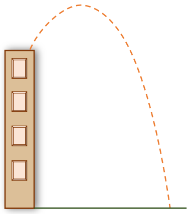
$$t_{1,2}=\frac{18\pm\sqrt{(-18)^2-4\cdot 4\cdot 7}}{2\cdot 4}=\frac{18\pm\sqrt{212}}{8}=\frac{18\pm 2\sqrt{53}}{8}=\frac{9\pm\sqrt{53}}{4}\approx\begin{cases}4.07\\ \cancel{0.43}\end{cases}$$

Since the time needed for Joanna cannot be shorter than 1 hr, we reject the 0.43 possibility. So, working alone, **Joanna** requires approximately 4.07 hours ≈ **4 hours 4 minutes**, while **Krista** can do the same job in approximately 3.07 hours ≈ **3 hours 4 minutes**.

Example 7 ► Solving a Projectile Problem Using a Quadratic Function

A ball is projected upward from the top of a 96-ft building at 32 ft/sec. Its height above the ground, h, in feet, can be modelled by the function $h(t) = -16t^2 + 32t + 96$, where t is the time in seconds after the ball was projected. To the nearest tenth of a second, when does the ball hit the ground?

Solution ►



The ball hits the ground when its height h above the ground is equal to zero. So, we look for the solutions to the equation

$$h(t) = 0$$

which is equivalent to

$$-16t^2 + 32t + 96 = 0$$

Before applying the Quadratic Formula, we may want to lower the coefficients by dividing both sides of the equation by -16. So, we have

$$t^2 - 2t - 6 = 0$$

and

$$t_{1,2} = \frac{2 \pm \sqrt{(-2)^2 + 4 \cdot 6}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7} \approx \begin{cases} 3.6 \\ \cancel{-1.6} \end{cases}$$

Thus, the ball hits the ground in about **3.6 seconds**.

Q.2 Exercises

1. Discuss the validity of the following solution to the equation $\left(\frac{1}{x-2}\right)^2 - \frac{1}{x-2} - 2 = 0$:

Since this equation is quadratic in form, we solve the related equation $a^2 - a - 2 = 0$ by factoring

$$(a-2)(a+1) = 0.$$

The possible solutions are $a = 2$ and $a = -1$. Since 2 is not in the domain of the original equation, the solution set as $\{-1\}$.

Solve each equation by treating it as a quadratic in form.

2. $x^4 - 6x^2 + 9 = 0$

3. $x^8 - 29x^4 + 100 = 0$

4. $x - 10\sqrt{x} + 9 = 0$

5. $2x - 9\sqrt{x} + 4 = 0$

6. $y^{-2} - 5y^{-1} - 36 = 0$

7. $2a^{-2} + a^{-1} - 1 = 0$

8. $\left(1+\sqrt{t}\right)^2+\left(1+\sqrt{t}\right)-6=0$

9. $\left(2+\sqrt{x}\right)^2-3\left(2+\sqrt{x}\right)-10=0$

10. $(x^2+5x)^2+2(x^2+5x)-24=0$

11. $(t^2-2t)^2-4(t^2-2t)+3=0$

12. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}-5=0$

13. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}-8=0$

14. $1-\frac{1}{2p+1}-\frac{1}{(2p+1)^2}=0$

15. $\frac{2}{(u+2)^2}+\frac{1}{u+2}=3$

16. $\left(\frac{x+3}{x-3}\right)^2-\left(\frac{x+3}{x-3}\right)=6$

17. $\left(\frac{y^2-1}{y}\right)^2-4\left(\frac{y^2-1}{y}\right)-12=0$

In problems 23-40, solve each formula for the indicated variable.

18. $F=\frac{mv^2}{r}$, for v

19. $V=\pi r^2h$, for r

20. $A=4\pi r^2$, for r

21. $V=\frac{1}{3}s^2h$, for s

22. $F=\frac{Gm_1m_2}{r^2}$, for r

23. $N=\frac{kq_1q_2}{s^2}$, for s

24. $a^2+b^2=c^2$, for b

25. $I=\frac{703W}{H^2}$, for H

26. $A=\pi r^2+\pi rs$, for r

27. $A=2\pi r^2+2\pi rh$, for r

28. $s=v_0t+\frac{gt^2}{2}$, for t

29. $t=\frac{a}{\sqrt{a^2+b^2}}$, for a

30. $P=\frac{A}{(1+r)^2}$, for r

31. $P=EI-RI^2$, for I

32. $s(6s-t)=t^2$, for s

33. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for v, assuming that $c>0$ and $m>0$

34. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for c, assuming that $v>0$ and $m>0$

35. $p=\frac{E^2R}{(r+R)^2}$, for E, assuming that $(r+R)>0$

36. The "**golden**" proportions have been considered visually pleasing for the past 2900 years. A rectangle with the width w and length l has "golden" proportions if

$$\frac{\boldsymbol{w}}{\boldsymbol{l}}=\frac{\boldsymbol{l}}{\boldsymbol{w}+\boldsymbol{l}}.$$

Solve this formula for $\boldsymbol{l}$. Then, find the value of the **golden ratio** $\frac{\boldsymbol{l}}{\boldsymbol{w}}$ up to three decimal places.

Answer each question.

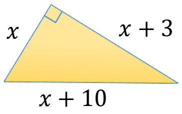
37. A boat moves r km/h in still water. If the rate of the current is c km/h,

a. give an expression for the rate of the boat moving upstream;
b. give an expression for the rate of the boat moving downstream.

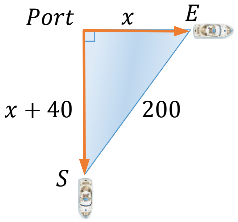
38. **a.** Vivian marks a class test in n hours. Give an expression representing Vivian's rate of marking, in the number of marked tests per hour.
b. How many tests will she have marked in h hours?

Solve each problem.

39. Find the exact length of each side of the triangle.

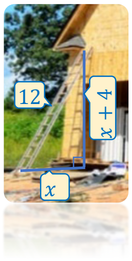


40. Two cruise ships leave a port at the same time, but they move at different rates. The faster ship is heading south, and the slower one is heading east. After a few hours, they are 200 km apart. If the faster ship went 40 km farther than the slower one, how far did each ship travelled?



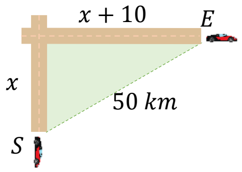
41. The length of a rectangular area carpet is 2 ft more than twice the width. Diagonally, the carpet measures 13 ft. Find the dimensions of the carpet.

42. The legs of a right triangle with 26 cm long hypotenuse differ by 14 cm. Find the lengths of the legs.



43. A 12-ft ladder is tilting against a house. The top of the ladder is 4 ft further from the ground than the bottom of the ladder is from the house. To the nearest inch, how high does the ladder reach?

44. Two cars leave an intersection, one heading south and the other heading east. In one hour the cars are 50 kilometers apart. If the faster car went 10 kilometers farther than the slower one, how far did each car travelled?

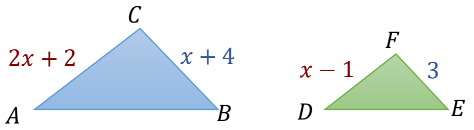


45. The length and width of a computer screen differ by 4 inches. Find the dimensions of the screen, knowing that its area is 117 square inches.

46. The length of an American flag is 1 inch shorter than twice the width. If the area of this flag is 190 square inches, find the dimensions of the flag.

47. The length of a Canadian flag is twice the width. If the area of this flag is 100 square meters, find the exact dimensions of the flag.

48. ***Thales Theorem*** states that corresponding sides of similar triangles are proportional. The accompanying diagram shows two similar triangles, $\varDelta ABC$ and $\varDelta DEF$. Given the information in the diagram, find the length AC.

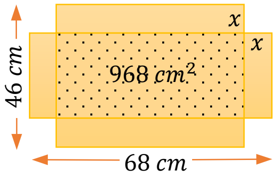


Solve each problem.

49. Sonia bought an area carpet for her 12 ft by 18 ft room. The carpet covers 135 ft^2, and when centered in the room, it leaves a strip of the bare floor of uniform width around the edges of the room. How wide is this strip?

50. Park management plans to create a rectangular 14 m by 20 m flower garden with a sidewalk of uniform width around the perimeter of the garden. There are enough funds to install 152 m^2 of a brick sidewalk. Find the width of the sidewalk.

51. Squares of equal area are cut from each corner of a 46 cm by 68 cm rectangular cardboard. Obtained this way flaps are folded up to create an open box with the area of the base equal to 968 cm^2. What is the height of the box?



52. The outside measurements of a picture frame are 22 cm and 28 cm. If the area of the exposed picture is 301 cm^2, find the width of the frame.

53. The length of a rectangle is one centimeter shorter than twice the width. The rectangle shares its longer side with a square of 169 cm^2 area. What are the dimensions of the rectangle?

54. A rectangular piece of cardboard is 15 centimeters longer than it is wide. 100 cm^2 squares are removed from each corner of the cardboard. Folding up the established flaps creates an open box of 13.5-litre volume. Find the dimensions of the original piece of cardboard. (*Hint:* 1 litre = 1000 cm^2)

55. Karin travelled 420 km by her motorcycle to visit a friend. When planning the return trip by the same road, she calculated that her driving time could be 1 hour shorter if she increases her average speed by 10 km/h. On average, how fast was she driving to her friend?

56. An average, an Airbus A380 flies 80 km/h faster than a Boeing 787 Dreamliner. Suppose an Airbus A380 flew 2600 km in half an hour shorter time than it took a Boeing 787 Dreamliner to fly 2880 mi. Determine the speed of each plane.

57. Two small planes, a Skyhawk and a Mooney Bravo, took off from the same place and at the same time. The Skyhawk flew 500 km. The Mooney Bravo flew 1050 km in one hour longer time and at a 100 km/h faster speed. If the planes fly faster than 200 km/h, find the average rate of each plane.

58. Gina drives 550 km to a conference. Due to heavier traffic, she returns at 10 km/h slower rate. If the round trip took her 10.5 hours, what was Gina's average rate of driving to the workshop?

59. A barge travels 25 km upriver and then returns in a total of 5 hours. If the current in the river is 3 km/hr, approximately how fast would this barge move in still water?

60. A canoeist travels 3 kilometers down a river with a 3 km/h current. For the return trip upriver, the canoeist chose to use a longer branch of the river with a 2 km/hr current. If the return trip is 4 km long and the time needed for travelling both ways is 3 hours, approximate the speed of the canoe in still water.

61. Two planes take off from the same airport and at the same time. The first plane flies with an average speed r km/h and is heading North. The second plane flies faster by 40 km/h and is heading East. In thirty minutes the planes are 580 kilometers apart from each other. Determine the average speed of each plane.

62. Jack flew 650 km to visit his relatives in Alaska. On the way to Alaska, his plane encounter a 40 km/h headwind. On the returning trip, the plane flew with a 20 km/h tailwind. If the total flying time (both ways) was 5 hours 45 minutes, what was the average speed of the plane in still air?

63. Two janitors, an experienced and a newly hired one, need 4 hours to clean a school building. The newly hired worker would need 1.5 hour longer time than the experienced one to clean the school on its own. To the nearest minute, how much time is required for the experienced janitor to clean the school working alone?

64. Two workers can weed out a vegetable garden in 2 hr. On its own, one worker can do the same job in half an hour shorter time than the other. To the nearest minute, how long would it take the faster worker to weed out the garden by himself?

65. Helen and Monica are planting flowers in their garden. On her own, Helen would need an hour longer than Monica to plant all the flowers. Together, they can finish the job in 8 hr. To the nearest minute, how long would it take each person to plant all the flowers if working alone?

66. To prepare the required number of pizza crusts for a day, the owner of Ricardo's Pizza needs 40 minutes shorter time than his worker Sergio. Together, they can make these pizza crusts in 2 hours. To the nearest minute, how long would it take each of them to do this job alone?

67. A fish tank can be filled with water with the use of one of two pipes of different diameters. If only the larger-diameter pipe is used, the tank can be filled in an hour shorter time than if only the smaller-diameter pipe is used. If both pipes are open, the tank can be filled in 1 hr 12 min. How much time is needed for each pipe to fill the tank if working alone?

68. Two roofers, Garry and Larry, can install new asphalt roof shingles in 6 hours 40 min. On his own, Garry can do this job in 3 hours shorter time than Larry can. How much time each or the roofers need to install these shingles alone?

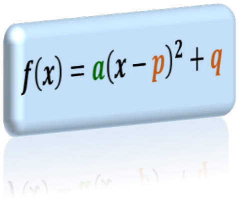
69. A ball is thrown down with the initial velocity of 6 m/sec from a balcony that is 100 m above the ground. Suppose that function $h(t) = -4.9t^2 - 6t + 100$ can be used to determine the height $h(t)$ of the ball t seconds after it was thrown down. Approximately in how many seconds the ball will be 5 meters above the ground?

70. A bakery's weekly profit, P (in dollars), for selling n poppyseed strudels can be modelled by the function $P(n) = -0.05x^2 + 7x - 200$. What is the minimum number of poppyseed strudels that must be sold to make a profit of $200?

71. If P dollars is invested in an account that pays the annual interest rate r (in decimal form), then the amount A of money in the account after 2 years can be determined by the formula $A = P(1+r)^2$. Suppose $3000 invested in this account for 2 years grew to $3257.29. What was the interest rate?

72. To determine the distance, d, of an object to the horizon we can use the equation $d = \sqrt{12800h + h^2}$, where h represents the distance of an object to the Earth's surface, and both, d and h, are in kilometers. To the nearest meter, how far above the Earth's surface is a plane if its distance to the horizon is 400 kilometers?

Q3 Properties and Graphs of Quadratic Functions



In this section, we explore an alternative way of graphing quadratic functions. It turns out that if a quadratic function is given in vertex form, $\boldsymbol{f(x) = a(x-p)^2 + q}$, its graph can be obtained by transforming the shape of the basic **parabola**, $f(x) = x^2$, by applying a **vertical dilation** by the factor of a, as well as a **horizontal translation** by p units and **vertical translation** by q units. This approach makes the graphing process easier than when using a table of values.

In addition, the vertex form allows us to identify the main characteristics of the corresponding graph such as **shape**, **opening**, **vertex**, and **axis of symmetry**. Then, the additional properties of a quadratic function, such as **domain** and **range**, or where the function increases or decreases can be determined by observing the obtained graph.

Properties and Graph of the Basic Parabola $\boldsymbol{f(x) = x^2}$

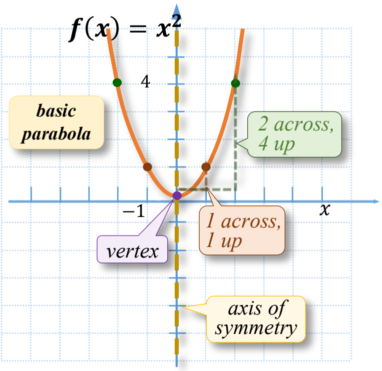
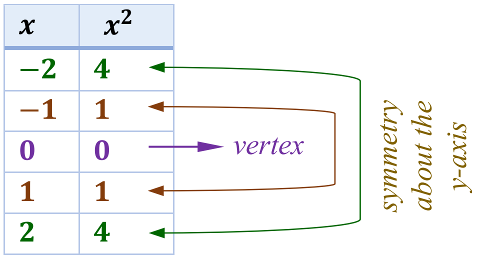


Figure 3.1

Recall the shape of the **basic parabola**, $f(x) = x^2$, as discussed in *Section P4*.



x	x^2
-2	4
-1	1
0	0
1	1
2	4

Observe the relations between the points listed in the table above. If we start with plotting the **vertex** $\boldsymbol{(0,0)}$, then the next pair of points, $\boldsymbol{(1,1)}$ and $\boldsymbol{(-1,1)}$, is plotted **1 unit across** from the vertex (both ways) and **1 unit up**. The following pair, $\boldsymbol{(2,4)}$ and $\boldsymbol{(-2,4)}$, is plotted **2 units across** from the vertex and **4 units up**. The graph of the parabola is obtained by connecting these 5 main points by a curve, as illustrated in *Figure 3.1*.

The graph of this parabola is symmetric in the y-axis, so the equation of the **axis of symmetry** is $\boldsymbol{x = 0}$.

The **domain** of the basic parabola is the set of all real numbers, $\mathbb{R}$, as $f(x) = x^2$ is a polynomial, and polynomials can be evaluated for any real x-value.

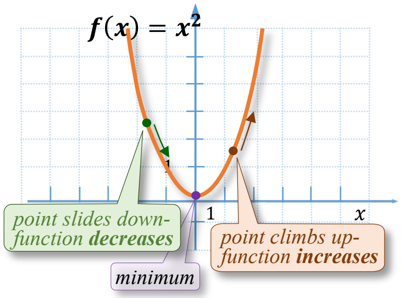


Figure 3.2

The **arms** of the parabola are directed **upwards**, which means that the vertex is the lowest point of the graph. Hence, the **range** of the basic parabola function, $f(x) = x^2$, is the interval $[\mathbf{0}, \infty)$, and the **minimum value** of the function is **0**.

Suppose a point 'lives' on the graph and travels from left to right. Observe that in the case of the basic parabola, if x-coordinates of the 'travelling' point are smaller than 0, the point slides down along the graph. Similarly, if x-coordinates are larger than 0, the point climbs up the graph. (See *Figure 3.2*) To describe this property in mathematical language, we say that the function $f(x) = x^2$ **decreases** in the interval $(-\infty, 0]$ and **increases** in the interval $[0, \infty)$.

Properties and Graphs of a Dilated Parabola $f(x) = ax^2$

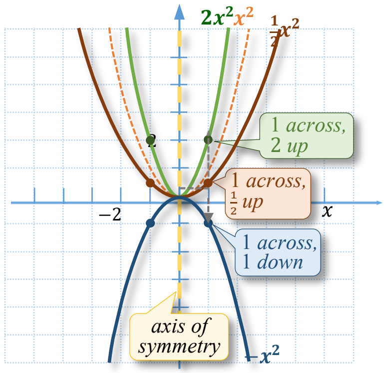


Figure 3.3

Figure 3.3 shows graphs of several functions of the form $f(x) = ax^2$. Observe how the shapes of these parabolas change for various values of a in comparison to the shape of the basic parabola $y = x^2$.

The common point for all of these parabolas is the vertex $(0,0)$. Additional points, essential for graphing such parabolas, are shown in the table below.

x	ax^2
-2	$4a$
-1	a
0	0
1	a
2	$4a$

vertex

*1 unit apart from zero, **a** units up*

*2 units apart from zero, **4a** units up*

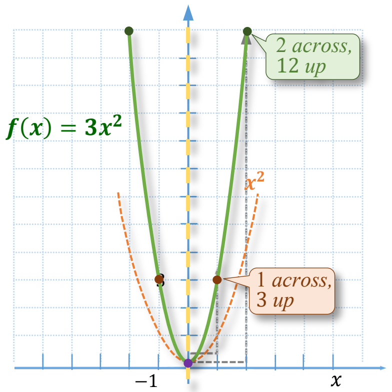


Figure 3.4

For example, to graph $f(x) = 3x^2$, it is convenient to plot the **vertex** first, which is at the point $(\mathbf{0}, \mathbf{0})$. Then, we may move the pen **1** unit **across** from the vertex (either way) and **3** units **up** to plot the points $(-1,3)$ and $(1,3)$. If the grid allows, we might want to plot the next two points, $(-2,12)$ and $(2,12)$, by moving the pen **2** units **across** from the vertex and $4 \cdot 3 = \mathbf{12}$ units **up**, as in *Figure 3.4*.

Notice that the obtained shape (in solid green) is **narrower** than the shape of the basic parabola (in dashed orange). However, similarly as in the case of the basic parabola, the shape of the dilated function is still **symmetrical about the y-axis, $x = 0$**.

Now, suppose we want to graph the function $f(x) = -\frac{1}{2}x^2$. As before, we may start by plotting the vertex at $(0,0)$. Then, we move the pen **1** unit **across** from the vertex (either way) and **half** a unit **down** to plot the points $\left(-1, -\frac{1}{2}\right)$ and $\left(1, -\frac{1}{2}\right)$, as in *Figure 3.5*. The next pair of points can be plotted by moving the pen **2** units **across** from the vertex and **2** units **down**, as the ordered pairs $(-2,-2)$ and $(2,-2)$ satisfy the equation $f(x) = -\frac{1}{2}x^2$.

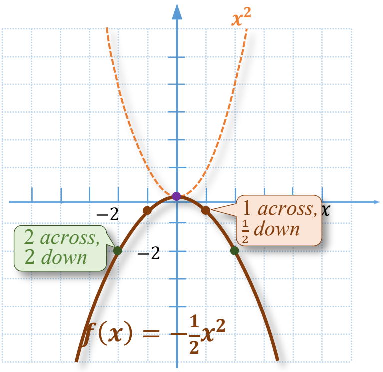


Figure 3.5

Notice that this time the obtained shape (in solid brown) is **wider** than the shape of the basic parabola (in dashed orange). Also, as a result of the **negative a-value**, the parabola opens **down**, and the **range** of this function is $(-\infty, \mathbf{0}]$.

Generally, the **shape** of a quadratic function of the form $\boldsymbol{f(x) = ax^2}$ is
- **narrower** than the shape of the basic parabola, if $|a| > \mathbf{1}$;
- **wider** than the shape of the basic parabola, if $\mathbf{0} < |a| < \mathbf{1}$; and
- **the same** as the shape of the **basic parabola**, $y = x^2$, if $|a| = \mathbf{1}$.

The parabola opens **up**, for $\boldsymbol{a > 0}$, and **down**, for $\boldsymbol{a < 0}$.
Thus the **vertex** becomes the **lowest point** of the graph, if $\boldsymbol{a > 0}$, and the **highest point** of the graph, if $\boldsymbol{a < 0}$.

The **range** of $\boldsymbol{f(x) = ax^2}$ is $[\mathbf{0}, \infty)$, if $\boldsymbol{a > 0}$, and $(-\infty, \mathbf{0}]$, if $\boldsymbol{a < 0}$.

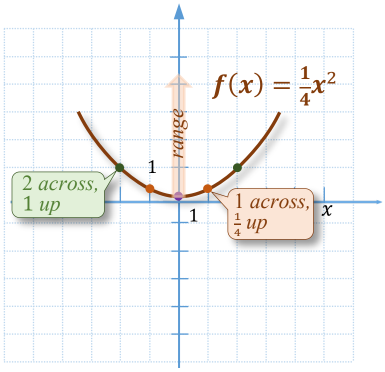
The **axis of symmetry** of the dilated parabola $\boldsymbol{f(x) = ax^2}$ remains the same as that of the basic parabola, which is $\boldsymbol{x = 0}$.

Example 1 ► Graphing a Dilated Parabola and Describing Its Shape, Opening, and Range

For each quadratic function, describe its shape and opening. Then graph it and determine its range.

a. $f(x) = \frac{1}{4}x^2$ **b.** $g(x) = -2x^2$

Solution ►



a. Since the leading coefficient of the function $f(x) = \frac{1}{4}x^2$ is positive, the parabola **opens up**. Also, since $0 < \frac{1}{4} < 1$, we expect the shape of the parabola to be **wider** than that of the basic parabola.

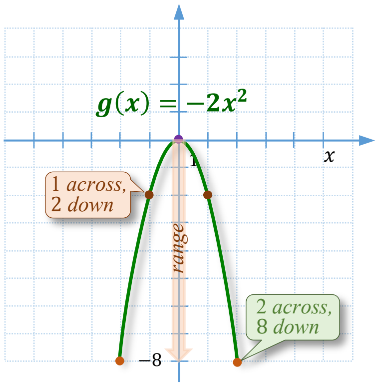
To graph $f(x) = \frac{1}{4}x^2$, first we plot the vertex at $(0,0)$ and then points $\left(\pm 1, \frac{1}{4}\right)$ and $\left(\pm 2, \frac{1}{4} \cdot 4\right) = (\pm 2, 1)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $[0, \infty)$.

b. Since the leading coefficient of the function $g(x) = -2x^2$ is negative, the parabola **opens down**. Also, since $|-2| > 1$, we expect the shape of the parabola to be **narrower** than that of the basic parabola.

To graph $g(x) = -2x^2$, first we plot the vertex at $(0,0)$ and then points $(\pm 1, -2)$ and $(\pm 2, -2 \cdot 4) = (\pm 2, -8)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $(-\infty, 0]$.



Properties and Graphs of the Basic Parabola with Shifts

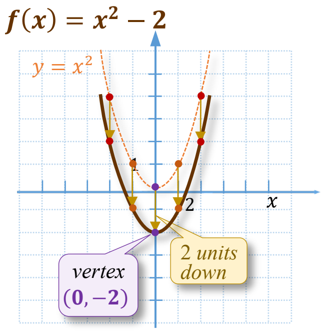


Figure 3.6

Suppose we would like to graph the function $f(x) = x^2 - 2$. We could do this via a table of values, but there is an easier way if we already know the shape of the basic parabola $y = x^2$.

Observe that for every x-value, the value of $x^2 - 2$ is obtained by subtracting 2 from the value of x^2. So, to graph $f(x) = x^2 - 2$, it is enough to **move each point** (x, x^2) of the basic parabola by **two units down**, as indicated in *Figure 3.6*.

The shift of y-values by 2 units down causes the **range** of the new function, $f(x) = x^2 - 2$, to become $[-2, \infty)$. Observe that this vertical shift also changes the minimum value of this function, from 0 to -2.

The **axis of symmetry** remains unchanged, and it is $\boldsymbol{x = 0}$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = x^2 + q}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{q}$ **steps up**, if $\boldsymbol{q > 0}$;
- **shifting** the graph of the basic parabola $|\boldsymbol{q}|$ **steps down**, if $\boldsymbol{q < 0}$.

The **vertex** of such parabola is at $(\boldsymbol{0, q})$. The **range** of it is $[\boldsymbol{q}, \infty)$.
The **minimum** (lowest) **value** of the function is $\boldsymbol{q}$.
The **axis of symmetry** is $\boldsymbol{x = 0}$.

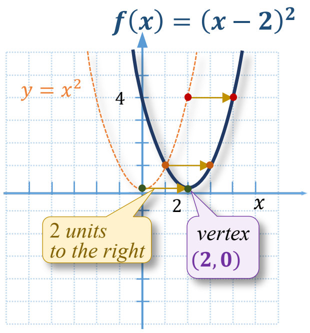


Figure 3.7

Now, suppose we wish to graph the function $f(x) = (x-2)^2$. We can graph it by joining the points calculated in the table below.

$\boldsymbol{x}$	$\boldsymbol{(x-2)^2}$
0	**4**
1	**1**
2	**0**
3	**1**
4	**4**

Observe that the parabola $f(x) = (x-2)^2$ assumes its lowest value at the vertex. The lowest value of the perfect square $(x-2)^2$ is zero, and it is attained at the x-value of 2. Thus, the vertex of this parabola is (2,0).

Notice that the **vertex** $(\boldsymbol{2, 0})$ of $f(x) = (x-2)^2$ is positioned 2 units to the right from the vertex (0,0) of the basic parabola.

This suggests that the graph of the function $f(x) = (x-2)^2$ can be obtained without the aid of a table of values. It is enough to shift the graph of the basic parabola **2 units** to the **right**, as shown in *Figure 3.7*.

Observe that the horizontal shift does not influence the **range** of the new parabola $f(x) = (x-2)^2$. It is still $[\boldsymbol{0}, \infty)$, the same as for the basic parabola. However, the **axis of symmetry** has changed to $\boldsymbol{x = 2}$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = (x-p)^2}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{p}$ **steps** to the **right**, if $\boldsymbol{p > 0}$;
- **shifting** the graph of the basic parabola $|\boldsymbol{p}|$ **steps** to the **left**, if $\boldsymbol{p < 0}$.

The **vertex** of such a parabola is at $(\boldsymbol{p, 0})$. The **range** of it is $[\boldsymbol{0}, \infty)$.

The **minimum value** of the function is $\boldsymbol{0}$.

The **axis of symmetry** is $\boldsymbol{x = p}$.

Example 2 ► Graphing Parabolas and Observing Transformations of the Basic Parabola

Graph each parabola by plotting its vertex and following the appropriate opening and shape. Then describe transformations of the basic parabola that would lead to the obtained graph. Finally, state the range and the equation of the axis of symmetry.

a. $f(x) = (x+3)^2$

b. $g(x) = -x^2 + 1$

Solution ►

a. The perfect square $(x+3)^2$ attains its lowest value at $x=-3$. So, the **vertex** of the parabola $f(x)=(x+3)^2$ is $(\mathbf{-3, 0})$. Since the leading coefficient is 1, the parabola takes the shape of $y=x^2$, and its **arms open up**.

The graph of the function f can be obtained by **shifting** the graph of the basic parabola **3 units to the left**, as shown in *Figure 3.8.*

The **range** of function f is $[\mathbf{0}, \infty)$, and the equation of the **axis of symmetry** is $\boldsymbol{x=-3}$.

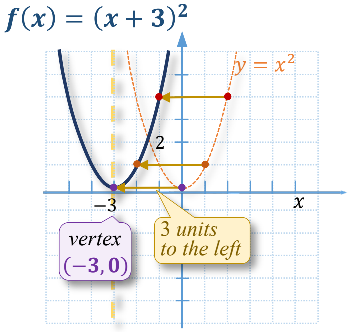


Figure 3.8

b. The expression $-x^2+1$ attains its highest value at $x=0$. So, the **vertex** of the parabola $g(x)=-x^2+1$ is $(\mathbf{0}, \mathbf{1})$. Since the leading coefficient is -1, the parabola takes the shape of $y=x^2$, but its **arms open down**.

The graph of the function g can be obtained by:

- first, **flipping the graph** of the basic parabola **over the *x*-axis**, and then
- **shifting** the graph of $y=-x^2$ **1 unit up**, as shown in *Figure 3.9.*

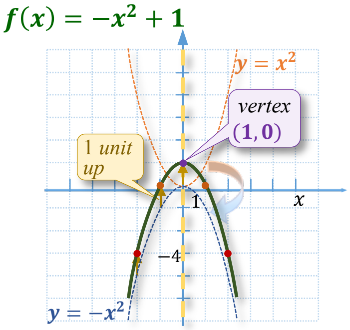


Figure 3.9

The **range** of the function g is $(-\infty, \mathbf{1}]$, and the equation of the **axis of symmetry** is $\boldsymbol{x=0}$.

Note: The order of transformations in the above example is essential. The reader is encouraged to check that **shifting** the graph of $y=x^2$ by 1 unit up first and then **flipping** it over the x-axis results in a different graph than the one in *Figure 3.9*.

Properties and Graphs of Quadratic Functions Given in the Vertex Form $f(x)=a(x-p)^2+q$

So far, we have discussed properties and graphs of quadratic functions that can be obtained from the graph of the basic parabola by applying mainly a single transformation. These transformations were: dilations (including flips over the x-axis), and horizontal and vertical shifts. Sometimes, however, we need to apply more than one transformation. We have already encountered such a situation in *Example 2b*, where a flip and a vertical shift were applied. Now, we will look at properties and graphs of any function of the form $\boldsymbol{f(x)=a(x-p)^2+q}$, referred to as the **vertex form** of a quadratic function.

Suppose we wish to graph $f(x) = 2(x+1)^2 - 3$. This can be accomplished by connecting the points calculated in a table of values, such as the one below, or by observing the coordinates of the vertex and following the shape of the graph of $y = 2x^2$. Notice that the vertex of our parabola is at $(-1, -3)$. This information can be taken directly from the equation $f(x) = 2(x+1)^2 - 3 = 2\big(x-(-1)\big)^2 - 3$,

opposite to the number in the bracket

the same last number

without the aid of a table of values.

x	$2(x+1)^2 - 3$
-3	5
-2	-1
-1	-3
0	-1
1	5

1 unit apart from zero, 2 units up

vertex

The rest of the points follow the pattern of the shape for the $y = 2x^2$ parabola: 1 across, 2 up; 2 across, $4 \cdot 2 = 8$ up. So, we connect the points as in *Figure 3.10*.

Notice that the graph of function f could also be obtained as a result of translating the graph of $y = 2x^2$ by 1 unit left and 3 units down, as indicated in *Figure 3.10* by the blue vectors.

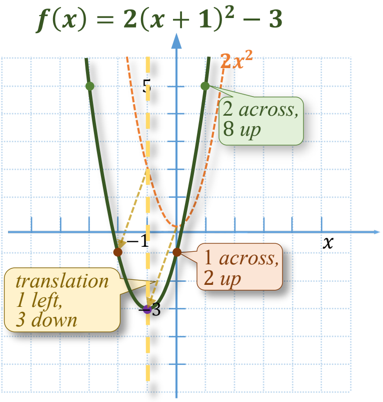


Figure 3.10

Here are the main properties of the graph of function f:
- It has a **shape** of $y = \mathbf{2x^2}$;
- It is a parabola that **opens up**;
- It has a **vertex** at $(\mathbf{-1, -3})$;
- It is **symmetrical** about the line $\boldsymbol{x = -1}$;
- Its **minimum value** is – **3**, and this minimum is attained at $x = -1$;
- Its domain is the set of all real numbers, and its **range** is the interval $[\mathbf{-3}, \infty)$;
- It **decreases** for $x \in (-\infty, \mathbf{-1}]$ and **increases** for $x \in [\mathbf{-1}, \infty)$.

The above discussion of properties and graphs of a quadratic function given in vertex form leads us to the following general observations:

Characteristics of Quadratic Functions Given in Vertex Form $f(x) = a(x-p)^2 + q$

1. The graph of a quadratic function given in **vertex form**

$$\boldsymbol{f(x) = a(x-p)^2 + q}, \text{ where } \boldsymbol{a \neq 0},$$

is a **parabola** with **vertex** $(\boldsymbol{p, q})$ and **axis of symmetry** $\boldsymbol{x = p}$.

2. The graph **opens up** if $\boldsymbol{a}$ is **positive** and **down** if $\boldsymbol{a}$ is **negative**.

3. If $\boldsymbol{a > 0}$, $\boldsymbol{q}$ is the **minimum value**. If $\boldsymbol{a < 0}$, $\boldsymbol{q}$ is the **maximum value**.

3. The graph is **narrower** than that of $y = x^2$ if $\boldsymbol{|a| > 1}$.
The graph is **wider** than that of $y = x^2$ if $\boldsymbol{0 < |a| < 1}$.

4. The **domain** of function f is the set of real numbers, $\mathbb{R}$.
The **range** of function f is $[\boldsymbol{q}, \infty)$ if $\boldsymbol{a}$ is **positive** and $(-\infty, \boldsymbol{q}]$ if $\boldsymbol{a}$ is **negative**.

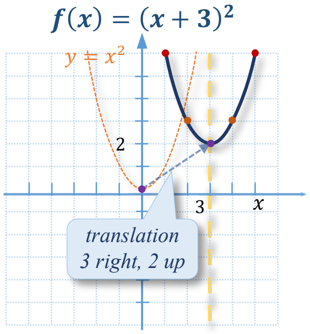
Example 3 ► Identifying Properties and Graphing Quadratic Functions Given in Vertex Form $\boldsymbol{f(x)=a(x-p)^2+q}$

For each function, identify its **vertex**, **opening**, **axis of symmetry**, and **shape**. Then graph the function and state its **domain** and **range**. Finally, describe **transformations** of the basic parabola that would lead to the obtained graph.

a. $f(x)=(x-3)^2+2$ **b.** $g(x)=-\frac{1}{2}(x+1)^2+3$

Solution ►

a. The vertex of the parabola $f(x)=(x-3)^2+2$ is $(\boldsymbol{3},\boldsymbol{2})$; the graph **opens up**, and the equation of the axis of symmetry is $\boldsymbol{x=3}$. To graph this function, we can plot the vertex first and then follow the shape of the basic parabola $y=\boldsymbol{x^2}$.



The domain of function f is $\mathbb{R}$, and the range is $[\boldsymbol{2},\infty)$.

The graph of f can be obtained by shifting the graph of the basic parabola **3 units to the right** and **2 units up**.

b. The vertex of the parabola $g(x)=-\frac{1}{2}(x+1)^2+3$ is $(\boldsymbol{-1},\boldsymbol{3})$; the graph **opens down**, and the equation of the axis of symmetry is $\boldsymbol{x=-1}$. To graph this function, we can plot the vertex first and then follow the shape of the parabola $y=-\frac{1}{2}\boldsymbol{x^2}$. This means that starting from the vertex, we move the pen one unit across (both ways) and drop half a unit to plot the next two points, $\left(0,\frac{5}{2}\right)$ and symmetrically $\left(-2,\frac{5}{2}\right)$. To plot the following two points, again, we start from the vertex and move our pen two units across and 2 units down (as $-\frac{1}{2}\cdot 4=-2$). So, the next two points are $(1,1)$ and symmetrically $(-4,1)$, as indicated in *Figure 3.11*.

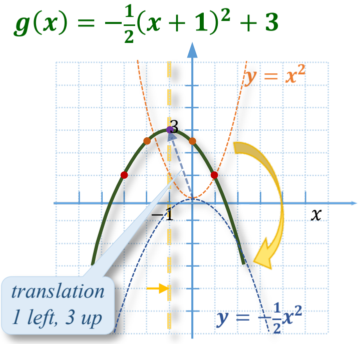


Figure 3.11

The domain of function g is $\mathbb{R}$, and the range is $(-\infty,\boldsymbol{3}]$.

The graph of g can be obtained from the graph of the basic parabola in two steps:

1. **Dilate** the basic parabola by multiplying its y-values by the factor of $-\frac{1}{2}$.
2. Shift the graph of the dilated parabola $y=-\frac{1}{2}\boldsymbol{x^2}$, **1 unit to the left** and **3 units up**, as indicated in *Figure 3.11*.

Aside from the main properties such as vertex, opening and shape, we are often interested in x- and y-intercepts of the given parabola. The next example illustrates how to find these intercepts from the vertex form of a parabola.

Example 4 ► Finding the Intercepts from the Vertex Form $\boldsymbol{f(x)=a(x-p)^2+q}$

Find the x- and y-intercepts of each parabola.

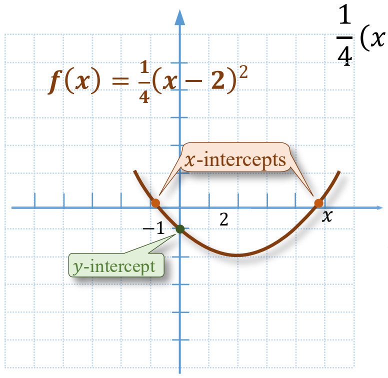
a. $f(x) = \frac{1}{4}(x-2)^2 - 2$ **b.** $g(x) = -2(x+1)^2 - 3$

Solution ► **a.** To find the y-intercept, we evaluate the function at zero. Since

$$f(0) = \frac{1}{4}(-2)^2 - 2 = 1 - 2 = -1,$$

then the y-intercept is $(\mathbf{0}, -\mathbf{1})$.

To find x-intercepts, we set $f(x) = 0$. So, we need to solve the equation

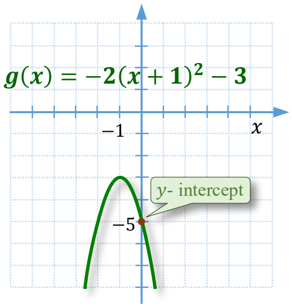


$$\begin{aligned} \frac{1}{4}(x-2)^2 - 2 &= 0 \\ \frac{1}{4}(x-2)^2 &= 2 \\ (x-2)^2 &= 8 \\ \sqrt{(x-2)^2} &= \sqrt{8} \\ |x-2| &= 2\sqrt{2} \\ x - 2 &= \pm 2\sqrt{2} \\ x = 2 \pm 2 &= \begin{cases} 2 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{cases} \end{aligned}$$

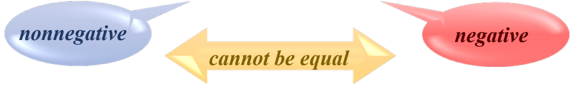
Hence, the two x-intercepts are: $\left(\mathbf{2} - \mathbf{2}\sqrt{\mathbf{2}}, \mathbf{0}\right)$ and $\left(\mathbf{2} + \mathbf{2}\sqrt{\mathbf{2}}, \mathbf{0}\right)$.

b. Since $g(0) = -2(1)^2 - 3 = -5$, then the y-intercept is $(\mathbf{0}, -\mathbf{5})$.

To find x-intercepts, we attempt to solve the equation



$$\begin{aligned} -2(x+1)^2 - 3 &= 0 \\ -2(x+1)^2 &= 3 \\ (x+1)^2 &= -\frac{3}{2} \end{aligned}$$



However, since the last equation doesn't have any solution, we conclude that function $g(x)$ has no x-intercepts.

Q.3 Exercises

1. Match each quadratic function **a.-d.** with its graph **I-IV**.

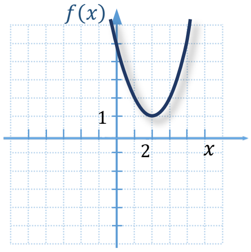
a. $f(x) = (x - 2)^2 - 1$

b. $f(x) = (x - 2)^2 + 1$

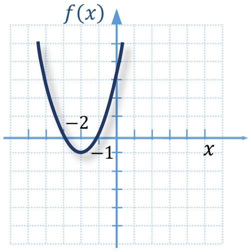
c. $f(x) = (x + 2)^2 + 1$

d. $f(x) = (x + 2)^2 - 1$

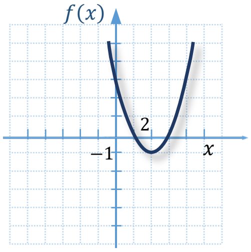
I



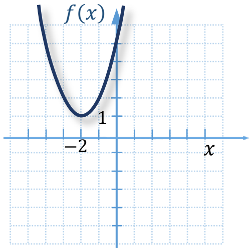
II



III



IV



2. Match each quadratic function **a.-d.** with its graph **I-IV**.

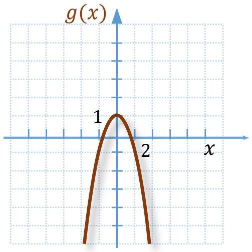
a. $g(x) = -(x - 2)^2 + 1$

b. $g(x) = x^2 - 1$

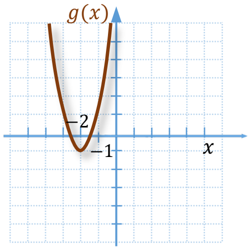
c. $g(x) = -2x^2 + 1$

d. $g(x) = 2(x + 2)^2 - 1$

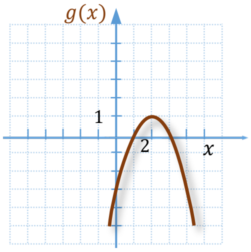
I



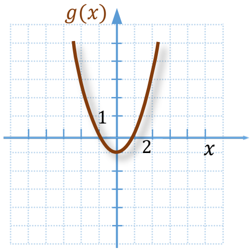
II



III



IV



3. Match each quadratic function with the characteristics of its parabolic graph.

a. $f(x) = 5(x - 3)^2 + 2$

b. $f(x) = -4(x + 2)^2 - 3$

c. $f(x) = -\frac{1}{2}(x - 3)^2 + 2$

d. $f(x) = \frac{1}{4}(x + 2)^2 - 3$

I vertex $(3,2)$, opens down

II vertex $(3,2)$, opens up

III vertex $(-2,-3)$, opens down

IV vertex $(-2,-3)$, opens up

For each quadratic function, describe the ***shape*** *(as* ***wider***, ***narrower***, *or the* ***same*** *as the shape of $y = x^2$) and* ***opening (up*** *or* ***down)*** *of its graph. Then* ***graph it*** *and determine its* ***range***.

4. $f(x) = 3x^2$ **5.** $f(x) = -\frac{1}{2}x^2$ **6.** $f(x) = -\frac{3}{2}x^2$

7. $f(x) = \frac{5}{2}x^2$ **8.** $f(x) = -x^2$ **9.** $f(x) = \frac{1}{3}x^2$

Graph *each parabola by plotting its vertex, and following its shape and opening. Then,* ***describe transformations*** *of the basic parabola that would lead to the obtained graph. Finally, state the* ***domain*** *and* ***range***, *and the equation of the* ***axis of symmetry***.

10. $f(x) = (x-3)^2$ **11.** $f(x) = -x^2 + 2$ **12.** $f(x) = x^2 - 5$

13. $f(x) = -(x+2)^2$ **14.** $f(x) = -2x^2 - 1$ **15.** $f(x) = \frac{1}{2}(x+2)^2$

For each parabola, state its ***vertex***, ***shape***, ***opening***, *and* ***x- and y-intercepts***. *Then,* ***graph*** *the function and describe* ***transformations*** *of the basic parabola that would lead to the obtained graph.*

16. $f(x) = 3x^2 - 1$ **17.** $f(x) = -\frac{3}{4}x^2 + 3$

18. $f(x) = -\frac{1}{2}(x+4)^2 + 2$ **19.** $f(x) = \frac{5}{2}(x-2)^2 - 4$

20. $f(x) = 2(x-3)^2 + \frac{3}{2}$ **21.** $f(x) = -3(x+1)^2 + 5$

22. $f(x) = -\frac{2}{3}(x+2)^2 + 4$ **23.** $f(x) = \frac{4}{3}(x-3)^2 - 2$

24. Four students, **A**, **B**, **C**, and **D**, tried to graph the function $f(x) = -2(x+1)^2 - 3$ by transforming the graph of the basic parabola, $y = x^2$. Here are the transformations that each student applied

*Student **A**:*
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of -2

*Student **B**:*
- dilation of y-values by the factor of -2
- shift 1 unit left
- shift 3 units down

*Student **C**:*
- flip over the x-axis
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of 2

*Student **D**:*
- shift 1 unit left
- dilation of y-values by the factor of 2
- shift 3 units down
- flip over the x-axis

With the assumption that all transformations were properly applied, discuss whose graph was correct and what went wrong with the rest of the graphs. Is there any other sequence of transformations that would result in a correct graph?

For each parabola, state the coordinates of its ***vertex*** *and then* ***graph*** *it. Finally, state the* ***extreme value*** *(****maximum*** *or* ***minimum***, *whichever applies) and the* ***range*** *of the function.*

25. $f(x) = 3(x-1)^2$ **26.** $f(x) = -\frac{5}{2}(x+3)^2$

27. $f(x) = (x+2)^2 - 3$

28. $f(x) = -3(x+4)^2 + 5$

29. $f(x) = -2(x-5)^2 - 2$

30. $f(x) = 2(x-4)^2 + 1$

31. $f(x) = \frac{1}{2}(x+1)^2 + \frac{3}{2}$

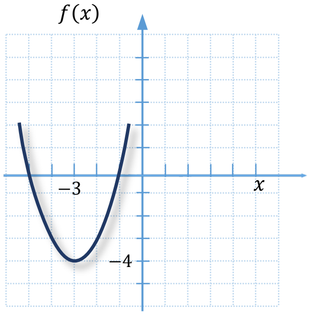
32. $f(x) = -\frac{1}{2}(x-1)^2 - 3$

33. $f(x) = -\frac{1}{4}(x-3)^2 + 4$

34. $f(x) = \frac{3}{4}\left(x+\frac{5}{2}\right)^2 - \frac{3}{2}$

Given the graph of a parabola, state the most probable ***equation*** *of the corresponding function. Hint: Use the vertex form of a quadratic function.*

35.



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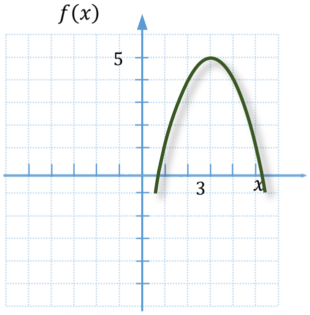

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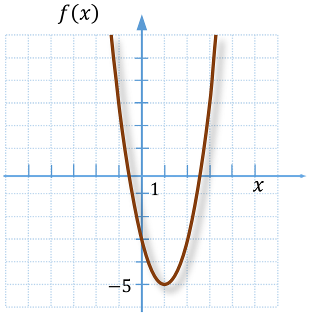

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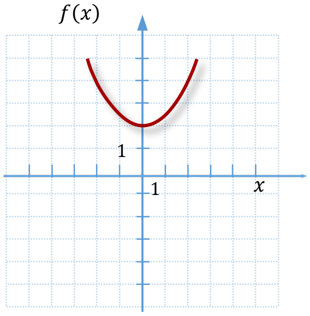

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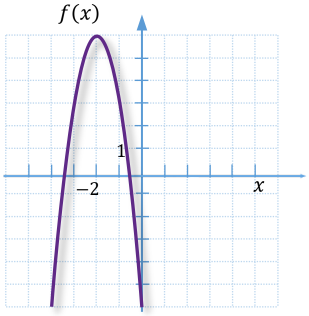

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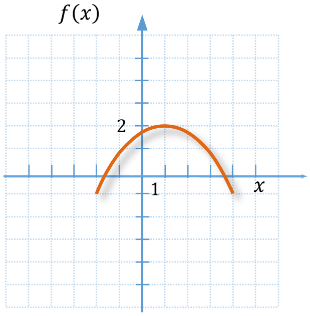


Q4 Properties of Quadratic Functions and Optimization Problems

In the previous section, we examined how to graph and read the characteristics of the graph of a quadratic function given in vertex form, $f(x) = a(x-p)^2 + q$. In this section, we discuss the ways of **graphing** and reading the **characteristics** of the graph of a quadratic function given in **standard form**, $\boldsymbol{f(x) = ax^2 + bx + c}$. One of these ways is to convert standard form of the function to vertex form by **completing the square** so that the information from the vertex form may be used for graphing. The other handy way of graphing and reading properties of a quadratic function is to **factor** the defining trinomial and use the **symmetry** of a parabolic function.

At the end of this section, we apply properties of quadratic functions to solve certain **optimization problems**. To solve these problems, we look for the **maximum** or **minimum** of a particular quadratic function satisfying specified conditions called **constraints**. Optimization problems often appear in geometry, calculus, business, computer science, etc.

Graphing Quadratic Functions Given in the Standard Form $f(x) = ax^2 + bx + c$

To graph a quadratic function given in standard form, $f(x) = ax^2 + bx + c$, we can use one of the following methods:

1. constructing a **table of values** (this would always work, but it could be cumbersome);
2. converting to **vertex form** by using the technique of completing the square (see *Examples 1-3*);
3. **factoring** and employing the properties of a parabolic function. (this is a handy method if the function can be easily factored – see *Examples 4 and 5*)

The table of values approach can be used for any function, and it was already discussed on various occasions throughout this textbook.

Converting to **vertex form** involves completing the square. For example, to convert the function $f(x) = 2x^2 + x - 5$ to its vertex form, we might want to start by dividing both sides of the equation by the leading coefficient 2, and then complete the square for the polynomial on the right side of the equation, as below.

$$\frac{f(x)}{2} = x^2 + \frac{1}{2}x - \frac{5}{2}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5 \cdot 8}{2 \cdot 8}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{41}{16}$$

Finally, the vertex form is obtained by multiplying both sides of the equation back by 2. So, we have

$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{41}{8}$$

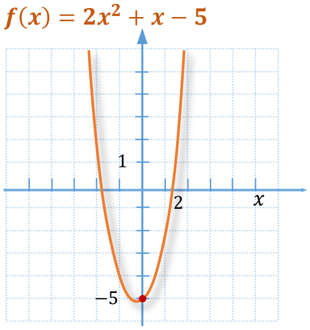


Figure 4.1

This form lets us identify the vertex, $\left(-\frac{1}{4}, -\frac{41}{8}\right)$, and the shape, $y = 2x^2$, of the parabola, which is essential for graphing it. To create an approximate graph of function f, we may want to round the vertex to approximately $(-0.25, -5.1)$ and evaluate $f(0) = 2 \cdot 0^2 + 0 - 5 = -5$. So, the graph is as in *Figure 4.1*.

Example 1 ► **Converting the Standard Form of a Quadratic Function to the Vertex Form**

Rewrite each function in its vertex form. Then, identify the vertex.

a. $f(x) = -3x^2 + 2x$ **b.** $g(x) = \frac{1}{2}x^2 + x + 3$

Solution ► **a.** To convert f to its vertex form, we follow the completing the square procedure. After dividing the equation by the leading coefficient,

$$f(x) = -3x^2 + 2x,$$

we have

$$\frac{f(x)}{-3} = x^2 - \frac{2}{3}x$$

Then, we complete the square for the right side of the equation,

$$\frac{f(x)}{-3} = \left(x - \frac{1}{3}\right)^2 - \frac{1}{9},$$

and finally, multiply back by the leading coefficient,

$$f(x) = -3\left(x - \frac{1}{3}\right)^2 + \frac{1}{3}.$$

Therefore, the vertex of this parabola is at the point $\left(\boldsymbol{\frac{1}{3}}, \boldsymbol{\frac{1}{3}}\right)$.

b. As in the previous example, to convert g to its vertex form, we first wish to get rid of the leading coefficient. This can be achieved by multiplying both sides of the equation $g(x) = \frac{1}{2}x^2 + x + 3$ by 2. So, we obtain

$$2g(x) = x^2 + 2x + 6$$
$$2g(x) = (x + 1)^2 - 1 + 6$$
$$2g(x) = (x + 1)^2 + 5,$$

which can be solved back for g,

$$g(x) = \frac{1}{2}(x + 1)^2 + \frac{5}{2}.$$

Therefore, the vertex of this parabola is at the point $\left(\boldsymbol{-1}, \boldsymbol{\frac{5}{2}}\right)$.

Completing the square allows us to derive a formula for the vertex of the graph of any quadratic function given in its standard form, $f(x) = ax^2 + bx + c$, where $a \neq 0$. Applying the same procedure as in *Example 1*, we calculate

$$f(x) = ax^2 + bx + c$$

$$\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}$$

Recall: This is the **discriminant** **Δ!**

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{-(b^2 - 4ac)}{4a}$$

Thus, the coordinates of the vertex $(\boldsymbol{p}, \boldsymbol{q})$ are $\boldsymbol{p} = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$ and $\boldsymbol{q} = \frac{-(\boldsymbol{b}^2 - \boldsymbol{4ac})}{\boldsymbol{4a}} = \frac{-\Delta}{\boldsymbol{4a}}$.

Observation: Notice that the expression for q can also be found by evaluating f at $x = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$.

So, the vertex of the parabola can also be expressed as $\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$.

VERTEX FORMULA

Summarizing, the **vertex** of a parabola defined by $f(x) = \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$, where $\boldsymbol{a} \neq 0$, can be calculated by following one of the formulas:

$$\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-(\boldsymbol{b}^2 - \boldsymbol{4ac})}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-\Delta}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$$

Example 2 ► **Using the Vertex Formula to Find the Vertex of a Parabola**

Use the vertex formula to find the vertex of the graph of $f(x) = -x^2 - x + 1$.

Solution ► The first coordinate of the vertex is equal to $-\frac{b}{2a} = -\frac{-1}{2\cdot(-1)} = -\frac{\mathbf{1}}{\mathbf{2}}$.

The second coordinate can be calculated by following the formula

$$\frac{-\Delta}{4a} = \frac{-((-1)^2 - 4\cdot(-1)\cdot 1)}{4\cdot(-1)} = \frac{\mathbf{5}}{\mathbf{4}},$$

or by evaluating $f\left(-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 1 = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{\mathbf{5}}{\mathbf{4}}$.

So, the vertex is $\left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{4}}\right)$.

Example 3 ► Graphing a Quadratic Function Given in Standard Form

Graph each function.

a. $g(x) = \frac{1}{2}x^2 + x + 3$ **b.** $f(x) = -x^2 - x + 1$

Solution ►

a. The shape of the graph of function g is the same as that of $y = \frac{1}{2}x^2$. Since the leading coefficient is positive, the arms of the parabola **open up**.
The **vertex**, $\left(\mathbf{-1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, was found in *Example 1b* as a result of completing the square. Since the vertex is in quadrant II and the graph opens up, we do not expect any x-intercepts. However, without much effort, we can find the y-intercept by evaluating $g(0) = \mathbf{3}$. Furthermore, since $(0,3)$ belongs to the graph, then by symmetry, $(-2,3)$ must also belong to the graph. So, we graph function g is as in *Figure 4.2*.

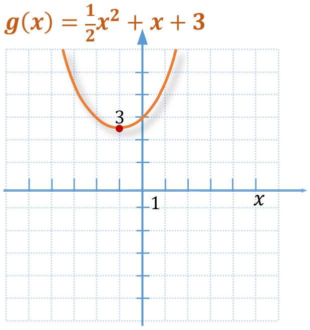


Figure 4.2

When plotting points with fractional coordinates, round the values to one place value.

b. The graph of function f has the shape of the basic parabola. Since the leading coefficient is negative, the arms of the parabola **open down**.
The **vertex**, $\left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{4}}\right)$, was found in *Example 2* by using the vertex formula. Since the vertex is in quadrant II and the graph opens down, we expect two x-intercepts. Their values can be found via the quadratic formula applied to the equation $-x^2 - x + 1 = 0$. So, the x-intercepts are $x_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \approx \mathbf{-1.6}$ *or* $\mathbf{0.6}$. In addition, the y-intercept of the graph is $f(0) = \mathbf{1}$.
Using all this information, we graph function f, as in *Figure 4.3*.

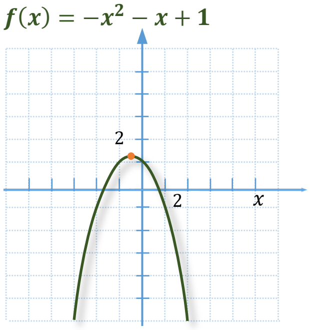


Figure 4.3

Graphing Quadratic Functions Given in the Factored Form $f(x) = a(x - r_1)(x - r_2)$

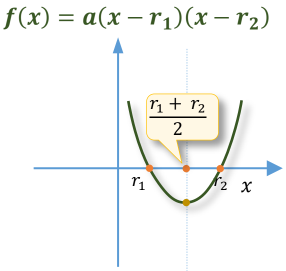


Figure 4.4

What if a quadratic function is given in factored form? Do we have to change it to vertex or standard form in order to find the vertex and graph it?

The factored form, $f(x) = a(x - r_1)(x - r_2)$, allows us to find the roots (or x-intercepts) of such a function. These are r_1 and r_2. A parabola is symmetrical about the axis of symmetry, which is the vertical line passing through its vertex. So, the first coordinate of the vertex is the same as the first coordinate of the midpoint of the line segment connecting the roots, r_1 with r_2, as indicated in *Figure 4.4*. Thus, the first coordinate of the vertex is the average of the two roots, $\frac{r_1 + r_2}{2}$. Then, the second coordinate of the vertex can be found by evaluating $f\left(\frac{r_1 + r_2}{2}\right)$.

Example 4 ► Graphing a Quadratic Function Given in a Factored Form

Graph function $g(x) = -(x-2)(x+1)$.

Solution ►

First, observe that the graph of function g has the same shape as the graph of the basic parabola, $f(x) = x^2$. Since the leading coefficient is negative, the arms of the parabola **open down**. Also, the graph intersects the x-axis at 2 and -1. So, the first coordinate of the vertex is the average of 2 and -1, which is $\frac{1}{2}$. The second coordinate is

$$g\left(\frac{1}{2}\right) = -\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} + 1\right) = -\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$$

Therefore, function g can be graphed by connecting the vertex, $\left(\frac{1}{2}, \frac{9}{4}\right)$, and the x-intercepts, $(-1,0)$ and $(2,0)$, with a parabolic curve, as in *Figure 4.5*. For a more precise graph, we may additionally plot the y-intercept, $g(0) = 2$, and the symmetrical point $g(1) = 2$.

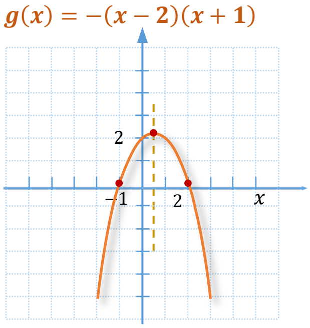


Figure 4.5

Example 5 ► Using Complete Factorization to Graph a Quadratic Function

Graph function $f(x) = 4x^2 - 2x - 6$.

Solution ►

Since the discriminant $\Delta = (-2)^2 - 4 \cdot 4 \cdot (-6) = 4 + 96 = 100$ is a perfect square number, the defining trinomial is factorable. So, to graph function f, we may want to factor it first. Notice that the GCF of all the terms is 2. So, $f(x) = 2(2x^2 - x - 3)$. Then, using factoring techniques discussed in *Section F2*, we obtain $f(x) = 2(2x-3)(x+1)$. This form allows us to identify the roots (or zeros) of function f, which are $\frac{3}{2}$ and -1. So, the first coordinate of the vertex is the average of $\frac{3}{2} = 1.5$ and -1, which is $\frac{1.5+(-1)}{2} = \frac{0.5}{2} = 0.25$. The second coordinate can be calculated by evaluating

$$f(0.25) = 2(2 \cdot 0.25 - 3)(0.25 + 1) = 2(0.5 - 3)(1.25) = 2(-2.5)(1.25) = -6.25$$

So, we can graph function f by connecting its vertex, $(\mathbf{0.25}, \mathbf{-6.25})$, and its x-intercepts, $(-1,0)$ and $(1.5,0)$, with a parabolic curve, as in *Figure 4.6*. For a more precise graph, we may additionally plot the y-intercept, $f(0) = -6$, and by symmetry, $f(0.5) = -6$.

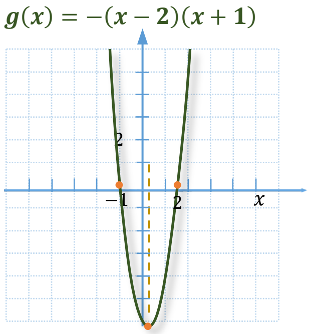


Figure 4.6

Observation: Since x-intercepts of a parabola are the solutions (zeros) of its equation, the equation of a parabola with x-intercepts at $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ can be written as

$$\boldsymbol{y = a(x - r_1)(x - r_2)},$$

for some real coefficient $\boldsymbol{a} \neq 0$.

Example 6 ► Finding an Equation of a Quadratic Function Given Its Solutions

a. Find an equation of a quadratic function whose graph passes the x-axis at -1 and 3.

b. Find an equation of a quadratic function whose graph passes the x-axis at -1 and 3 and the y-axis at -4.

c. Write a quadratic equation with integral coefficients knowing that the solutions of this equation are $\frac{1}{2}$ and $-\frac{2}{3}$.

Solution

► **a.** x-intercepts of a function are the zeros of this function. So, -1 and 3 are the zeros of the quadratic function. This means that the defining formula for such function should include factors $\big(x-(-1)\big)$ and $(x-3)$. So, it could be

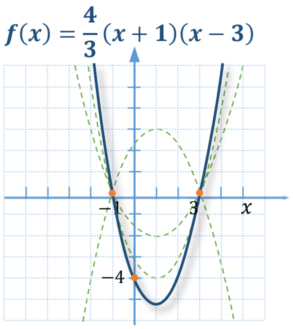
$$f(x) = (x+1)(x-3).$$

Notice that this is indeed a quadratic function with x-intercepts at -1 and 3. Hence, it satisfies the conditions of the problem.

b. Using the solution to *Example 6a*, notice that any function of the form

$$f(x) = a(x+1)(x-3),$$

where a is a nonzero real number, is a quadratic function with x-intercepts at -1 and 3. To guarantee that the graph of our function passes through the point $(0,-4)$, we need to find the particular value of the coefficient a. This can be done by substituting $x=0$ and $f(x)=-4$ into the function's equation and solving it for a. Thus,



$$-4 = a(0+1)(0-3)$$

$$-4 = -3a$$

$$a = \frac{4}{3},$$

and the desired function is $f(x) = \frac{4}{3}(x+1)(x-3)$.

c. First, observe that $\frac{1}{2}$ is a solution to the linear equation $2x-1=0$. Similarly, $-\frac{2}{3}$ is a solution to the equation $3x+2=0$. Multiplying these two equations side by side, we obtain a quadratic equation

$(2x-1)(3x+2)=0$

that satisfies the conditions of the problem.

> ***Note:*** Here, we could create the desired equation by writing
>
> $$\left(x-\tfrac{1}{2}\right)\left(x-\left(-\tfrac{2}{3}\right)\right)=0$$
>
> and then multiplying it by the $LCD = \mathbf{6} = \mathbf{2}\cdot\mathbf{3}$
>
> $$\mathbf{2}\left(x-\tfrac{1}{2}\right)\mathbf{3}\left(x+\tfrac{2}{3}\right)=\mathbf{6}\cdot 0$$
>
> $$(2x-1)(3x+2)=0$$

Optimization Problems

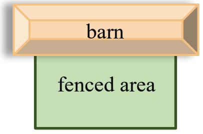
In many applied problems we are interested in **maximizing** or **minimizing** some quantity under specific conditions, called **constraints**. For example, we might be interested in finding the greatest area that can be fenced in by a given length of fence, or minimizing the cost of producing a container of a given shape and volume. These types of problems are called **optimization problems**.

Since the vertex of the graph of a quadratic function is either the highest or the lowest point of the parabola, it can be used in solving optimization problems that can be modeled by a quadratic function.

The vertex of a parabola provides the following information.

- The y-value of the vertex gives the maximum or minimum value of y.
- The x-value tells where the maximum or minimum occurs.

Example 7 ► Maximizing Area of a Rectangular Region



John has 60 meters of fencing to enclose a rectangular field by his barn. Assuming that the barn forms one side of the rectangle, find the maximum area he can enclose and the dimensions of the enclosed field that yield this area.

Solution ►

Let l and w represent the length and width of the enclosed area correspondingly, as indicated in *Figure 4.7*. The 60 meters of fencing is used to cover the distance of twice along the width and once along the length. So, we can form the constraint equation

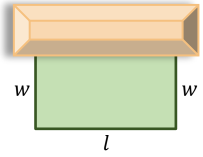


Figure 4.7

$$2w + l = 60 \qquad (1)$$

To analyse the area of the field,

$$A = lw, \qquad (2)$$

we would like to express it as a function of one variable, for example w. To do this, we can solve the constraint equation (1) for l and substitute the obtained expression into the equation of area, (2). Since $l = 60 - 2w$, then

$$A = lw = (60 - 2w)w$$

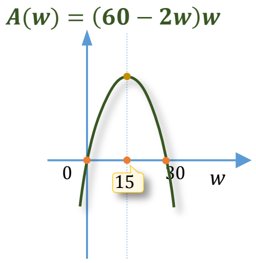


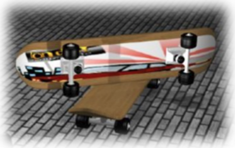
Figure 4.8

Observe that the graph of the function $A(w) = (60 - 2w)w$ is a parabola that opens down and intersects the x-axis at 0 and 30. This is because the leading coefficient of $(60 - 2w)w$ is negative and the roots to the equation $(60 - 2w)w = 0$ are 0 and 30. These roots are symmetrical in the axis of symmetry, which also passes through the vertex of the parabola, as illustrated in *Figure 4.8*. So, the first coordinate of the vertex is the average of the two roots, which is $\frac{0+30}{2} = 15$. Thus, the width that would maximize the enclosed area is $w_{max} = \mathbf{15}$ **meters**. Consequently, the length that would maximize the enclosed area is $l_{max} = 60 - 2w_{max} = 60 - 2 \cdot 15 = \mathbf{30}$ **meters**. The maximum area represented by the second coordinate of the vertex can be obtained by evaluating the function of area at the width of 15 meters.

$$A(15) = (60 - 2 \cdot 15)15 = 30 \cdot 15 = 450 \text{ m}^2$$

So, the maximum area that can be enclosed by 60 meters of fencing is **450 square meters**, and the dimensions of this rectangular area are **15 by 30 meters**.

Example 8 ► **Minimizing Average Unit Cost**



A company producing skateboards has determined that when x hundred skateboards are produced, the average cost of producing one skateboard can be modelled by the function

$$C(x) = 0.15x^2 - 0.75x + 1.5,$$

where $C(x)$ is in hundreds of dollars. How many skateboards should be produced to minimize the average cost of producing one skateboard? What would this cost be?

Solution ► Since $C(x)$ is a quadratic function, to find its minimum, it is enough to find the vertex of the parabola $C(x) = 0.15x^2 - 0.75x + 1.5$. This can be done either by completing the square or by using the formula for the vertex, $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$. We will do the latter. So, the vertex is

$$\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = \left(\frac{0.75}{0.3}, \frac{-(0.75^2 - 4 \cdot 0.15 \cdot 1.5)}{0.6}\right) = \left(2.5, \frac{-(0.5625 - 1.35)}{0.6}\right)$$
$$= \left(2.5, \frac{0.3375}{0.6}\right) = (\mathbf{2.5}, \mathbf{0.5625}).$$

This means that the lowest average unit cost can be achieved when 250 skateboards are produced, and that the lowest average cost of producing a skateboard would be $56.25.

Q.4 Exercises

*Convert each quadratic function to its **vertex form**. Then, state the coordinates of the **vertex**.*

1. $f(x) = x^2 + 6x + 10$
2. $f(x) = x^2 - 4x - 5$
3. $f(x) = x^2 + x - 3$
4. $f(x) = x^2 - x + 7$
5. $f(x) = -x^2 + 7x + 3$
6. $f(x) = 2x^2 - 4x + 1$
7. $f(x) = -3x^2 + 6x + 12$
8. $f(x) = -2x^2 - 8x + 10$
9. $f(x) = \frac{1}{2}x^2 + 3x - 1$

*Use the vertex formula, $\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$, to find the coordinates of the **vertex** of each parabola.*

10. $f(x) = x^2 + 6x + 3$
11. $f(x) = -x^2 + 3x - 5$
12. $f(x) = \frac{1}{2}x^2 - 4x - 7$
13. $f(x) = -3x^2 + 6x + 5$
14. $f(x) = 5x^2 - 7x$
15. $f(x) = 3x^2 + 6x - 20$

*For each parabola, state its **vertex**, **opening** and **shape**. Then **graph** it and state the **domain** and **range**.*

16. $f(x) = x^2 - 5x$
17. $f(x) = x^2 + 3x$
18. $f(x) = x^2 - 2x - 5$
19. $f(x) = -x^2 + 6x - 3$
20. $f(x) = -x^2 - 3x + 2$
21. $f(x) = 2x^2 + 12x + 18$

22. $f(x) = -2x^2 + 3x - 1$ **23.** $f(x) = -2x^2 + 4x + 1$ **24.** $f(x) = 3x^2 + 4x + 2$

For each quadratic function, state its ***zeros*** *(roots), coordinates of the* ***vertex****,* ***opening*** *and* ***shape****. Then* ***graph*** *it and identify its* ***extreme*** *(minimum or maximum)* ***value*** *as well as where it occurs.*

25. $f(x) = (x - 2)(x + 2)$ **26.** $f(x) = -(x + 3)(x - 1)$ **27.** $f(x) = x^2 - 4x$

28. $f(x) = x^2 + 5x$ **29.** $f(x) = x^2 - 8x + 16$ **30.** $f(x) = -x^2 - 4x - 4$

31. $f(x) = -3(x^2 - 1)$ **32.** $f(x) = \frac{1}{2}(x + 3)(x - 4)$ **33.** $f(x) = -\frac{3}{2}(x - 1)(x - 5)$

Find an equation of a quadratic function satisfying the given conditions.

34. passes the x-axis at -2 and 5 **35.** has x-intercepts at 0 and $\frac{2}{5}$

36. passes the x-axis at -3 and -1 and y-axis at 2 **37.** $f(1) = 0, f(4) = 0, f(0) = 3$

Write a quadratic equation with the indicated solutions using only integral coefficients.

38. -5 and 6 **39.** 0 and $\frac{1}{3}$ **40.** $-\frac{2}{5}$ and $\frac{3}{4}$ **41.** 2

42. Suppose the x-intercepts of the graph of a parabola are $(x_1, 0)$ and $(x_2, 0)$. What is the equation of the axis of symmetry of this graph?

43. How can we determine the number of x-intercepts of the graph of a quadratic function without graphing the function?

True or false? Explain.

44. The domain and range of a quadratic function are both the set of real numbers.

45. The graph of every quadratic function has exactly one y-intercept.

46. The graph of $y = -2(x - 1)^2 - 5$ has no x-intercepts.

47. The maximum value of y in the function $y = -4(x - 1)^2 + 9$ is 9.

48. The value of the function $f(x) = x^2 - 2x + 1$ is at its minimum when $x = 0$.

49. The graph of $f(x) = 9x^2 + 12x + 4$ has one x-intercept and one y-intercept.

50. If a parabola opens down, it has two x-intercepts.

Solve each problem.

51. A ball is projected from the ground straight up with an initial velocity of 24.5 m/sec. The function $h(t) = -4.9t^2 + 24.5t$ allows for calculating the height $h(t)$, in meters, of the ball above the ground after t seconds.

What is the maximum height reached by the ball? In how many seconds should we expect the ball to come back to the ground?

52. A firecracker is fired straight up and explodes at its maximum height above the ground. The function $h(t) = -4.9t^2 + 98t$ allows for calculating the height $h(t)$, in meters, of the firecracker above the ground t seconds after it was fired. In how many seconds after firing should we expect the firecracker to explode and at what height?

53. Antonio prepares and sells his favourite desserts at a market stand. Suppose his daily cost, C, in dollars, to sell n desserts can be modelled by the function $C(n) = 0.5n^2 - 30n + 350$. How many of these desserts should he sell to minimize the cost and what is the minimum cost?

54. Chris has a hot-dog stand. His daily cost, C, in dollars, to sell n hot-dogs can be modelled by the function $C(n) = 0.1n^2 - 15n + 700$. How many hotdogs should he sell to minimize the cost and what is the minimum cost?

55. Find two positive numbers with a sum of 32 that would produce the maximum product.

56. Find two numbers with a difference of 32 that would produce the minimum product.

57. Luke uses 16 meters of fencing to enclose a rectangular area for his baby goats. The enclosure shares one side with a large barn, so only 3 sides need to be fenced. If Luke wishes to enclose the greatest area, what should the dimensions of the enclosure be?

58. Ryan uses 60 meters of fencing to enclose a rectangular area for his livestock. He plans to subdivide the area by placing additional fence down the middle of the rectangle to separate different types of livestock. What dimensions of the overall rectangle will maximize the total area of the enclosure?

59. Julia works as a tour guide. She charges \$58 for an individual tour. When more people come for a tour, she charges \$2 less per person for each additional person, up to 25 people.

- **a.** Express the price per person P as a function of the number of people n, for $n \in \{1,2, \dots ,25\}$.
- **b.** Express her revenue, R, as a function of the number of people on tour.
- **c.** How many people on tour would maximize Julia's revenue?
- **d.** What is the highest revenue she can achieve?

60. One-day adult passes for The Mission Folk Festival cost \$50. At this price, the organizers of the festival expect about 1300 people to purchase the pass. Suppose that the organizers observe that every time they increase the cost per pass by 5\$, the number of passes sold decrease by about 100.

- **a.** Express the number of passes sold, N, as a function of the cost, c, of a one-day pass.
- **b.** Express the revenue, R, as a function of the cost, c, of a one-day pass.
- **c.** How much should a one-day pass costs to maximize the revenue?
- **d.** What is the maximum revenue?

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Trigonometry

Trigonometry is the branch of mathematics that studies the relations between the sides and angles of triangles. The word "**trigonometry**" comes from the Greek **trigōnon** (triangle) and **metron** (measure.) It was first studied by the Babylonians, Greeks, and Egyptians, and used in surveying, navigation, and astronomy. Trigonometry is a powerful tool that allows us to find the measures of angles and sides of triangles, without physically measuring them, and areas of plots of land. We begin our study of trigonometry by studying angles and their degree measures.

T1 Angles and Degree Measure

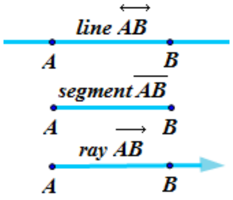


Figure 1a

Two distinct points $\boldsymbol{A}$ and $\boldsymbol{B}$ determine a line denoted $\overleftrightarrow{AB}$. The portion of the line between $\boldsymbol{A}$ and $\boldsymbol{B}$, including the points $\boldsymbol{A}$ and $\boldsymbol{B}$, is called a **line segment** (or simply, a **segment**) $\overline{AB}$. The portion of the line $\overleftrightarrow{AB}$ that starts at $\boldsymbol{A}$ and continues past $\boldsymbol{B}$ is called the **ray** $\overrightarrow{AB}$ (see *Figure 1a*.) Point $\boldsymbol{A}$ is the **endpoint** of this ray.

Two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ sharing the same endpoint $\boldsymbol{A}$, cut the plane into two separate regions. The union of the two rays and one of those regions is called an **angle**, the common endpoint $\boldsymbol{A}$ is called a **vertex**, and the two **rays** are called **sides** or **arms** of this angle. Customarily, we draw a small arc connecting the two rays to indicate which of the two regions we have in mind.

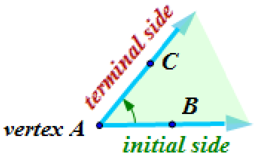


Figure 1b

In trigonometry, an **angle** is often identified with its **measure**, which is the **amount of rotation** that a ray in its initial position (called the **initial side**) needs to turn about the vertex to come to its final position (called the **terminal side**), as in *Figure 1b*. If the rotation from the initial side to the terminal side is *counterclockwise*, the angle is considered to be *positive*. If the rotation is *clockwise*, the angle is *negative* (see *Figure 1c*).

An angle is named either after its vertex, its rays, or the amount of rotation between the two rays. For example, an angle can be denoted $\angle \boldsymbol{A}$, $\angle \boldsymbol{BAC}$, or $\angle \boldsymbol{\theta}$, where the sign $\angle$ (or ∡) simply means *an angle*. Notice that in the case of naming an angle with the use of more than one letter, like $\angle \boldsymbol{BAC}$, the middle letter ($\boldsymbol{A}$) is associated with the vertex and the angle is oriented from the ray containing the first point ($\boldsymbol{B}$) to the ray containing the third point ($\boldsymbol{C}$). Customarily, angles (often identified with their measures) are denoted by Greek letters such as $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\theta}$, etc.

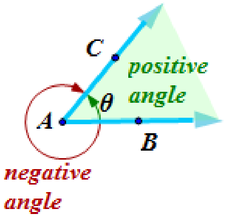


Figure 1c

An angle formed by rotating a ray counterclockwise (in short, **ccw**) exactly one **complete revolution** around its vertex is defined to have a measure of 360 degrees, which is abbreviated as **360°**.

Definition 1.1 ►

One **degree** ($\mathbf{1^\circ}$) is the measure of an angle that is $\frac{1}{360}$ part of a complete revolution.

One **minute** ($\mathbf{1'}$), is the measure of an angle that is $\frac{1}{60}$ part of a degree.

One **second** ($\mathbf{1''}$) is the measure of an angle that is $\frac{1}{60}$ part of a minute.

Therefore $\mathbf{1^\circ = 60'}$ and $\mathbf{1' = 60''}$.

A fractional part of a degree can be expressed in decimals (e.g. 29.68°) or in minutes and seconds (e.g. 29°40'48"). We say that the first angle is given in **decimal form,** while the second angle is given in **DMS** (**D**egree, **M**inute, **S**econd) **form**.

Example 1 ► Converting Between Decimal and DMS Form

Convert as indicated.

a. 29.68° to DMS form

b. $46°18'21''$ to decimal degree form

Solution ►

a. 29.68° can be converted to DMS form, using any calculator with **DMS** or **° ′ ″** key. To do it by hand, separate the fractional part of a degree and use the conversion factor $1° = 60'$.

$$\begin{aligned}\mathbf{29.68°} &= 29° + 0.68° \\ &= 29° + 0.68 \cdot 60' = 29° + 40.8'\end{aligned}$$

Similarly, to convert the fractional part of a minute to seconds, separate it and use the conversion factor $1' = 60''$. So we have

$$29.68° = 29° + 40' + 0.8 \cdot 60'' = \mathbf{29°40'48''}$$

b. Similarly, $46°18'21''$ can be converted to the decimal form, using the **DMS** or **° ′ ″** key. To do it by hand, we use the conversions $1' = \left(\frac{1}{60}\right)^{\circ}$ and $1'' = \left(\frac{1}{3600}\right)^{\circ}$.

$$\mathbf{46°18'21''} = \left[46 + 18 \cdot \frac{1}{60} + 21 \cdot \frac{1}{3600}\right]^{\circ} \cong \mathbf{46.3058°}$$

Example 2 ► Adding and Subtracting Angles in DMS Form

Perform the indicated operations.

a. $36°58'21'' + 5°06'45''$

b. $36°17' - 15°46'15''$

Solution ►

a. First, we add degrees, minutes, and seconds separately. Then, we convert each $60''$ into $1'$ and each $60'$ into 1°. Finally, we add the degrees, minutes, and seconds again.

$$\begin{aligned}36°58'21'' + 5°06'45'' &= 41° + 64' + 66'' \\ &= 41° + 1°04' + 1'06'' = \mathbf{42°05'06''}\end{aligned}$$

b. We can subtract within each denomination, degrees, minutes, and seconds, even if the answer is negative. Then, if we need more minutes or seconds to perform the remaining subtraction, we convert 1° into $60'$ or $1'$ into $60''$ to finish the calculation.

$$\begin{aligned}36°17' - 15°46'15'' &= 21° - 29' - 15'' \\ &= 20° + 60' - 29' - 15'' = 20° + 31' - 15'' \\ &= 20° + 30' + 60'' - 15'' = \mathbf{20°30'45''}\end{aligned}$$

Angles in Standard Position

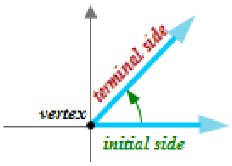


Figure 2

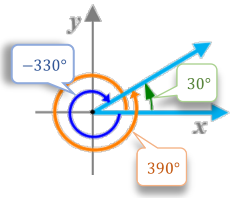


Figure 3

In trigonometry, we often work with angles in **standard position**, which means angles located in a rectangular system of coordinates with the vertex at the origin and the initial side on the positive x-axis, as in *Figure 2*. With the notion of angle as an amount of rotation of a ray to move from the initial side to the terminal side of an angle, the standard position allows us to represent infinitely many angles with the same terminal side. Those are the angles produced by rotating a ray from the initial side by full revolutions beyond the terminal side, either in a positive or negative direction. Such angles share the same initial and terminal sides and are referred to as **coterminal** angles.

For example, angles $-330°, 30°, 390°, 750°$, and so on, are coterminal.

Definition 1.2 ► Angles α and β are **coterminal**, if and only if there is an integer k, such that

$$\alpha = \beta + k \cdot 360°$$

Example 3 ► **Finding Coterminal Angles**

Find one positive and one negative angle that is closest to 0° and coterminal with

a. $80°$

b. $-530°$

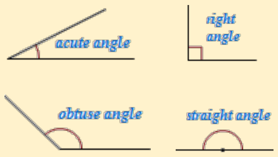
Solution ►

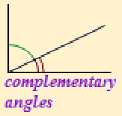
a. To find the closest to 0° positive angle coterminal with 80° we add one complete revolution, so we have $80° + 360° = \mathbf{440}°$.
Similarly, to find the closest to 0° negative angle coterminal with 80° we subtract one complete revolution, so we have $80° - 360° = \mathbf{-280}°$.

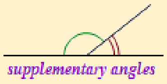
b. This time, to find the closest to 0° positive angle coterminal with $-530°$ we need to add two complete revolutions: $-530° + 2 \cdot 360° = \mathbf{190}°$.
To find the closest to 0° negative angle coterminal with $-530°$, it is enough to add one revolution: $-530° + 360° = \mathbf{-170}°$.

Definition 1.3 ► Let α be the measure of an angle. Such an angle is called

acute, if $\alpha \in (0°, 90°)$;
right, if $\alpha = 90°$; (right angle is marked by the symbol ∟)
obtuse, if $\alpha \in (90°, 180°)$; and
straight, if $\alpha = 180°$.







Angles that sum to $90°$ are called **complementary**.
Angles that sum to $180°$ are called **supplementary**.

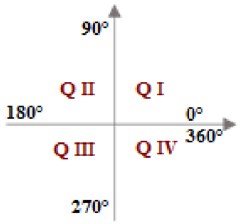


Figure 4

The two axes divide the plane into 4 regions, called **quadrants**. They are numbered counterclockwise, starting with the top right one, as in *Figure 4*.
An angle in standard position is said to lie in the quadrant in which its terminal side lies. For example, an **acute** angle is in *quadrant I* and an **obtuse** angle is in *quadrant II*.
Angles in standard position with their terminal sides along the x-axis or y-axis, such as $\mathbf{0°, 90°, 180°, 270°}$, and so on, are called **quadrantal angles**.

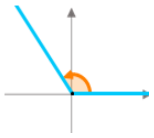
Example 4 ► Classifying Angles by Quadrants

Draw each angle in standard position. Determine the quadrant in which each angle lies or classify the angle as quadrantal.

a. $125°$ **b.** $-50°$ **c.** $270°$ **d.** $210°$

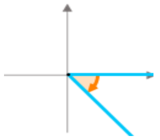
Solution ►

a.



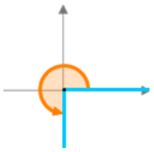
$125°$ is in **QII**

b.



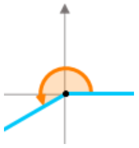
$-50°$ is in **QIV**

c.



quadrantal angle

d.



$210°$ is in **QIII**

Example 5 ► Finding Complementary and Supplementary Angles

Find the complement and the supplement of $57°$.

Solution ►

Since complementary angles add to $90°$, the complement of $57°$ is $90° - 57° = 33°$.
Since supplementary angles add to $180°$, the supplement of $57°$ is $180° - 57° = 123°$.

T.1 Exercises

*Convert each angle measure to **decimal degrees**. Round the answer to the nearest thousandth of a degree.*

1. $20°04'30''$
2. $71°45'$
3. $274°18'15''$
4. $34°41'07''$
5. $15°10'05''$
6. $64°51'35''$

*Convert each angle measure to **degrees, minutes, and seconds**. Round the answer to the nearest second.*

7. $18.0125°$
8. $89.905°$
9. $65.0015°$

10. 184.3608° **11.** 175.3994° **12.** 102.3771°

Perform each calculation.

13. $62^\circ 18' + 21^\circ 41'$ **14.** $71^\circ 58' + 47^\circ 29'$ **15.** $65^\circ 15' - 31^\circ 25'$

16. $90^\circ - 51^\circ 28'$ **17.** $15^\circ 57' 45'' + 12^\circ 05' 18''$ **18.** $90^\circ - 36^\circ 18' 47''$

Give the complement and the supplement of each angle.

19. 30° **20.** 60° **21.** 45° **22.** 86.5° **23.** $15^\circ 30'$

24. Give an expression representing the complement of a $\boldsymbol{\theta}^\circ$ angle.

25. Give an expression representing the supplement of a $\boldsymbol{\theta}^\circ$ angle.

Sketch each angle in standard position. Draw an arrow representing the correct amount of rotation. Give the quadrant of each angle or identify it as a quadrantal angle.

26. 75° **27.** 135° **28.** -60° **29.** 270° **30.** 390°

31. 315° **32.** 510° **33.** -120° **34.** 240° **35.** -180°

Find the angle of least positive measure coterminal with each angle.

36. -30° **37.** 375° **38.** -203° **39.** 855° **40.** 1020°

Give an expression that generates all angles coterminal with the given angle. Use $\boldsymbol{k}$ *to represent any integer.*

41. 30° **42.** 45° **43.** 0° **44.** 90° **45.** α°

Find the degree measure of the smaller angle formed by the hands of a clock at the following times.

46. **47.** $3:15$ **48.** $1:45$

T2 Trigonometric Ratios of an Acute Angle and of Any Angle

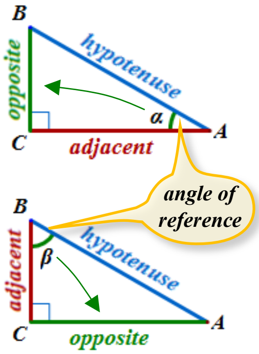


Figure 2.1

Generally, trigonometry studies ratios between sides in right angle triangles. When working with right triangles, it is convenient to refer to the side **opposite** to an angle, the side **adjacent** to (next to) an angle, and the **hypotenuse**, which is the longest side, opposite to the right angle. Notice that the opposite and adjacent sides depend on the **angle of reference** (one of the two acute angles.) However, the hypotenuse stays the same, regardless of the choice of the angle of reference. See *Figure 2.1.*

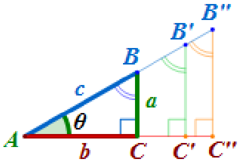


Figure 2.2

Notice that any two right triangles with the same acute angle $\boldsymbol{\theta}$ are **similar**. See *Figure 2.2.* **Similar** means that their corresponding angles are **congruent** and their corresponding sides are **proportional**. For instance, assuming notation as on *Figure 2.2*, we have

$$\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'},$$

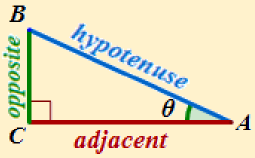
or equivalently

$$\frac{BC}{AB} = \frac{B'C'}{AB'}, \qquad \frac{AC}{AB} = \frac{AC'}{AB'}, \qquad \frac{BC}{AC} = \frac{B'C'}{AC'}.$$

Therefore, the ratios of any two sides of a right triangle does not depend on the size of the triangle but only on the size of the angle of reference. See the following demonstration. This means that we can study those **ratios** of sides as **functions** of an acute angle.

Trigonometric Functions of Acute Angles

Definition 2.1 ►



Given a **right angle triangle** with an **acute** angle $\boldsymbol{\theta}$, the three **primary trigonometric ratios** of the angle $\boldsymbol{\theta}$, called ***sine***, ***cosine***, and ***tangent*** (abbreviation: *sin, cos, tan*), are defined as follows:

$$\sin\theta = \frac{Opposite}{Hypotenuse}, \qquad \cos\theta = \frac{Adjacent}{Hypotenuse}, \qquad \tan\theta = \frac{Opposite}{Adjacent}$$

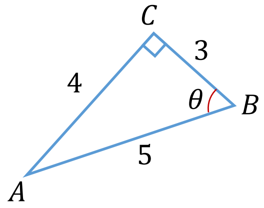
For easier memorization, we can use the acronym **SOH – CAH – TOA** (read: *so – ka – toe – ah*), formed from the first letter of the function and the corresponding ratio.

The three **reciprocal trigonometry ratios** of the angle $\boldsymbol{\theta}$, called ***cosecant***, ***secant***, and ***cotangent*** (abbreviation: *csc, sec, cot*), are reciprocals of the sine, cosine, and tangent ratios, respectively, and are defined as follows:

$$\csc\theta = \frac{Hypotenuse}{Opposite}, \qquad \sec\theta = \frac{Hypotenuse}{Adjacent}, \qquad \cot\theta = \frac{Adjacent}{Opposite}$$

Example 1 ► **Identifying Sides of a Right Triangle to Form Trigonometric Ratios**

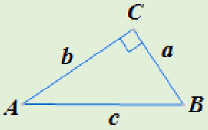
Identify the hypotenuse, opposite, and adjacent side of angle θ and state values of the six trigonometric ratios.



Solution ► Side AB is the hypotenuse, as it lies across from the right angle.
Side BC is the adjacent, as it is part of the angle θ other than hypotenuse.
Side AC is the opposite, as it lies across from angle θ.
Therefore, $\sin\theta = \frac{opp.}{hyp.} = \frac{\mathbf{4}}{\mathbf{5}}$, $\cos\theta = \frac{adj.}{hyp.} = \frac{\mathbf{3}}{\mathbf{5}}$, $\tan\theta = \frac{opp.}{adj.} = \frac{\mathbf{4}}{\mathbf{3}}$, $\csc\theta = \frac{hyp.}{opp.} = \frac{\mathbf{5}}{\mathbf{4}}$, $\sec\theta = \frac{hyp.}{adj.} = \frac{\mathbf{5}}{\mathbf{3}}$, and $\cot\theta = \frac{adj.}{opp.} = \frac{\mathbf{3}}{\mathbf{4}}$.

The three **primary trigonometric ratios** together with the **Pythagorean Theorem** allow us to **solve** any right-angle triangle. That means that given the measurements of two sides, or one side and one angle, with a little help of algebra, we can find the measurements of all remaining sides and angles of any right triangle. See *Section T4*.

Pythagorean Theorem ► A **triangle ABC is right** with $\angle C = 90°$ if and only if $\boldsymbol{a^2 + b^2 = c^2}$.

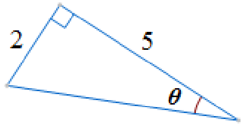


Convention: *The **side opposite** the given **vertex** (or angle) is named after the vertex, except that by a lower case rather than a capital letter. For example, the side opposite vertex **A** is called **a**.*

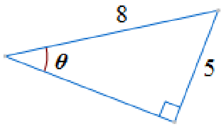
Example 2 ► **Finding Values of Trigonometric Ratios With the Aid of Pythagorean Theorem**

Given the triangle, find the exact values of the sine, cosine, and tangent ratios for angle θ.

a.



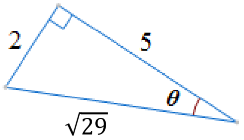
b.



Solution ► **a.** Let $\boldsymbol{h}$ denote the hypotenuse. By the **Pythagorean Theorem**, we have

$$h^2 = 2^2 + 5^2$$
$$h = \sqrt{4+25} = \sqrt{29}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{\mathbf{2\sqrt{29}}}{\mathbf{29}}$$

$$\cos\theta = \frac{5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{\mathbf{5\sqrt{29}}}{\mathbf{29}}$$

$$\tan\theta = \frac{\mathbf{2}}{\mathbf{5}}$$

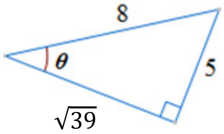
Note:
It is customary to rationalize the denominator.

b. Let $\boldsymbol{a}$ denote the adjacent side. By the **Pythagorean Theorem**, we have

$$a^2 + 5^2 = 8^2$$

$$a = \sqrt{8^2 - 5^2} = \sqrt{64 - 25} = \sqrt{39}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \boldsymbol{\frac{5}{8}}$$

$$\cos\theta = \boldsymbol{\frac{\sqrt{39}}{8}}$$

$$\tan\theta = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \boldsymbol{\frac{5\sqrt{39}}{39}}$$

Trigonometric Functions of Any Angle

Notice that any angle of a right triangle, other than the right angle, is acute. Thus, the “SOH – CAH – TOA” definition of the trigonometric ratios refers to acute angles only. However, we can extend this definition to include all angles. This can be done by observing our right triangle within the Cartesian Coordinate System.

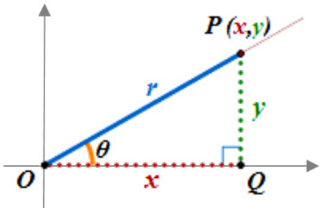


Figure 2.3

Let triangle OPQ with $\angle Q = 90°$ be placed in the coordinate system so that O coincides with the origin, Q lies on the positive part of the x-axis, and P lies in the first quadrant. See *Figure 2.3*. Let (x, y) be the coordinates of the point P, and let θ be the measurement of $\angle QOP$. This way, angle θ is in standard position and the triangle OPQ is obtained by **projecting** point P perpendicularly onto the x-axis. Thus in this setting, the position of point P actually determines both the angle θ and the $\triangle OPQ$. Observe that the coordinates of point P ($\boldsymbol{x}$ and $\boldsymbol{y}$) really represent the length of the **adjacent** and the **opposite** side, correspondingly. Since the length of the **hypotenuse** represents the distance of the point P from the origin, it is often denoted by $\boldsymbol{r}$ (from *radius*.)

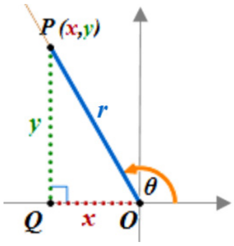


Figure 2.4

By rotating the radius $\boldsymbol{r}$ and projecting the point P perpendicularly onto x-axis (follow the green dotted line from P to Q in *Figure 2.4*), we can obtain a right triangle corresponding to any angle θ, not only an acute angle. Since the coordinates of a point in a plane can be negative, to establish a correcpondence between the coordinates x and y of the point P, and the distances OQ and QP, it is convenient to think of **directed distances** rather than just distances. Distance becomes directed if we assign a sign to it. So, lets assign a positive sign to horizontal or vertical distances that follow the directions of the corresponding number lines, and a negative sign otherwise. For example, the directed distance $OQ = x$ in *Figure 2.3* is positive because the direction from O to Q follows the order on the x-axis while the directed distance $OQ = x$ in *Figure 2.4* is negative because the direction from O to Q is against the order on the x-axis. Likewise, the directed distance $QP = y$ is positive for angles in the first and second quadrant (as in *Figure 2.3* and *2.4*), and it is negative for angles in the third and fourth quadrant (convince yourself by drawing a diagram).

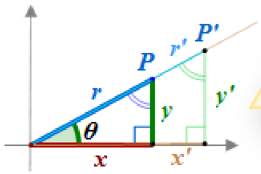
Definition 2.2 ► Let $P(\boldsymbol{x},\boldsymbol{y})$ be any point, different than the origin, on the terminal side of an angle $\boldsymbol{\theta}$ in standard position. Also, let $\boldsymbol{r}=\sqrt{\boldsymbol{x}^2+\boldsymbol{y}^2}$ be the distance of the point P from the origin. We define

$$\sin\boldsymbol{\theta}=\frac{\boldsymbol{y}}{\boldsymbol{r}},\qquad \cos\boldsymbol{\theta}=\frac{\boldsymbol{x}}{\boldsymbol{r}},\qquad \tan\boldsymbol{\theta}=\frac{\boldsymbol{y}}{\boldsymbol{x}}\ \ (\text{for } x\neq 0)$$

$$\csc\boldsymbol{\theta}=\frac{\boldsymbol{r}}{\boldsymbol{y}}\ (\text{for } y\neq 0),\qquad \sec\boldsymbol{\theta}=\frac{\boldsymbol{r}}{\boldsymbol{x}}\ (\text{for } x\neq 0),\qquad \cot\boldsymbol{\theta}=\frac{\boldsymbol{x}}{\boldsymbol{y}}\ (\text{for } y\neq 0)$$

Observations:

- For acute angles, *Definition 2.2* agrees with the "**SOH – CAH – TOA**" *Definition 2.1*.



$$\frac{y}{r}=\frac{y'}{r'}$$

$$\frac{x}{r}=\frac{x'}{r'}$$

$$\frac{y}{x}=\frac{y'}{x'}$$

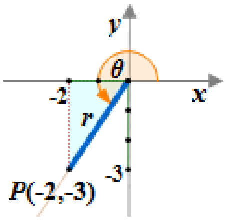
- Proportionality of similar triangles guarantees that each point of the same terminal ray defines the same trigonometric ratio. This means that the above definition assigns a unique value to each trigonometric ratio for any given angle regardless of the point chosen on the terminal side of this angle. Thus, the above trigonometric ratios are in fact **functions of any real angle** and these functions are properly defined in terms of x, y, and r.
- Since $r>0$, the first two trigonometric functions, **sine** and **cosine**, are defined for any real angle θ.
- The reamining trigonometric functions, **tangent**, **cosecant**, **secant**, and **cotangent**, are defined for all real angles θ, except for angles that create a zero in the ratio's denominator. For example, tangent is defined for all angles except those with terminal sides on the y-axis. This is because the x-coordinate of any point on the y-axis equals zero, which cannot be used to create the ratio $\frac{y}{x}$. Thus, tangent is a function of all real angles, except for 90°, 270°, and so on (generally, except for angles of the form $\boldsymbol{90°}+\boldsymbol{k}\cdot\boldsymbol{180°}$, where $\boldsymbol{k}$ is an integer).

Example 3 ► **Evaluating Trigonometric Functions of any Angle in Standard Position**

Find the exact value of the six trigonometric functions of an angle θ in standard position whose terminal side contains the point

a. $P(-2,-3)$ **b.** $P(0,1)$

Solution ► **a.** To ilustrate the situation, lets sketch the least positive angle θ in standard position with the point $P(-2,-3)$ on its terminal side.



To find values of the trigonometric functions, first, we will determine the length of r:

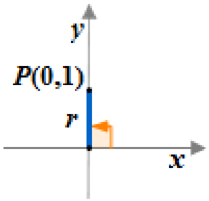
$$r=\sqrt{(-2)^2+(-3)^2}=\sqrt{4+9}=\sqrt{13}$$

Now, we can state the exact values of the trigonometric functions:

$$\sin\theta=\frac{y}{r}=\frac{-3}{\sqrt{13}}=\frac{\boldsymbol{-3\sqrt{13}}}{\boldsymbol{13}},\qquad \csc\theta=\frac{r}{y}=\frac{\sqrt{13}}{-3}=-\frac{\boldsymbol{\sqrt{13}}}{\boldsymbol{3}}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} = \frac{\mathbf{-2\sqrt{13}}}{\mathbf{13}} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{13}}{-2} = -\frac{\mathbf{\sqrt{13}}}{\mathbf{2}},$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{-2} = \frac{\mathbf{3}}{\mathbf{2}}, \qquad \cot\theta = \frac{x}{y} = \frac{-2}{-3} = \frac{\mathbf{2}}{\mathbf{3}}.$$



b. Since $x = 0,\ y = 1,\ r = \sqrt{0^2 + 1^2} = 1$, then

$$\sin\theta = \frac{y}{r} = \frac{1}{1} = \mathbf{1}, \qquad \csc\theta = \frac{r}{y} = \frac{1}{1} = \mathbf{1},$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = \mathbf{0}, \qquad \sec\theta = \frac{r}{x} = \frac{1}{0} = \textbf{undefined},$$

$$\tan\theta = \frac{y}{x} = \frac{1}{0} = \textbf{undefined}, \qquad \cot\theta = \frac{x}{y} = \frac{0}{1} = \mathbf{0}.$$

we can't divide by zero!

Notice that the measure of the least positive angle θ in standard position with the point $P(0,1)$ on its terminal side is 90°. Therefore, we have

$$\sin 90° = 1\,, \qquad \cos 90° = 0, \qquad \tan 90° = \text{undefined}$$
$$\csc 90° = 1, \qquad \sec 90° = DNE, \qquad \cot 90° = 0$$

The values of trigonometric functions of other commonly used quadrantal angles, such as $0°, 180°, 270°$, and $360°$, can be found similarly as in *Example 3b*. These values for the primary functions are summarized in the table below. The reader is encouraged to extend the table for the reciprocal functions.

Table 2.1 **Function Values of Quadrantal Angles**

θ / function	**0°**	**90°**	**180°**	**270°**	**360°**
sin θ	0	1	0	-1	0
cos θ	1	0	-1	0	1
tan θ	0	undefined	0	undefined	0

Example 4 ► Evaluating Trigonometric Functions Using Basic Identities

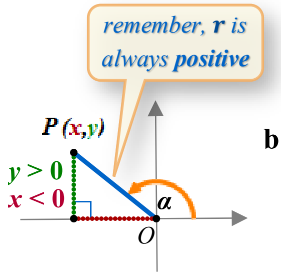
Knowing that $\cos\alpha = -\frac{3}{4}$ and the angle α is in quadrant II, find

a. $\sin\alpha$ **b.** $\tan\alpha$

Solution ►

a. We know that $\cos\alpha = -\frac{3}{4} = \frac{x}{r}$. Hence, the terminal side of angle $\alpha \in QII$ contains a point $P(x, y)$ satisfying the condition $\frac{x}{r} = -\frac{3}{4}$. Since r must be positive, we will assign $x = -3$ and $r = 4$, to model the situation. Using the Pythagorean equation and the fact that the y-coordinate of any point in the second quadrant is positive, we determine the corresponding y-value to be

$$y = \sqrt{r^2 - x^2} = \sqrt{4^2 - (-3)^2} = \sqrt{16 - 9} = \sqrt{7}.$$



Now, we are ready to use *Definition 2.2* to state the sine value of angle α:

$$\sin\alpha = \frac{y}{r} = \frac{\sqrt{7}}{4}.$$

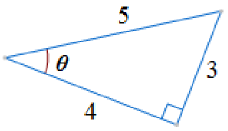
b. To find the value of $\tan\alpha$, since we already know the values of x, y, and r, we can again use *Definition 2.2*:

$$\tan\alpha = \frac{y}{x} = \frac{\sqrt{7}}{-3} = -\frac{\sqrt{7}}{3}.$$

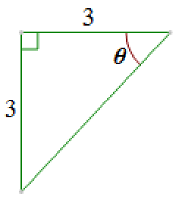
T.2 Exercises

*Find the **exact values** of the six trigonometric functions for the indicated angle θ. Rationalize denominators when applicable.*

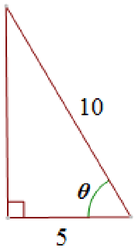
1.



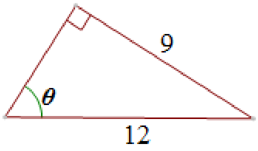
2.



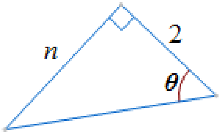
3.



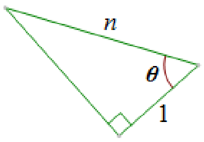
4.



5.



6.



Sketch an angle θ in standard position such that θ has the least positive measure, and the given point is on the terminal side of θ. Then find the values of the three primary trigonometric functions for each angle. Rationalize denominators when applicable.

7. $(-3,4)$
8. $(-4,-3)$
9. $(5,-12)$
10. $(0,3)$
11. $(-4,0)$
12. $(1,\sqrt{3})$
13. $(3,5)$
14. $(0,-8)$
15. $(-2\sqrt{3},-2)$
16. $(5,0)$

17. If the terminal side of an angle θ is in quadrant III, what is the sign of each of the trigonometric function values of θ?

*Suppose that the point (x,y) is in the indicated quadrant. Decide whether the given ratio is **positive** or **negative**.*

18. $QI, \frac{y}{x}$
19. $QII, \frac{y}{x}$
20. $QII, \frac{y}{r}$
21. $QIII, \frac{x}{r}$
22. $QIV, \frac{y}{x}$
23. $QIII, \frac{y}{x}$
24. $QIV, \frac{y}{r}$
25. $QI, \frac{y}{r}$
26. $QIV, \frac{x}{r}$
27. $QII, \frac{x}{r}$

Use the definition of trigonometric functions in terms of ***x***, ***y***, *and* ***r*** *to determine each value. If it is undefined, say so.*

28. $\sin 90°$
29. $\cos 0°$
30. $\tan 180°$
31. $\cos 180°$
32. $\cot 270°$
33. $\cos 270°$
34. $\csc 270°$
35. $\sec 90°$
36. $\sin 0°$
37. $\cot 90°$

Determine the values of the remaining two primary trigonometric functions of the angle satisfying the given conditions. Rationalize denominators when applicable.

38. $\sin\alpha = \frac{\sqrt{2}}{4};\ \alpha \in Q\text{II}$
39. $\sin\beta = -\frac{2}{3};\ \beta \in Q\text{III}$
40. $\cos\theta = \frac{2}{5};\ \theta \in Q\text{IV}$

T3 Evaluation of Trigonometric Functions

In the previous section, we defined sine, cosine, tangent, secant, cosecant, and cotangent as functions of real angles. In this section, we will take interest in finding values of these functions for angles $\theta \in [0°,\ 360°)$. As shown before, one can find exact values of trigonometric functions of an angle θ with the aid of a right triangle with the acute angle θ and given side lengths, or by using coordinates of a given point on the terminal side of the angle θ in standard position. What if such data is not given? Then, one could consider approximating trigonometric function values by measuring sides of a right triangle with the desired angle θ and calculating corresponding ratios. However, this could easily prove to be a cumbersome process, with inaccurate results. Luckily, we can rely on calculators, which are programmed to return approximated values of the three primary trigonometric functions for any angle.

Attention: *In this section, any calculator instruction will refer to scientific calculators.*

Example 1 ► **Evaluating Trigonometric Functions Using a Calculator**

Find each function value up to four decimal places.

When evaluating functions of angles in degrees, the calculator must be set to the **degree mode**.

a. $\sin 39°12'\,10"$ **b.** $\tan 102.6°$

Solution ► **a.** Before entering the expression into the calculator, we need to check if the calculator is in degree mode by pressing the DRG key until DEG appears at the top of the screen. Now we can enter $\sin 39°12'10''$ by pressing

sin 39 *D°M'S* 12 *D°M'S* 10 =

Thus $\sin 39°12'10'' \approx \mathbf{0.6321}$ when rounded to four decimal places.

b. When evaluating trigonometric functions of angles in decimal degrees, it is not necessary to write the degree (°) sign when in degree mode. We simply key in

tan 102.6 =

to obtain $\tan 102.6° \approx \mathbf{-4.4737}$ when rounded to four decimal places.

Special Angles

It has already been discussed how to find the **exact values** of trigonometric functions of **quadrantal angles** using the definitions in terms of x, y, and r. *See Section T2, Example 3b, and Table 2.1.*

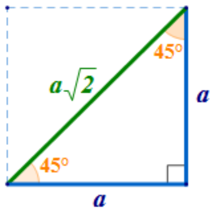


Figure 3.1

Are there any other angles for which the trigonometric functions can be evaluated exactly? Yes, we can find the exact values of trigonometric functions of any angle that can be modelled by a right triangle with known sides. For example, angles such as 30°, 45°, or 60° can be modeled by half of a square or half of an equilateral triangle. In each triangle, the relations between the lengths of sides are easy to establish.

In the case of half a square *(see Figure 3.1)*, we obtain a right triangle with two acute angles of $\mathbf{45°}$, and two equal sides of certain length $\boldsymbol{a}$.

Hence, by The Pythagorean Theorem, the diagonal $d = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \boldsymbol{a\sqrt{2}}$.

Summary: The sides of any $\mathbf{45° - 45° - 90°}$ triangle are in the relation $\boldsymbol{a - a - a\sqrt{2}}$.

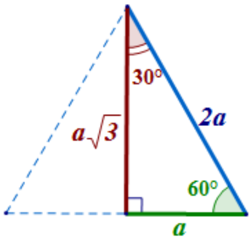


Figure 3.2

By dividing an equilateral triangle *(see Figure 3.1)* along its height, we obtain a right triangle with acute angles of $\mathbf{30°}$ and $\mathbf{60°}$. If the length of the side of the original triangle is denoted by $\boldsymbol{2a}$, then the length of half a side is $\boldsymbol{a}$, and the length of the height can be calculated by applying The Pythagorean Theorem, $h = \sqrt{(2a)^2 - a^2} = \sqrt{3a^2} = \boldsymbol{a\sqrt{3}}$.

Summary: The sides of any $\mathbf{30° - 60° - 90°}$ triangle are in the relation $\boldsymbol{a - 2a - a\sqrt{3}}$.

Since the trigonometric ratios do not depend on the size of a triangle, for simplicity, we can assume that $a = 1$ and work with the following **special triangles**:

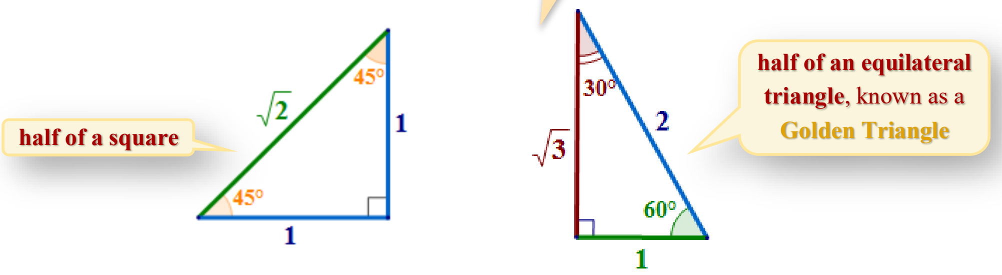


Figure 3.3

Special angles such as $\mathbf{30°}, \mathbf{45°}$, and $\mathbf{60°}$ are frequently seen in applications. We will often refer to the exact values of trigonometric functions of these angles. Special triangles give us a tool for finding those values.

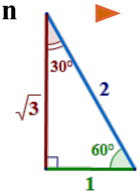
Advice: *Make sure that you can* ***recreate the special triangles*** *by taking half of a square or half of an equilateral triangle anytime you wish to* ***recall the relations*** *between their sides.*

Example 2 ► **Finding Exact Values of Trigonometric Functions of Special Angles**

Find the *exact* value of each expression.

a. $\cos 60°$ **b.** $\tan 30°$ **c.** $\sin 45°$ **d.** $\tan 45°$

Solution ►

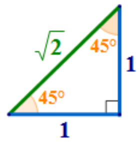


a. Refer to the $30° - 60° - 90°$ triangle and follow the SOH-CAH-TOA definition of sine:

$$\cos 60° = \frac{adj.}{hyp.} = \frac{1}{2}$$

b. Refer to the same triangle as above:

$$\tan 30° = \frac{opp.}{adj.} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$



c. Refer to the $45° - 45° - 90°$ triangle:

$$\sin 45° = \frac{opp.}{hyp.} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

d. Refer to the $45° - 45° - 90°$ triangle:

$$\tan 45° = \frac{opp.}{adj.} = \frac{1}{1} = 1$$

The exact values of trigonometric functions of special angles are summarized in the table below.

Table 3.1 Function Values of Special Angles

θ / function	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Observations:

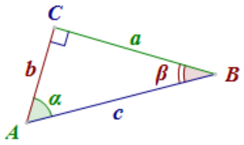


Figure 3.4

- Notice that $\sin 30° = \cos 60°$, $\sin 60° = \cos 30°$, and $\sin 45° = \cos 45°$. Is there any general rule to explain this fact? Lets look at a right triangle with acute angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (*see Figure 3.4*). Since the sum of angles in any triangle is 180° and $\angle C = 90°$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} = 90°$, therefore they are **complementary angles**. From the definition, we have $\sin\boldsymbol{\alpha} = \frac{\boldsymbol{a}}{\boldsymbol{b}} = \cos\boldsymbol{\beta}$. Since angle $\boldsymbol{\alpha}$ was chosen arbitrarily, this rule applies to any pair of acute complementary angles. It happens that this rule actually applies to all complementary angles. So we have the following claim:

$\mathbf{sin}\ \boldsymbol{\alpha} = \mathbf{cos}\ (\mathbf{90°} - \boldsymbol{\alpha})$
$\mathbf{sec}\ \boldsymbol{\alpha} = \mathbf{csc}\ (\mathbf{90°} - \boldsymbol{\alpha})$
$\mathbf{tan}\ \boldsymbol{\alpha} = \mathbf{cot}\ (\mathbf{90°} - \boldsymbol{\alpha})$

The **cofunctions** (like sine and cosine, secant and cosecant, or tangent and cotangent) of **complementary** angles are equal.

Example 3 ► Using the Cofunction Relationship

Rewrite $\cos 75°$ in terms of the cofunction of the complementary angle.

Solution ► Since the complement of 75° is $90° - 75° = 15°$, then $\cos 75° = \mathbf{sin\ 15°}$.

Reference Angles

Can we determine exact values of trigonometric functions of nonquadrantal angles that are larger than 90°?

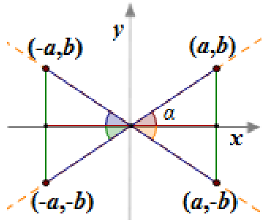
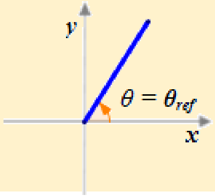


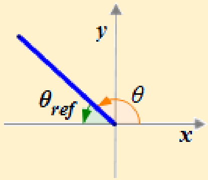
Figure 3.5

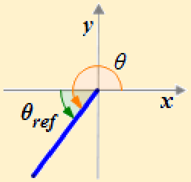
Assume that point (a, b) lies on the terminal side of acute angle α. By *Definition 2.2*, the values of trigonometric functions of angles with terminals containing points $(-a, b), (-a, -b)$, and $(a, -b)$ are the same as the values of corresponding functions of the angle α, except for their signs.

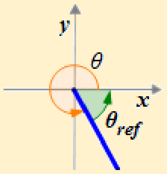
Therefore, to find the value of a trigonometric function of any angle θ, it is enough to evaluate this function at the corresponding acute angle θ_{ref}, called the **reference angle**, and apply the sign appropriate to the quadrant of the terminal side of θ.

Definition 3.1 ► Let θ be an angle in standard position. The acute angle $\boldsymbol{\theta_{ref}}$ formed by the terminal side of the angle θ and the x-axis is called the **reference angle**.









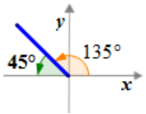
Attention: *Think of a **reference angle** as the smallest rotation of the terminal arm required to line it up with the **x-axis**.*

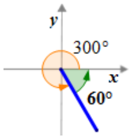
Example 4 ► **Finding the Reference Angle**

Find the **reference angle** for each of the given angles.

a. 40° **b.** 135° **c.** 210° **d.** 300°

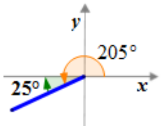
Solution ► **a.** Since **40°** $\in QI$, this is already the reference angle.





b. Since 135° $\in QII$, the reference angle equals 180° − 135° = **45°**.

c. Since 205° $\in QIII$, the reference angle equals 205° − 180° = **25°**.



d. Since 300° $\in QIV$, the reference angle equals 360° − 300° = **60°**.

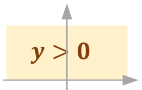
CAST Rule

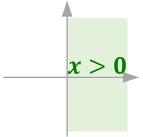
Using the x, y, r definition of trigonometric functions, we can determine and summarize the signs of those functions in each of the quadrants.

Since $\sin\theta = \frac{y}{r}$ and r is positive, then the sign of the sine ratio is the same as the sign of the y-value. This means that the values of sine are positive only in quadrants where y is positive, thus in QI and QII.

Since $\cos\theta = \frac{x}{r}$ and r is positive, then the sign of the cosine ratio is the same as the sign of the x-value. This means that the values of cosine are positive only in quadrants where x is positive, thus in QI and QIV.

Since $\tan\theta = \frac{y}{x}$, then the values of the tangent ratio are positive only in quadrants where both x and y have the same signs, thus in QI and QIII.





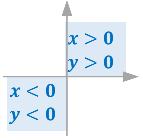


Table 3.2 Signs of Trigonometric Functions in Quadrants

$\theta \in$ / function	QI	QII	QIII	QIV
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−

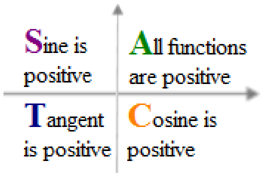


Figure 3.6

Since we will be making frequent decisions about signs of trigonometric function values, it is convenient to have an acronym helping us memorizing these signs in different quadrants. The first letters of the names of functions that are positive in particular quadrants, starting from the fourth quadrant and going counterclockwise, spells **CAST**, which is very helpful when working with trigonometric functions of any angles.

Example 5 ► **Identifying the Quadrant of an Angle**

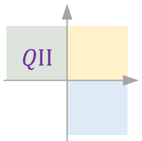
Identify the quadrant or quadrants for each angle satisfying the given conditions.

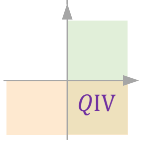
a. $\sin\theta > 0$; $\tan\theta < 0$ **b.** $\cos\theta > 0$; $\sin\theta < 0$

Solution ►

a. Using **CAST**, we have $\sin\theta > 0$ in QI(**A**ll) and QII(**S**ine) and $\tan\theta < 0$ in QII and QIV. Therefore both conditions are met only in **quadrant II**.

b. $\cos\theta > 0$ in QI(**A**ll) and QIV(**C**osine) and $\sin\theta < 0$ in QIII and QIV. Therefore both conditions are met only in **quadrant IV**.





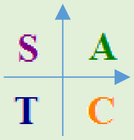
Example 6 ► **Identifying Signs of Trigonometric Functions of Any Angle**

Using the **CAST rule**, identify the sign of each function value.

a. $\cos 150°$ **b.** $\tan 225°$

Solution ► **a.** Since $150° \in QII$ and cosine is negative in QII, then $\cos 150°$ is **negative**.

b. Since $225° \in QIII$ and tangent is positive in $QIII$, then $\tan 225°$ is **positive**.



To find the exact value of a trigonometric function $\boldsymbol{T}$ of an angle $\boldsymbol{\theta}$ with the reference angle $\boldsymbol{\theta_{ref}}$ being a special angle, we follow the rule:

$$\boldsymbol{T(\theta) = \pm T(\theta_{ref})},$$

where the final sign is determined according to the quadrant of angle θ and the **CAST** rule.

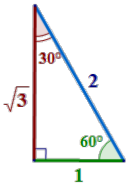
Example 7 ► **Finding Exact Function Values Using Reference Angles**

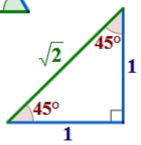
Find the exact values of the following expressions.

a. $\sin 240°$ **b.** $\cos 315°$

Solution ► **a.** The reference angle of $240°$ is $240° - 180° = 60°$. Since $240° \in QIII$ and sine in the third quadrant is **negative**, we have

$$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{\mathbf{2}}$$





b. The reference angle of $315°$ is $360° - 315° = 45°$. Since $315° \in QIV$ and cosine in the fourth quadrant is **positive**, we have

$$\cos 315° = \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\mathbf{2}}$$

Finding Special Angles in Various Quadrants when Given Trigonometric Function Value

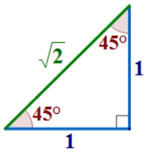
Now that it has been shown how to find exact values of trigonometric functions of angles that have a reference angle of one of the special angles ($\mathbf{30°}, \mathbf{45°}$, or $\mathbf{60°}$), we can work at reversing this process. Familiarity with values of trigonometric functions of the special angles, in combination with the ideas of reference angles and quadrantal sign analysis, should help us in solving equations of the type $\boldsymbol{T(\theta) = exact\ value}$, where $\boldsymbol{T}$ represents any trigonometric function.

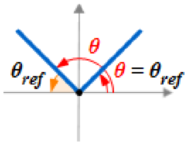
Example 8 ► **Finding Angles with a Given Exact Function Value, in Various Quadrants**

Find all angles θ satisfying the following conditions.

a. $\sin\theta = \frac{\sqrt{2}}{2};\ \theta \in [0°, 180°)$ **b.** $\cos\theta = -\frac{1}{2};\ \theta \in [0°, 360°)$

Solution ► **a.** Refering to the half of a square triangle, we recognize that $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ represents the ratio of sine of $45°$. Thus, the reference angle $\theta_{ref} = 45°$. Moreover, we are searching for an angle θ from the interval $[0°, 180°)$ and we know that $\sin\theta > 0$. Therefore, θ must lie in the first or second quadrant and have the reference angle of $45°$. Each quadrant gives us one solution, as shown in the figure on the right.

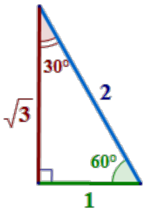




If θ is in the first quadrant, then $\theta = \theta_{ref} = 45°$.
If θ is in the second quadrant, then $\theta = 180° - 45° = 135°$.

So the solution set of the above problem is $\{\mathbf{45°, 135°}\}$.

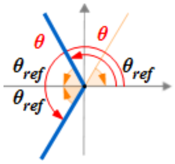
here we can disregard the sign of the given value as we are interested in the reference angle only



b. Refering to the half of an equlateral triangle, we recognize that $\frac{1}{2}$ represents the ratio of cosine of 60°. Thus, the reference angle $\theta_{ref} = 60°$. We are searching for an angle θ from the interval $[0°, 360°)$ and we know that $\cos\theta < 0$. Therefore, θ must lie in the second or third quadrant and have the reference angle of 60°.

If θ is in the second quadrant, then $\theta = 180° - 60° = 120°$.
If θ is in the third quadrant, then $\theta = 180° + 60° = 240°$.

So the solution set of the above problem is $\{\mathbf{120°, 240°}\}$.



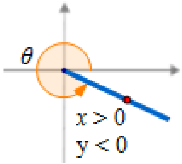
Finding Other Trigonometric Function Values

Example 9 ► **Finding Other Function Values Using a Known Value, Quadrant Analysis, and the x, y, r Definition of Trigonometric Ratios**

Find values of the remaining primary trigonometric functions of the angle satisfying the given conditions.

a. $\sin\theta = -\frac{7}{13}$; $\theta \in QIV$ **b.** $\tan\theta = \frac{15}{8}$; $\theta \in$ QIII

Solution ►



a. We know that $\sin\theta = -\frac{7}{13} = \frac{y}{r}$. Hence, the terminal side of angle $\theta \in QIV$ contains a point $P(x, y)$ satisfying the condition $\frac{y}{r} = -\frac{7}{13}$. Since r must be positive, we will assign $y = -7$ and $r = 13$, to model the situation. Using the Pythagorean equation and the fact that the x-coordinate of any point in the fourth quadrant is positive, we determine the corresponding x-value to be

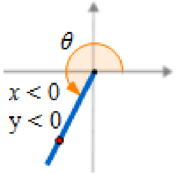
$$x = \sqrt{r^2 - y^2} = \sqrt{13^2 - (-7)^2} = \sqrt{169 - 49} = \sqrt{120} = 2\sqrt{30}.$$

Now, we are ready to state the remaining function values of angle θ:

$$\cos\theta = \frac{x}{r} = \mathbf{\frac{2\sqrt{30}}{13}}$$

and

$$\tan\theta = \frac{y}{x} = \frac{-7}{2\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = \mathbf{\frac{-7\sqrt{30}}{60}}.$$



b. We know that $\tan\theta = \frac{15}{8} = \frac{y}{x}$. Similarly as above, we would like to determine x, y, and r values that would model the situation. Since angle $\theta \in QIII$, both x and y values must be negative. So we assign $y = -15$ and $x = -8$. Therefore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-15)^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Now, we are ready to state the remaining function values of angle θ:

$$\sin\theta = \frac{y}{r} = \frac{\mathbf{-15}}{\mathbf{17}}$$

and

$$\cos\theta = \frac{x}{r} = \frac{\mathbf{-8}}{\mathbf{17}}.$$

T.3 Exercises

*Use a calculator to **approximate** each value to **four** decimal places.*

1. $\sin 36°52'\,05''$

2. $\tan 57.125°$

3. $\cos 204°25'$

*Give the **exact** function value, **without** the aid of a calculator. Rationalize denominators when applicable.*

4. $\cos 30°$

5. $\sin 45°$

6. $\tan 60°$

7. $\sin 60°$

8. $\tan 30°$

9. $\cos 60°$

10. $\sin 30°$

11. $\tan 45°$

*Give the equivalent expression using the **cofunction** relationship.*

12. $\cos 50°$

13. $\sin 22.5°$

14. $\sin 10°$

*For each angle, find the **reference angle**.*

15. $98°$

16. $212°$

17. $13°$

18. $297°$

19. $186°$

*Identify the **quadrant** or **quadrants** for each angle satisfying the given conditions.*

20. $\cos\alpha > 0$

21. $\sin\beta < 0$

22. $\tan\gamma > 0$

23. $\sin\theta > 0;\ \cos\theta < 0$

24. $\cos\alpha < 0;\ \tan\alpha > 0$

25. $\sin\alpha < 0;\ \tan\alpha < 0$

*Identify the **sign** of each function value by quadrantal analysis.*

26. $\cos 74°$

27. $\sin 245°$

28. $\tan 129°$

29. $\sin 183°$

30. $\tan 298°$

31. $\cos 317°$

32. $\sin 285°$

33. $\tan 215°$

*Using reference angles, quadrantal analysis, and special triangles, find the **exact values** of the expressions. Rationalize denominators when applicable.*

34. $\cos 225°$

35. $\sin 120°$

36. $\tan 150°$

37. $\sin 150°$

38. $\tan 240°$

39. $\cos 210°$

40. $\sin 330°$

41. $\tan 225°$

Find all values of $\theta \in [0°, 360°)$ *satisfying the given condition.*

42. $\sin\theta = -\frac{1}{2}$

43. $\cos\theta = \frac{1}{2}$

44. $\tan\theta = -1$

45. $\sin\theta = \frac{\sqrt{3}}{2}$

46. $\tan\theta = \sqrt{3}$

47. $\cos\theta = -\frac{\sqrt{2}}{2}$

48. $\sin\theta = 0$

49. $\tan\theta = -\frac{\sqrt{3}}{3}$

Find values of the remaining primary trigonometric functions of the angle satisfying the given conditions.

50. $\sin\theta = \frac{\sqrt{5}}{7}$; $\theta \in Q\text{II}$

51. $\cos\alpha = \frac{3}{5}$; $\alpha \in Q\text{IV}$

52. $\tan\beta = \sqrt{3}$; $\beta \in Q\text{III}$

T4 Applications of Right Angle Trigonometry

Solving Right Triangles

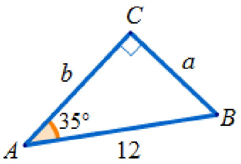
Geometry of right triangles has many applications in the real world. It is often used by carpenters, surveyors, engineers, navigators, scientists, astronomers, etc. Since many application problems can be modelled by a right triangle and trigonometric ratios allow us to find different parts of a right triangle, it is essential that we learn how to apply trigonometry to solve such triangles first.

Definition 4.1 ► To **solve a triangle** means to find the measures of all the unknown **sides** and **angles** of the triangle.

Example 1 ► **Solving a Right Triangle Given an Angle and a Side**

Given the information, solve triangle ABC, assuming that $\angle C = 90°$.

a.



b. $\angle B = 11.4°,\ b = 6\ cm$

Solution ► **a.** To find the length a, we want to relate it to the given length of 12 and the angle of 35°. Since a is opposite angle 35° and 12 is the length of the hypotenuse, we can use the ratio of sine:

$$\frac{a}{12} = \sin 35°$$

Then, after multiplying by 12, we have

$$a = 12 \sin 35° \simeq \mathbf{6.9}$$

round **lengths** to **one decimal place**

Attention: To be more accurate, if possible, **use the given data** rather than the previously calculated ones, which are most likely already rounded.

Since we already have the value of a, the length b can be determined in two ways: by applying the Pythagorean Theorem, or by using the cosine ratio. For better accuracy, we will apply the cosine ratio:

$$\frac{b}{12} = \cos 35°$$

which gives

$$b = 12 \cos 35° \simeq \mathbf{9.8}$$

Finally, since the two acute angles are complementary, $\angle B = 90° - 35° = \mathbf{55}°$.

We have found the three missing measurements, $a \simeq 6.9$, $b \simeq 9.8$, and $\angle B = 55°$, so the triangle is solved.

b. To visualize the situation, let's sketch a right triangle with $\angle B = 11.4°$ and $b = 6$ (see *Figure 1*). To find side a, we we would like to set up an equation that relates 6, a, and 11.4°. Since $b = 6$ is the opposite and a is the adjacent with respect to $\angle B = 11.4°$, we will use the ratio of tangent:

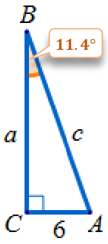


Figure 1

$$\tan 11.4° = \frac{6}{a}$$

To solve for a, we may want to multiply both sides of the equation by a and divide by $\tan 11.4°$. Observe that this will cause a and $\tan 11.4°$ to interchange (swap) their positions. So, we obtain

$$a = \frac{6}{\tan 11.4°} \simeq \mathbf{29.8}$$

To find side c, we will set up an equation that relates 6, c, and 11.4°. Since $b = 6$ is the opposite to $\angle B = 11.4°$ and c is the hypothenuse, the ratio of sine applies. So, we have

$$\sin 11.4° = \frac{6}{c}$$

Similarly as before, to solve for c, we can simply interchange the position of $\sin 11.4°$ and c to obtain

$$c = \frac{6}{\sin 11.4°} \simeq \mathbf{30.4}$$

Finally, $\angle A = 90° - 11.4° = \mathbf{78.6°}$, which completes the solution.

In summary, $\angle A = 78.6°$, $a \simeq 29.8$, and $c \simeq 30.4$.

Observation: Notice that after approximated length a was found, we could have used the Pythagorean Theoreom to find length c. However, this could decrease the accuracy of the result. For this reason, it is advised that we use the given rather than approximated data, if possible.

Finding an Angle Given a Trigonometric Function Value

So far we have been evaluating trigonometric functions for a given angle. Now, what if we wish to reverse this process and try to recover an angle that corresponds to a given trigonometric function value?

Example 2 ► **Finding an Angle Given a Trigonometric Function Value**

Find an angle θ, satisfying the given equation. *Round to one decimal place, if needed.*

a. $\sin\theta = 0.7508$ **b.** $\cos\theta = -0.5$

Solution ► **a.** Since 0.7508 is not a special value, we will not be able to find θ by relating the equation to a special triangle as we did in *Section T3, Example 8*. This time, we will need to rely on a calculator. To find θ, we want to "undo" the sine. The function that can "undo" the sine is called **arcsine**, or **inverse sine**, and it is often abbreviated by $\mathbf{sin^{-1}}$. By applying the $\sin^{-1}$ to both sides of the equation

$$\sin\theta = 0.7508,$$

we have

$$\cancel{\sin^{-1}}(\cancel{\sin}\,\theta) = \sin^{-1}(0.7508)$$

Since $\sin^{-1}$ "undoes" the sine function, we obtain

$$\theta = \sin^{-1} 0.7508 \simeq \mathbf{48.7}°$$

round **angles** to **one decimal place**

On most calculators, to find this value, we follow the sequence of keys:

2nd or **INV** or **Shift**, **SIN**, 0.7508, **ENTER** or **=**

b. In this example, the absolute value of cosine is a special value. This means that θ can be found by referring to the **golden triangle** properties and the **CAST** rule of signs as in *Section T3, Example 8b*. The other way of finding θ is via a calculator

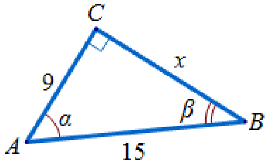
$$\theta = \cos^{-1}(-0.5) = \mathbf{120}°$$

> ***Note:*** *Calculators are programed to return*
> $\mathbf{sin^{-1}}$ *and* $\mathbf{tan^{-1}}$ *as angles from the interval* $[-\mathbf{90}°, \mathbf{90}°]$ *and*
> $\mathbf{cos^{-1}}$ *as angles from the interval* $[\mathbf{0}°, \mathbf{180}°]$.
> *That implies that when looking for an obtuse angle, it is easier to work with* $\mathbf{cos^{-1}}$, *if possible, as our calculator will return the actual angle. When using* $\mathbf{sin^{-1}}$ *or* $\mathbf{tan^{-1}}$, *we might need to search for a corresponding angle in the second quadrant on our own.*

More on Solving Right Triangles

Example 3 ► **Solving a Right Triangle Given Two Sides**

Solve the triangle.



Solution ► Since $\triangle ABC$ is a right triangle, to find the length x, we can use the Pythagorean Theorem.

$$x^2 + 9^2 = 15^2$$

so

$$\boldsymbol{x} = \sqrt{225 - 81} = \sqrt{144} = \mathbf{12}$$

To find the angle α, we can relate either $x = 12$, 9, and α, or 12, 15, and α. We will use the second triple and the ratio of sine. Thus, we have

$$\sin\alpha = \frac{12}{15},$$

therefore

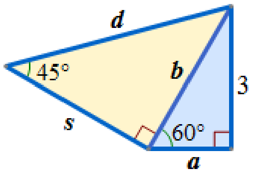
$$\boldsymbol{\alpha} = \sin^{-1}\frac{12}{15} \simeq \mathbf{53.1}°$$

Finally, $\boldsymbol{\beta} = 90° - \alpha \simeq 90° - 53.1° = \mathbf{36.9°}$.

In summary, $\alpha = 53.1°$, $\beta \simeq 36.9°$, and $x = 12$.

Example 4 ► **Using Relationships Between Sides of Special Triangles**

Find the **exact** value of each unknown in the figure.



Solution ► First, consider the blue right triangle. Since one of the acute angles is 60°, the other must be 30°. Thus the blue triangle represents half of an equilateral triangle with the side $\boldsymbol{b}$ and the height of 3 units. Using the relation $h = a\sqrt{3}$ between the height h and half a side a of an equilateral triangle, we obtain

$$\boldsymbol{a}\sqrt{3} = 3,$$

which gives us $\boldsymbol{a} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \boldsymbol{\sqrt{3}}$. Consequently, $\boldsymbol{b} = 2\boldsymbol{a} = \boldsymbol{2\sqrt{3}}$.

Now, considering the yellow right triangle, we observe that both acute angles are equal to 45° and therefore the triangle represents half of a square with the side $\boldsymbol{s} = \boldsymbol{b} = \boldsymbol{2\sqrt{3}}$.

Finally, using the relation between the diagonal and a side of a square, we have

$$\boldsymbol{d} = \boldsymbol{s}\sqrt{2} = 2\sqrt{3}\sqrt{2} = \boldsymbol{2\sqrt{6}}.$$

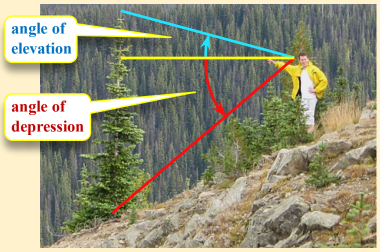
Angles of Elevation or Depression in Applications

The method of solving right triangles is widely adopted in solving many applied problems. One of the critical steps in the solution process is sketching a triangle that models the situation, and labeling the parts of this triangle correctly.

In trigonometry, many applied problems refer to angles of **elevation** or **depression**, or include some navigation terminology, such as **direction** or **bearing**.

Definition 4.2 ► **Angle of elevation** (or **inclination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **above** the horizontal line.

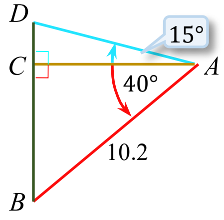
Angle of depression (or **declination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **below** the horizontal line.



Example 5 ► Applying Angles of Elevation or Depression

Find the height of the tree in the picture given next to *Definition 4.2*, assuming that the observer sees the top of the tree at an angle of elevation of 15°, the base of the tree at an angle of depression of 40°, and the distance from the base of the tree to the observer's eyes is 10.2 meters.

Solution ► First, let's draw a diagram to model the situation, label the vertices, and place the given data. Then, observe that the height of the tree BD can be obtained as the sum of distances BC and CD.



BC can be found from ⊿ABC, by using the ratio of sine of 40°.

From the equation

$$\frac{BC}{10.2} = \sin 40°,$$

we have

$$BC = 10.2 \sin 40° \simeq \mathbf{6.56}$$

To calculate the length DC, we would need to have another piece of information about ⊿ADC first. Notice that the side AC is common for the two triangles. This means that we can find it from ⊿ABC, and use it for ⊿ADC in subsequent calculations.

From the equation

$$\frac{CA}{10.2} = \cos 40°,$$

we have

$$CA = 10.2 \cos 40° \simeq 7.8137$$

since we use this result in further calculations, four decimals of accuracy is advised

Now, employing tangent of 15° in ⊿ADC, we have

$$\frac{CD}{7.8137} = \tan 15°$$

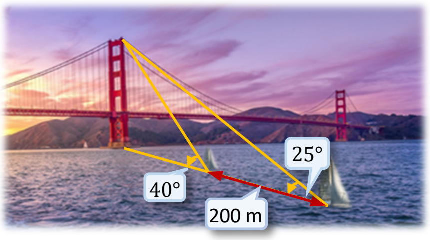
which gives us

$$CD = 7.8137 \cdot \tan 15° \simeq \mathbf{2.09}$$

Hence the height of the tree is $BC \simeq 6.56 + 2.09 = 8.65 \simeq \mathbf{8.7}$ meters.

Example 6 ► Using Two Angles of Elevation at a Given Distance to Determine the Height

When Ricky and Sonia were sailing their boat on a river, they observed the tip of a bridge tower at a 25° elevation angle. After sailing 200 meters closer to the tower, they noticed that the tip of the tower was visible at 40° elevation angle. Approximate the height of the tower to the nearest meter.



Solution ► To model the situation, let us draw the diagram and adopt the notation as in *Figure 2*. We look for height $\boldsymbol{h}$, which is a part of the two right triangles $\triangle ABC$ and $\triangle DBC$.

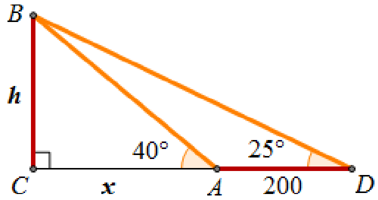


Figure 2

Since trigonometric ratios involve two sides of a triangle, and we already have length AD, a part of the side CD, it is reasonable to introduce another unknown, call it $\boldsymbol{x}$, to represent the remaining part CA. Then, applying the ratio of tangent to each of the right triangles, we produce the following system of equations:

$$\begin{cases} \dfrac{h}{x} = \tan 40° \\ \dfrac{h}{x+200} = \tan 25° \end{cases}$$

To solve the above system, we first solve each equation for h

$$\begin{cases} h \simeq 0.8391x \\ h \simeq 0.4663(x+200)\,, \end{cases}$$

and then by equating the right sides, we obtain

substitute to the top equation

$$0.8391x = 0.4663(x+200)$$

$$0.8391x - 0.4663x = 93.26$$

$$0.3728x = 93.26$$

$$x = \frac{93.26}{0.3728} \simeq 250.16$$

Therefore, $\boldsymbol{h} \simeq 0.8391 \cdot 250.16 \simeq \mathbf{210}\ \boldsymbol{m}$.

The height of the tower is approximately **210** meters.

Direction or Bearing in Applications

A large group of applied problems in trigonometry refer to **direction** or **bearing** to describe the location of an object, usually a plane or a ship. The idea comes from following the behaviour of a compass. The magnetic needle in a compass points North. Therefore, the location of an object is described as a clockwise deviation from the SOUTH-NORTH line.

There are two main ways of describing directions:

- One way is by stating the angle θ that starts from the North and opens clockwise until the line of sight of an object. For example, we can say that the point ***B*** is seen in the **direction** of 108° from the point ***A***, as in *Figure 2a*.
- Another way is by stating the acute angle formed by the South-North line and the line of sight. Such an angle starts either from the North (**N**) or the South (**S**) and opens either towards the East (**E**) or the West (**W**). For instance, the position of the point ***B*** in *Figure 2b* would be described as being at a **bearing** of **S72°E** (*read:* South 72° towards the East) from the point ***A***.

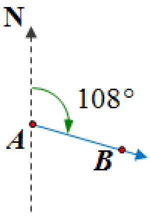


Figure 2a

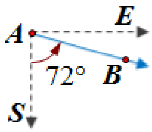


Figure 2b

This, for example, means that:

the direction of 195° can be seen as the bearing **S15°W** and the direction of 290° means the same as **N70°W**.

Example 7 ► Using Direction in Applications Involving Navigation

An airplane flying at a speed of 400 mi/hr flies from a point A in the direction of 153° for one hour and then flies in the direction of 63° for another hour.

a. How long will it take the plane to get back to the point A?

b. What is the direction that the plane needs to fly in order to get back to the point A?

Solution ►

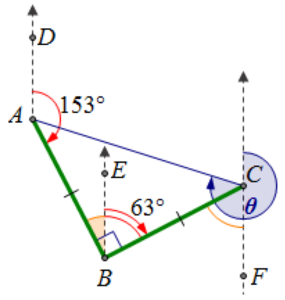


Figure 3

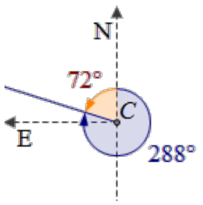
a. First, let's draw a diagram modeling the situation. Assume the notation as in *Figure 3*. Since the plane flies at 153° and the South-North lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ are parallel, by the property of interior angles, we have $\angle ABE = 180° - 153° = 27°$. This in turn gives us $\angle ABC = \angle ABE + \angle EBC = 27° + 63° = 90°$. So the $\triangle ABC$ is right angled with $\angle B = 90°$ and the two legs of length $AB = BC = 400\ mi$. This means that the $\triangle ABC$ is in fact a special triangle of the type $45° - 45° - 90°$.

Therefore $AC = AB\sqrt{2} = 400\sqrt{2} \simeq 565.7\ mi$.

Now, solving the well-known motion formula $R \cdot T = D$ for the time T, we have

$$T = \frac{D}{R} \simeq \frac{\cancel{400}\sqrt{2}}{\cancel{400}} = \sqrt{2} \simeq 1.4142\ hr \simeq \mathbf{1\ hr\ 25\ min}$$

Thus, it will take the plane approximately 1 hour and 25 minutes to return to the starting point A.



b. To direct the plane back to the starting point, we need to find angle θ, marked in blue, rotating clockwise from the North to the ray $\overrightarrow{CA}$. By the property of alternating angles, we know that $\angle FCB = 63°$. We also know that $\angle BCA = 45°$, as $\angle ABD$ is the "half of a square" special triangle. Therefore,

$$\theta = 180° + 63° + 45° = \mathbf{288°}.$$

Thus, to get back to the point A, the plane should fly in the direction of 288°. Notice that this direction can also be stated as **N72°W**.

T.4 Exercises

Using a calculator, find an angle θ satisfying the given equation. Leave your answer in decimal degrees rounded to the nearest tenth of a degree if needed.

1. $\sin\theta = 0.7906$

2. $\cos\theta = 0.7906$

3. $\tan\theta = 2.5302$

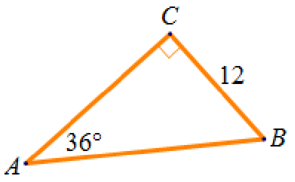
4. $\cos\theta = -0.75$

5. $\tan\theta = \sqrt{3}$

6. $\sin\theta = \frac{3}{4}$

Given the data, solve each triangle ABC with $\angle C = 90°$.

7.



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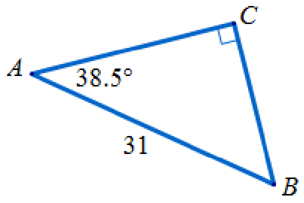

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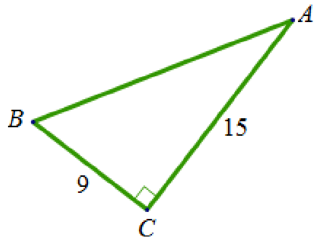


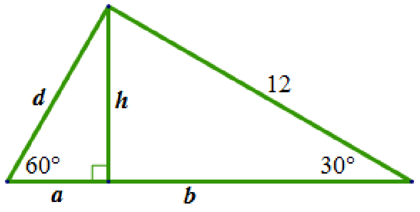
10. $\angle A = 42°,\ b = 17$

11. $a = 9.45,\ c = 9.81$

12. $\angle B = 63°12',\ b = 19.1$

Find the ***exact*** *value of each unknown in the figure.*

13.



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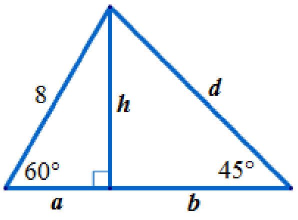

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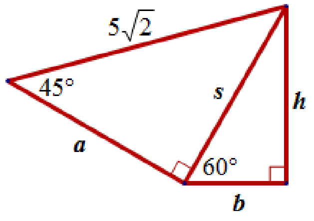

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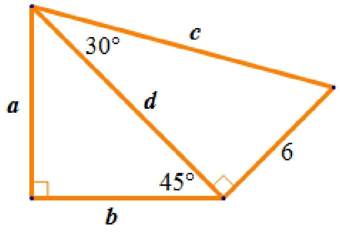


17. A circle of radius 8 centimeters is inscribed in a regular hexagon. Find the exact perimeter of the hexagon.

18. A regular pentagon is inscribed in a circle with 10 meters diameter. To the nearest centimeter, find the perimeter of the pentagon.

19. A 25 meters long supporting rope connects the top of a 23 meters high mast of a sailboat with the deck of the boat. To the nearest degree, find the angle between the rope and the mast.

20. A 16 meters long guy wire is attached to the top of a utility pole. The angle between the guy wire and the ground is 54°. To the nearest tenth of a meter, how tall is the pole?

21. From the top of a 52 m high cliff, the angle of depression to a boat is $4°15'$. To the nearest meter, how far is the boat from the base of the cliff?

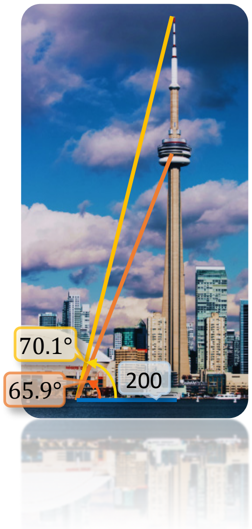
22. A spotlight reflector mounted to a ceiling of a 3.5 meters high hall is directed onto a piece of art displayed 1.5 meters above the floor. To the nearest degree, what angle of depression should be used to direct the light onto the piece of art if the reflector is 3.8 meters away from it?

23. To determine the height of the Eiffel Tower, a 1.8 meters tall tourist standing 50 meters from the center of the base of the tower measures the angle of elevation to the top of the tower to be 81°. Using this information, determine the height of the Eiffel Tower to the nearest meter.

24. To the nearest meter, find the height of an isosceles triangle with 25.2 meters long base and $35°40'$ angle by the base.

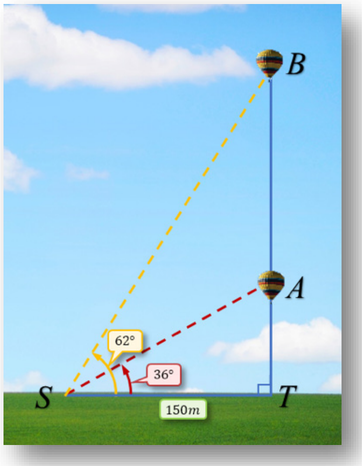
25. A plane flies 700 kilometers at a bearing of **N**56°**E** and then 850 kilometers at a bearing of **S**34°**E**. How far and in what direction is the plane from the starting point? Round the answers to the nearest kilometer and the nearest degree.

26. A plane flies at 420 km/h for 30 minutes in the direction of 142°. Then, it changes its direction to 232° and flies for 45 minutes. To the nearest kilometer, how far is the plane at that time from the starting point? To the nearest degree, in what direction should the plane fly to come back to the starting point?

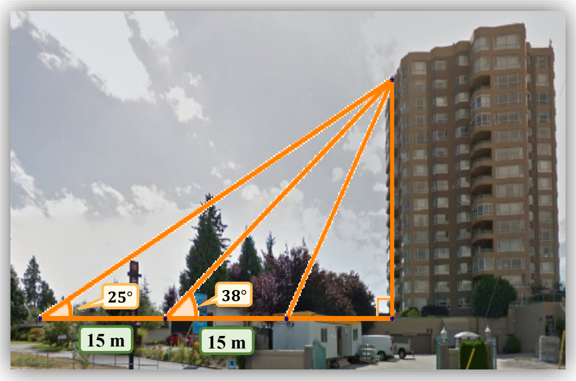


27. Standing 200 meters from the base of the CN Tower, a tourist sees the pinnacle of the tower at 70.1° elevation angle. The tower has a built-in restaurant as in the accompanying picture. The tourist can see this restaurant at 65.9° elevation angle. To the nearest meter, how tall is the CN Tower, including its pinnacle? How high above the ground is the restaurant?

28. A hot air balloon rises vertically at a constant rate, as shown in the accompanying figure. A hundred fifty meters away from the balloon's lift-off place, a spectator notices the balloon at 36° angle of elevation. A minute later, the spectator records that the angle of elevation of the balloon is 62°. To the nearest meter per second, determine the rate of the balloon.



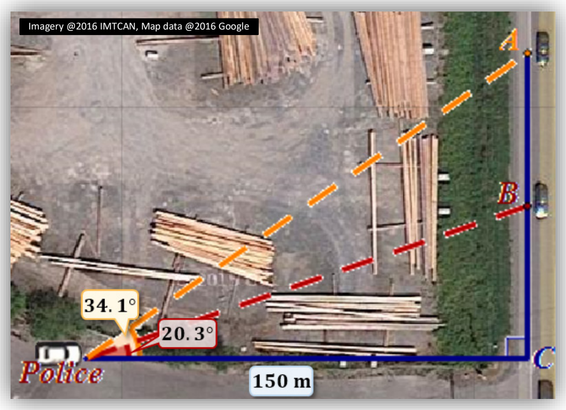
29. Two people observe an eagle nest on a tall tree in a park. One person sees the nest at the angle of elevation of 60° while the other at the angle of elevation of 75°. If the people are 25 meters apart from each other and the tree is between them, determine the altitude at which the nest is situated. Round your answer to the nearest tenth of a meter.



30. A person approaching a tall building records the angle of elevation to the top of the building to be 32°. Fifteen meters closer to the building, this angle becomes 40°. To the nearest meter, how tall is the building? What would the angle of elevation be in another 15 meters?

31. Suppose that the length of the shadow of The Palace of Culture and Science in Warsaw increases by 15.5 meters when the angle of elevation of the sun decreases from 48° to 46°. Based on this information, determine the height of the palace. Round your answer to the nearest meter.

32. A police officer observes a road from 150 meters distance as in the accompanying diagram. A car moving on the road covers the distance between two chosen by the officer points, A and B, in 1.5 seconds. If the angles between the lines of sight to points A and B and the line perpendicular to the observed road are respectively 34.1° and 20.3°, what was the speed of the car? State your answer in kilometers per hour rounded up to one decimal.



T5 The Laws of Sines and Cosines and Their Applications

The concepts of solving triangles developed in *Section T4* can be extended to all triangles. A triangle that is not right-angled is called an **oblique triangle**. Many application problems involve solving oblique triangles, yet we can not use the SOH-CAH-TOA rules when solving those triangles since **SOH-CAH-TOA** definitions **apply only to right triangles!** So, we need to search for other rules that will allow us to solve oblique triangles.

The Sine Law

Observe that all triangles can be classified with respect to the size of their angles as **acute** (with all acute angles), **right** (with one right angle), or **obtuse** (with one obtuse angle). Therefore, oblique triangles are either acute or obtuse.

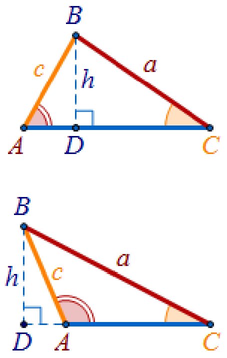


Figure 1

Let's consider both cases of an oblique $\triangle ABC$, as in *Figure 1*. In each case, let's drop the height h from vertex B onto the line $\overleftrightarrow{AC}$, meeting this line at point D. This way, we obtain two more right triangles, $\triangle ADB$ with hypotenuse c, and $\triangle BDC$ with hypotenuse a. Applying the ratio of sine to both of these triangles, we have:

$$\sin \angle A = \frac{h}{c}, \text{ so } h = c \sin \angle A$$

and

$$\sin \angle C = \frac{h}{a}, \text{ so } h = a \sin \angle C.$$

Thus,

$$a \sin \angle C = c \sin \angle A,$$

and we obtain

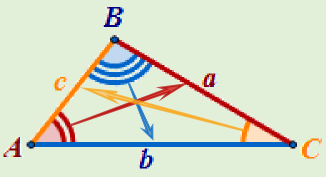
$$\boldsymbol{\frac{a}{\sin \angle A} = \frac{c}{\sin \angle C}}.$$

Similarly, by dropping heights from the other two vertices, we can show that

$$\boldsymbol{\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B}} \text{ and } \boldsymbol{\frac{b}{\sin \angle B} = \frac{c}{\sin \angle C}}.$$

This result is known as the Law of Sines.

The Sine Law ► In any triangle ABC, the lengths of the **sides are proportional to the sines of the opposite angles**. This fact can be expressed in any of the following, equivalent forms:



$$\boldsymbol{\frac{a}{b} = \frac{\sin \angle A}{\sin \angle B}, \quad \frac{b}{c} = \frac{\sin \angle B}{\sin \angle C}, \quad \frac{c}{a} = \frac{\sin \angle C}{\sin \angle A}}$$

or

$$\boldsymbol{\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C}}$$

or

$$\boldsymbol{\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c}}$$

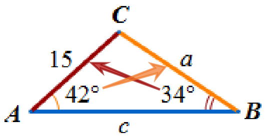
Observation: *As with any other proportion, to solve for one variable, we need to know the three remaining values. Notice that when using the Sine Law proportions, the three known values must include* ***one pair of opposite data****: a side and its opposite angle.*

Example 1 ► **Solving Oblique Triangles with the Aid of The Sine Law**

Given the information, solve each triangle ABC.

a. $\angle A = 42°,\ \angle B = 34°,\ b = 15$ **b.** $\angle A = 35°,\ a = 12,\ b = 9$

Solution ►



a. First, we will sketch a triangle ABC that models the given data. Since the sum of angles in any triangle equals 180°, we have

$$\angle \boldsymbol{C} = 180° - 42° - 34° = \mathbf{104°}.$$

Then, to find length a, we will use the pair $(a, \angle A)$ of opposite data, side a and $\angle A$, and the given pair $(b, \angle B)$. From the Sine Law proportion, we have

$$\frac{a}{\sin 42°} = \frac{15}{\sin 34°},$$

which gives

$$\boldsymbol{a} = \frac{15 \cdot \sin 42°}{\sin 34°} \simeq \mathbf{17.9}$$

To find length c, we will use the pair $(c, \angle C)$ and the given pair of opposite data $(b, \angle B)$. From the Sine Law proportion, we have

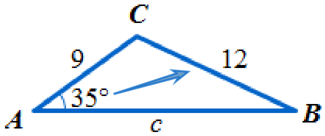
$$\frac{c}{\sin 104°} = \frac{15}{\sin 34°},$$

which gives

$$\boldsymbol{c} = \frac{15 \cdot \sin 104°}{\sin 34°} \simeq \mathbf{26}$$

for easier calculations, keep the unknown in the numerator

So the triangle is solved.



b. As before, we will start by sketching a triangle ABC that models the given data. Using the pair $(9, \angle B)$ and the given pair of opposite data $(12, 35°)$, we can set up a proportion

$$\frac{\sin \angle B}{9} = \frac{\sin 35°}{12}.$$

Then, solving it for $\sin \angle B$, we have

$$\sin \angle B = \frac{9 \cdot \sin 35°}{12} \simeq 0.4302,$$

which, after applying the inverse sine function, gives us

$$\angle \boldsymbol{B} \simeq \mathbf{25.5°}$$

Now, we are ready to find $\angle \boldsymbol{C} = 180° - 35° - 25.5° = \mathbf{119.5°}$, and finally, from the proportion

$$\frac{c}{\sin 119.5°} = \frac{12}{\sin 35°},$$

we have

$$\boldsymbol{c} = \frac{12 \cdot \sin 119.5°}{\sin 35°} \simeq \mathbf{18.2}$$

Thus, the triangle is solved.

Ambiguous Case

Observe that the size of one angle and the length of two sides does not always determine a unique triangle. For example, there are two different triangles that can be constructed with $\angle A = 35°,\ a = 9,\ b = 12$.

Such a situation is called an **ambiguous case**. It occurs when the opposite side to the given angle is shorter than the other given side but long enough to complete the construction of an oblique triangle, as illustrated in *Figure 2*.

In application problems, if the given information does not determine a unique triangle, both possibilities should be considered in order for the solution to be complete.

On the other hand, not every set of data allows for the construction of a triangle. For example (see *Figure 3*), if $\angle A = 35°,\ a = 5,\ b = 12$, the side a is too short to complete a triangle, or if $a = 2,\ b = 3,\ c = 6$, the sum of lengths of a and b is smaller than the length of c, which makes impossible to construct a triangle fitting the data.

Note that in any triangle, the **sum of lengths of any two sides** is always **bigger than the length of the third side**.

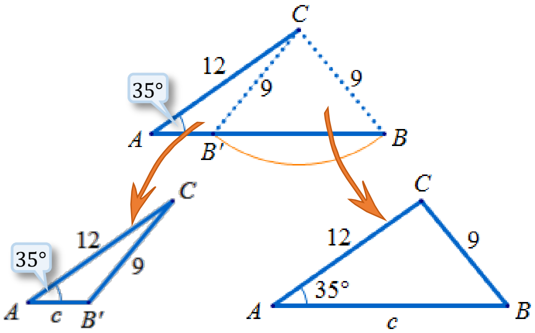


Figure 2

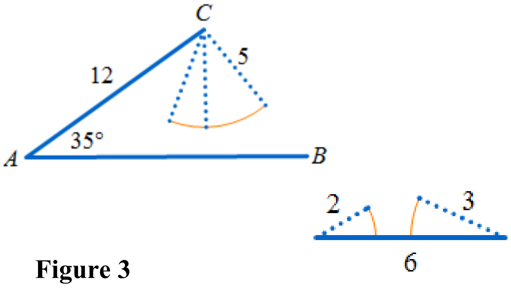
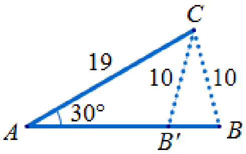


Figure 3

Example 2 ► **Using the Sine Law in an Ambiguous Case**

Solve triangle ABC, knowing that $\angle A = 30°,\ a = 10,\ b = 19$.

Solution ►



When sketching a diagram, we notice that there are two possible triangles, ΔABC and $\Delta AB'C$, complying with the given information. ΔABC can be solved in the same way as the triangle in *Example 1b*. In particular, one can calculate that in ΔABC, we have $\angle \boldsymbol{B} \simeq \mathbf{71.8}°$, $\angle \boldsymbol{C} \simeq \mathbf{78.2}°$, and $\boldsymbol{c} \simeq \mathbf{19.6}$.

Let's see how to solve $\Delta AB'C$ then. As before, to find $\angle B'$, we will use the proportion

$$\frac{\sin \angle B'}{19} = \frac{\sin 30°}{10},$$

which gives us $\sin\angle B' = \frac{19\cdot\sin 30°}{10} = 0.95$. However, when applying the inverse sine function to the number 0.95, a calculator returns the approximate angle of 71.8°. Yet, we know that angle B' is obtuse. So, we should look for an angle in the second quadrant, with the reference angle of 71.8°. Therefore, $\angle \boldsymbol{B'} = 180° - 71.8° = \mathbf{108.2}°$.

Now, $\angle \boldsymbol{C} = 180° - 30° - 108.2° = \mathbf{41.8}°$

and finally, from the proportion

$$\frac{c}{\sin 41.8°} = \frac{10}{\sin 30°},$$

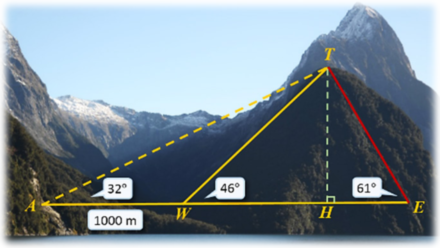
we have

$$\boldsymbol{c} = \frac{10\cdot\sin 41.8°}{\sin 30°} \simeq \mathbf{13.3}$$

Thus, $\triangle AB'C$ is solved.

Example 3 ► **Solving an Application Problem Using the Sine Law**

Refer to the accompanying diagram. Round all your answers to the nearest tenth of a meter.



From a distance of 1000 meters from the west base of a mountain, the top of the mountain is visible at a 32° angle of elevation. At the west base, the average slope of the mountain is estimated to be 46°.

a. Determine the distance WT from the west base to the top of the mountain.

b. What is the distance ET from the east base to the top of the mountain, if the average slope of the mountain there is 61°?

c. Find the height HT of the mountain.

Solution ► **a.** To find distance WT, consider $\triangle AWT$. Observe that one can easily find the remaining angles of this triangle, as shown below:

$$\angle AWT = 180° - 46° = 134°$$ supplementary angles

and

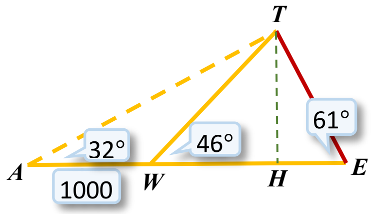
$$\angle ATW = 180° - 32° - 134° = 14°$$ sum of angles in a ⊿

Therefore, applying the Law of Sines, we have

$$\frac{WT}{\sin 32°} = \frac{1000}{\sin 14°},$$

which gives

$$\boldsymbol{WT} = \frac{1000\sin 32°}{\sin 14°} \simeq \mathbf{2190.5}\ \boldsymbol{m}.$$



b. To find distance ET, we can apply the Law of Sines to ΔWET using the pair $(2190.5, 61°)$. From the equation

$$\frac{ET}{\sin 46°} = \frac{2190.5}{\sin 61°},$$

we have

$$\boldsymbol{ET} = \frac{2190.5 \sin 46°}{\sin 61°} \simeq \mathbf{1801.6}\ \boldsymbol{m}.$$

c. To find the height HT of the mountain, we can use the right triangle WHT. By the definition of sine, we have

$$\frac{HT}{2190.5} = \sin 46°,$$

so $\boldsymbol{HT} = 2190.5 \sin 46° \simeq \mathbf{1575.7}\ \boldsymbol{m}$.

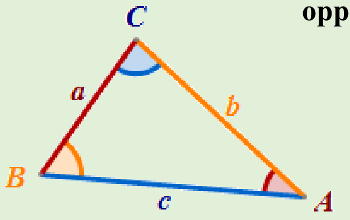
The Cosine Law

The above examples show how the **Sine Law** can help in solving oblique triangles when one **pair of opposite data** is given. However, the Sine Law is not enough to solve a triangle if the given information is

- the length of the **three sides** (but no angles), or
- the length of **two sides** and the **enclosed angle**.

Both of the above cases can be solved with the use of another property of a triangle, called the Cosine Law.

The Cosine Law ► In any triangle ABC, **the square of a side of a triangle is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the opposite angle**.



$$\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos\angle A}$$

$$\boldsymbol{b^2 = a^2 + c^2 - 2ac\cos\angle B}$$

$$\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos\angle C}$$

note the opposite side and angle

Observation: *If the angle of interest in any of the above equations is right, since* $\cos 90° = 0$, *the equation becomes Pythagorean. So the* ***Cosine Law*** *can be seen as an* ***extension of the Pythagorean Theorem****.*

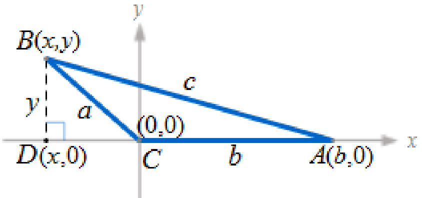


Figure 3

To derive this law, let's place an oblique triangle ABC in the system of coordinates so that vertex C is at the origin, side AC lies along the positive x-axis, and vertex B is above the x-axis, as in *Figure 3*.

Thus $C = (0,0)$ and $A = (b, 0)$. Suppose point B has coordinates (x, y). By *Definition 2.2*, we have

$$\cos\angle C = \frac{x}{a},$$

which gives us

$$x = a\cos\angle C\,.$$

Let $D = (x, 0)$ be the perpendicular projection of the vertex B onto the x-axis. After applying the Pythagorean equation to the right triangle ABD, with $\angle D = 90°$, we obtain

$$\begin{aligned} \boldsymbol{c^2} &= y^2 + (b-x)^2 \\ &= y^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx \\ &= a^2 + b^2 - 2b(a\cos\angle C) \\ &= \boldsymbol{a^2 + b^2 - 2ab\cos\angle C} \end{aligned}$$

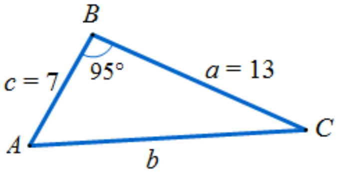
here we apply the Pythagorean equation to ΔBDC and replace $y^2 + x^2$ with a^2

Similarly, by placing the vertices A or B at the origin, one can develop the remaining two forms of the Cosine Law.

Example 4 ► Solving Oblique Triangles Given Two Sides and the Enclosed Angle

Solve triangle ABC, given that $\angle B = 95°$, $a = 13$, and $c = 7$.

Solution ►



First, we will sketch an oblique triangle ABC to model the situation. Since there is no pair of opposite data given, we cannot use the Law of Sines. However, applying the Law of Cosines with respect to side b and $\angle B$ allows for finding the length b. From

$$b^2 = a^2 + c^2 - 2ac\cos\angle B = 13^2 + 7^2 - 2\cdot 13\cdot 7\cos 95° \simeq 233.86,$$

we have $\boldsymbol{b \simeq 15.3}$.

watch the order of operations here!

Now, since we already have the pair of opposite data $(15.3, 95°)$, we can apply the Law of Sines to find, for example, $\angle C$. From the proportion

$$\frac{\sin\angle C}{7} = \frac{\sin 95°}{15.3},$$

we have

$$\sin\angle C = \frac{7\cdot\sin 95°}{15.3} \simeq 0.4558,$$

thus $\angle \boldsymbol{C} = \sin^{-1} 0.4558 \simeq \boldsymbol{27.1}°$.
Finally, $\angle \boldsymbol{A} = 180° - 95° - 27.1° = \boldsymbol{57.9}°$ and the triangle is solved.

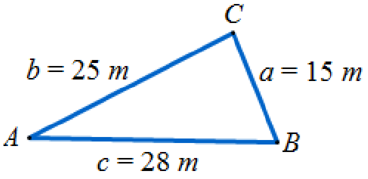
When applying the Law of Cosines in the above example, there was no other choice but to start with the pair of opposite data $(b, \angle B)$. However, in the case of three given sides, one could apply the Law of Cosines corresponding to any pair of opposite data. Is there any preference as to which pair to start with? Actually, yes. Observe that after using the Law of Cosines, we often use the **Law of Sines** to complete the solution since the **calculations are usually easier** to perform this way. Unfortunately, when solving a sine proportion for an obtuse angle, one would need to change the angle obtained from a calculator to its supplementary one. This is because calculators are programmed to return angles from the first quadrant when applying $\sin^{-1}$ to positive ratios. If we look for an obtuse angle, we need to employ the fact that $\sin\alpha = \sin(180° - \alpha)$ and take the supplement of the calculator's answer. To avoid this ambiguity, it is recommended to **apply the Cosine Law** to the pair of the **longest side and largest angle** first. This will guarantee that the Law of Sines will be used to find only acute angles and thus it will not cause ambiguity.

Recommendations:
- ***apply the Cosine Law only when it is absolutely necessary*** *(SAS or SSS)*
- ***apply the Cosine Law to find the largest angle first****, if applicable*

Example 5 ► Solving Oblique Triangles Given Three Sides

Solve triangle ABC, given that $a = 15\ m$, $b = 25\ m$, and $c = 28\ m$.

Solution ►



First, we will sketch a triangle ABC to model the situation. As before, there is no pair of opposite data given, so we cannot use the Law of Sines. So, we will apply the Law of Cosines with respect to the pair $(28, \angle C)$, as the side $c = 28$ is the longest. To solve the equation

$$\boxed{28^2 = 15^2 + 25^2 - 2 \cdot 15 \cdot 25 \cos \angle C}$$

for $\cos \angle C$, we will first solve it for $\cos \angle C$, and have

watch the order of operations when solving for cosine

$$\cos \angle C = \boxed{\frac{28^2 - 15^2 - 25^2}{-2 \cdot 15 \cdot 25}} = \frac{-66}{-750} = 0.088,$$

which, after applying $\cos^{-1}$, gives $\angle \boldsymbol{C} \simeq \mathbf{85}°$.

Since now we have the pair of opposite data $(28, 85°)$, we can apply the Law of Sines to find, for example, $\angle A$. From the proportion

$$\frac{\sin \angle A}{15} = \frac{\sin 85°}{28},$$

we have

$$\sin \angle A = \frac{15 \cdot \sin 85°}{28} \simeq 0.5337,$$

thus $\angle \boldsymbol{A} = \sin^{-1} 0.5337 \simeq \mathbf{32.3}°$.
Finally, $\angle \boldsymbol{B} = 180° - 85° - 32.3° = \mathbf{62.7}°$ and the triangle is solved.

Example 6 ► Solving an Application Problem Using the Cosine Law

Two planes leave an airport at the same time and fly in different directions. Plane A flies in the direction of 155° at 390 km/h and plane B flies in the direction of 260° at 415 km/h. To the nearest kilometer, how far apart are the planes after two hours?

Solution ► As usual, we start the solution by sketching a diagram appropriate to the situation. Assume the notation as in *Figure 3*.

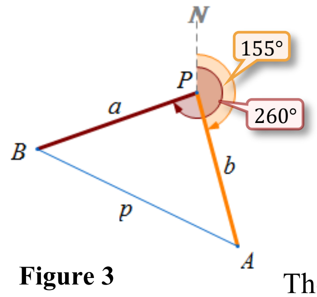


Figure 3

Since plane A flies at 390 km/h for two hours, we can find the distance

$$b = 2 \cdot 390 = 780\ km.$$

Similarly, since plane B flies at 415 km/h for two hours, we have

$$a = 2 \cdot 415 = 830\ km.$$

The measure of the enclosed angle APB can be obtained as a difference between the given directions. So we have

$$\angle APB = 260° - 155° = 105°.$$

Now, we are ready to apply the Law of Cosines in reference to the pair $(p, 105°)$. From the equation

$$p^2 = 830^2 + 780^2 - 2 \cdot 830 \cdot 780 \cos 105° \simeq 1632418.9,$$

we have $p \simeq \sqrt{1632418.9} \simeq 1278\ km.$

So we know that after two hours, the two planes are about **1278 kilometers** apart.

Area of a Triangle

The method used to derive the Law of Sines can also be used to derive a handy formula for finding the area of a triangle, without knowing its height.

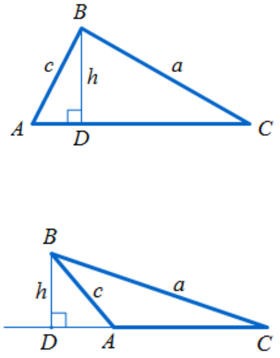


Figure 4

Let ABC be a triangle with height h dropped from the vertex B onto the line $\overleftrightarrow{AC}$, meeting $\overleftrightarrow{AC}$ at the point D, as shown in *Figure 4*. Using the right $\triangle ABD$, we have

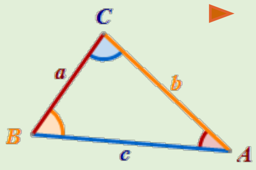
$$\sin \angle A = \frac{h}{c},$$

and equivalently $h = c \sin \angle A$, which after substituting into the well known formula for area of a triangle $[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bh}$, gives us

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle \boldsymbol{A}$$

Starting the proof with dropping a height from a different vertex would produce two more versions of this formula, as stated below.

The Sine Formula for Area of a Triangle

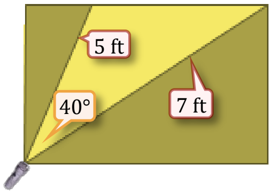


The **area** $[\boldsymbol{ABC}]$ of a triangle ABC can be calculated by taking **half of a product of the lengths of two sides and the sine of the enclosed angle.** We have

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle\boldsymbol{A}, \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ac}\ \mathbf{sin}\ \angle\boldsymbol{B}, \quad \text{or} \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ab}\ \mathbf{sin}\ \angle\boldsymbol{C}.$$

Example 7 ► Finding Area of a Triangle Given Two Sides and the Enclosed Angle

In a search for her lost earring, Irene used a flashlight to illuminate part of the floor under her bed. If the flashlight emitted the light at 40° angle and the length of the outside rays of light was 5 ft and 7 ft as indicated in the accompanying diagram, how many square feet of the floor were illuminated?



Solution ► We start with sketching an appropriate diagram. Assume the notation as in *Figure 5*.

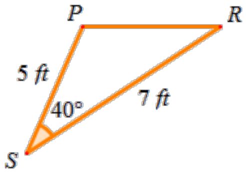


Figure 5

From the sine formula for area of a triangle, we have

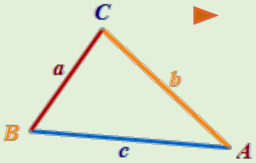
$$[PRS] = \frac{1}{2} \cdot 5 \cdot 7 \sin 40^\circ \simeq 11.2 \text{ ft}^2.$$

The area of the illuminated part of the floor under the bed was about **11 square feet**.

Heron's Formula

The Law of Cosines can be used to derive a formula for the area of a triangle when only the lengths of the three sides are known. This formula is known as Heron's formula (as mentioned in *Section RD1*), named after the Greek mathematician Heron of Alexandria.

Heron's Formula for Area of a Triangle



► The **area** $[ABC]$ of a triangle ABC with sides a, b, c, and **semiperimeter** $\boldsymbol{s} = \frac{\boldsymbol{a+b+c}}{\boldsymbol{2}}$ can be calculated using the formula

$$[\boldsymbol{ABC}] = \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}$$

Example 8 ► Finding Area of a Triangle Given Three Sides

The city of Abbotsford plans to convert a triangular lot into public parking. In square meters, what would the area of the parking be if the three sides of the lot are 45 m, 57 m, and 60 m long?

Solution ► To find the area of the triangular lot with given sides, we would like to use Heron's Formula. For this reason, we first calculate the semiperimeter

$$s = \frac{45 + 57 + 60}{2} = 81.$$

Then, the area equals

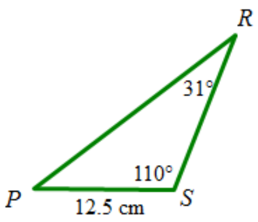
$$\sqrt{81(81-45)(81-57)(81-60)} = \sqrt{1469664} \simeq 1212.3\ m^2.$$

Thus, the area of the parking lot would be approximately **1212 square meters**.

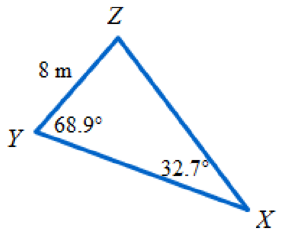
T.5 Exercises

Use the Law of Sines to solve each triangle.

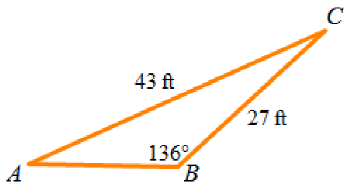
1.



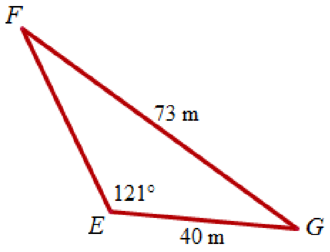
2.



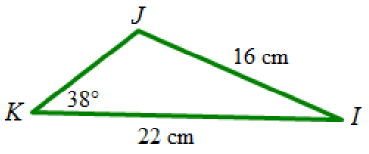
3.



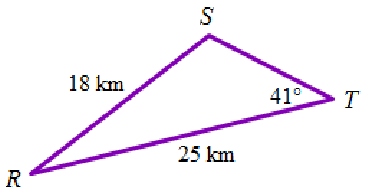
4.



5.



6.



7. $\angle A = 30°,\ \angle B = 30°,\ a = 10$

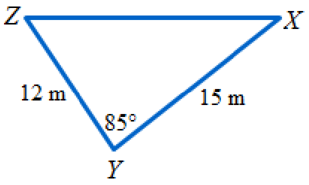
8. $\angle A = 150°,\ \angle C = 20°,\ a = 200$

9. $\angle C = 145°,\ b = 4,\ c = 14$

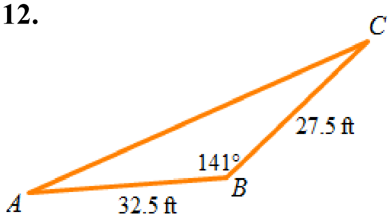
10. $\angle A = 110°15',\ a = 48,\ b = 16$

Use the Law of Cosines to solve each triangle.

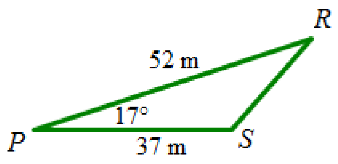
11.



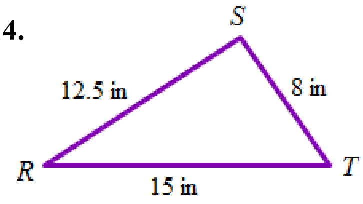
12.



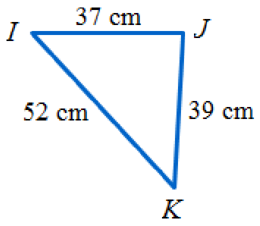
13.



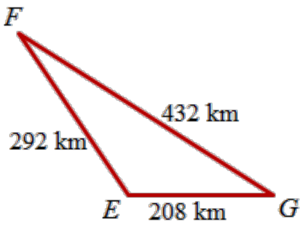
14.



15.



16.



17. $\angle C = 60°,\ a = 3,\ b = 10$

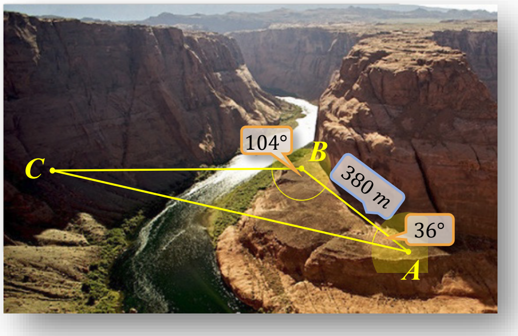
18. $\angle B = 112°,\ a = 23,\ c = 31$

19. $a = 2,\ b = 3,\ c = 4$

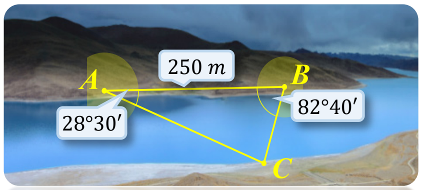
20. $a = 34,\ b = 12,\ c = 17.5$

21. In a triangle ABC, $\angle A$ is twice as large as $\angle B$. Does this mean that side a is twice as long as side b?

Use the appropriate law to solve each application problem.



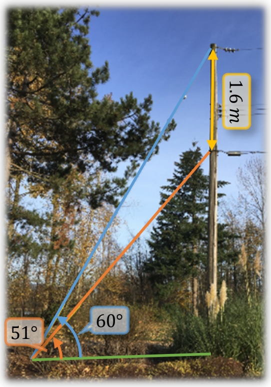
22. To approximate the distance across the Colorado River Canyon at the Horseshoe Bend, a hiker designates three points, A, B, and C, as in the accompanying figure. Then, he records the following measurements: $AB = 380$ meters, $\angle CAB = 36°$ and $\angle ABC = 104°$. How far is from B to C?



23. To find the width of a river, Peter designates three spots: A and B along one side of the river 250 meters apart from each other, and C, on the opposite side of the river (*see the accompanying figure*). Then, he finds that $\angle A = 28°30'$, and $\angle B = 82°40'$. To the nearest meter, what is the width of the river?

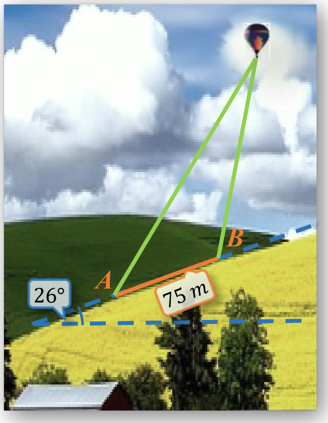
24. The captain of a ship sailing south spotted a castle tower at the distance of approximately 8 kilometers and the bearing of $\boldsymbol{S}47.5°\boldsymbol{E}$. In half an hour, the bearing of the tower was $\boldsymbol{N}35.7°\boldsymbol{E}$. What was the speed of the ship in km/h?

25. The captain of a ship sailing south saw a lighthouse at the bearing of $\boldsymbol{N}52.5°\boldsymbol{W}$. In 4 kilometers, the bearing of the lighthouse was $\boldsymbol{N}35.8°\boldsymbol{E}$. To the nearest tenth of a kilometer, how far was the ship from the lighthouse at each location?



26. Sam and Dan started sailing their boats at the same time and from the same spot. Sam followed the bearing of N12°W while Dan directed his boat at N5°E. After 3 hours, Sam was exactly west of Dan. If both sailors were 4 kilometers away from each other at that time, determine the distance sailed by Sam. *Round your answer to the nearest meter.*

27. A pole is anchored to the ground by two metal cables, as shown in the accompanying figure. The angles of inclination of the two cables are 51° and 60° respectively. Approximately how long is the top cable if the bottom one is attached to the pole 1.6 meters lower than the top one? *Round your answer to the nearest tenth of a meter.*



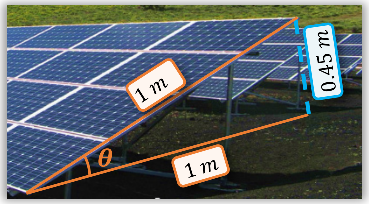
28. Two forest rangers were observing the forest from different lookout towers. At a certain moment, they spotted a group of lost hikers. The ranger on tower A saw the hikers at the direction of 46.7° and ranger on tower B saw the hikers at the direction of 315.8°. If tower A was 3.25 kilometers west of tower B, how far were the hikers from tower A? *Round your answer to the nearest hundredth of a kilometer.*

29. A hot-air balloon rises above a hill that inclines at 26°, as indicated in the accompanying diagram. Two spectators positioned on the hill at points A and B (*refer to the diagram*) observe the movement of the balloon. They notice that at a particular moment, the angle of elevation of the balloon from point A is 64° and from point B is 73°. If the spectators are 75 meters from each other, how far is the balloon from each of them? *Round your answers to the nearest meter.*

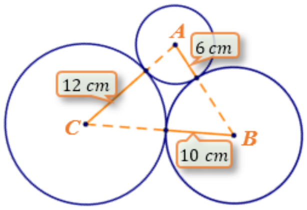
30. To the nearest centimeter, how long is the chord subtending a central angle of 25° in a circle of radius 30 cm?

31. An airplane takes off from city A and flies in the direction of $32°15'$ to city B, which is 500 km from A. After an hour of layover, the plane is heading in the direction of $137°25'$ to reach city C, which is 740 km from A. How far and in what direction should the plane fly to go back to city A?

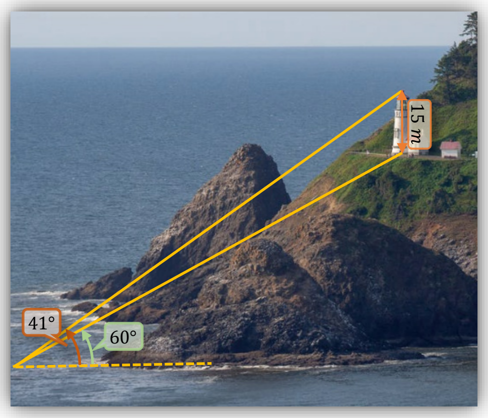
32. Find the area of a triangular hang-glider with two 7.5-meter sides that enclose the angle of 142°. *Round your answer to the nearest tenth of a square meter.*



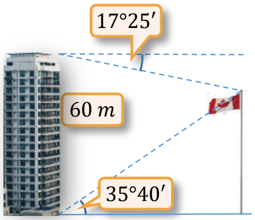
33. One-meter-wide solar panels were installed on a flat surface by tilting them up at an angle θ, as shown in the accompanying figure. If the distance between the top corner of a panel in the flat and tilted position is 0.45 meters, determine the measure of angle θ.



34. Three pipes with centres at points A, B, and C are tangent to each other. A perpendicular cross-section of the arrangement is shown in the accompanying figure. To the nearest tenth of a degree, determine the angles of triangle ABC, if the radii of the pipes are 6 cm, 10 cm, and 12 cm, respectively.



35. A 15-meters tall lighthouse is standing on a cliff. A person observing the lighthouse from a boat approaching the shore notices that the angle of elevation to the top of the lighthouse is 41° and to the bottom is 36°. Disregarding the person's height, estimate the height of the cliff.

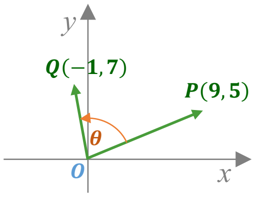


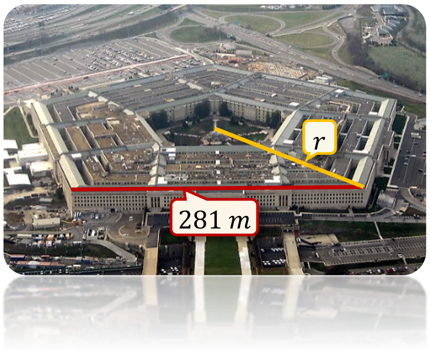
36. The top of a flag pole is visible from the top of a 60 meters high building at $17°25'$ angle of depression. From the bottom of this building, the tip of the flag pole can be seen at $35°40'$ angle of elevation. To the nearest centimeter, how tall is the flag pole?

37. Find the area of a triangular parcel having two sides of lengths 51.4 m and 62.1 m, and 48.7° angle between them.

38. A city plans to pave a triangular area with sides of length 82 meters, 78 meters, and 112 meters. A pallet of bricks chosen for the job can cover 10 square meters of area. How many pallets should be ordered?

39. Suppose points P and Q are located respectively at $(9,5)$ and $(-1,7)$. If point O is the origin of the Cartesian coordinate system, determine the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. *Round your answer to the nearest degree.*





40. The building of The Pentagon in Washington D.C. is in a shape of a regular pentagon with about 281 meters long side, as shown in the accompanying figure. To the nearest meter, determine the radius of the **circumcircle** of this pentagon (*the circle that passes through all the vertices of the polygon*).

41. The locations A, B, and C of three FM radio transmitters form a triangle with sides $AB = 75$ m, $BC = 85$ m, and $AC = 90$ m. The transmitters at A, B, and C have a circular range of radius 35 m, 40 m, and 50 m, correspondingly. Assuming that no area can receive a signal from more than one transmitter, determine the area of the ABC triangle that does not receive any signal from any of the three FM radio transmitters. Round your answer to the nearest tenth of a square meter.

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p.**430** The Pentagon by David B. Gleason / CC BY-SA 2.0

Polynomials and Polynomial Functions - ANSWERS

P1 Exercises

1. yes

3. no

5. 4; 1

7. 2; $\sqrt{2}$

9. $-\frac{2}{5}x^3 + 3x^2 - x + 5$; 3; $-\frac{2}{5}$

11. $x^5 + 8x^4 + 2x^3 - 3x$; 5; 1

13. $3q^4 + q^2 - 2q + 1$; 4; 3

15. first degree binomial

17. zero degree monomial

19. seventh degree monomial

21. -8

23. -12

25. -5

27. $2a - 3$

29. -21

31. $6a - 9$

33. $-x + 13y$

35. $4xy + 3x$

37. $6p^3 - 3p^2 + p + 2$

39. $3m + 11$

41. $-x - 4$

43. $-5x^2 + 4y^2 - 11z^2$

45. $-4x^2 - 3x - 5$

47. $5r^6 - r^5 - 7r^2 + 5$

49. $-5a^4 - 6a^3 + 9a^2 - 11$

51. $5x^2y^2 - 7y^3 + 17xy$

53. $-z^2 + x + 4m$

55. $10z^2 - 16z$

57. **a.** $(f + g)(x) = 8x - 8$ **b.** $(f - g)(x) = 2x - 4$

59. **a.** $(f + g)(x) = -2x^2 - 3x + 1$ **b.** $(f - g)(x) = 8x^2 - 7x - 1$

61. **a.** $(f + g)(x) = -6x^{2n} - 2x^n - 1$ **b.** $(f - g)(x) = 10x^{2n} - 4x^n + 7$

63. $(P - Q)(-2) = -1$

65. $(R - Q)(0) = -7$

67. $(P + Q)(a) = a^2 + 2a + 1$

69. $(P + R)(2k) = 4k^2 + 2k - 6$

71. ~9.3 cm

73. **a.** $R(n) = 56n$ **b.** $P(n) = 24n - 1500$ **c.** $P(100) = 900$;
The profit from selling 100 dresses is $900.

P2 Exercises

1. **a.** no; $x^2 \cdot x^4 = x^6$ **b.** no; $-2x^2$ is in the simplest form **c.** yes **d.** yes **e.** no; $(a^2)^3 = a^6$
f. no; $4^5 \cdot 4^2 = 4^7$ **g.** no; $\frac{6^5}{3^2} = 2^5 \cdot 3^3$ **h.** no; $xy^0 = x$ **i.** yes

3. $-8y^8$

5. $14x^3y^8$

7. $-27x^6y^3$

9. $\frac{-5x^3}{y^2}$

11. $\frac{64a^6}{b^2}$

13. $\frac{-125p^3}{q^9}$

15. $12a^5b^5$

17. $\frac{16y}{x^3}$

19. $64x^{18}y^6$

21. x^{2n-1}

23. 5^{2ab}

25. $-2x^2$

27. $x^{a^2-b^2}$

29. $-16x^7y^4$

31. $-6x^2+10x$

33. $-12x^5y+9x^4y^2$

35. $15k^4-10k^3+20k^2$

37. x^2+x-30

39. $6x^2+5x-6$

41. $6u^4-8u^3-30u^2$

43. $6x^3-7x^2-13x+15$

45. $6m^4-13m^2n^2+5n^4$

47. a^2-4b^2

49. $a^2-4ab+4b^2$

51. y^3+27

53. $2x^4-4x^3y-x^2y^2+3xy^3-2y^4$

55. true

57. true

59. false; $(2-1)^3 \neq 2^3-1^3$

61. $25x^2-16$

63. $\frac{1}{4}x^2-9y^2$

65. x^4-49y^6

67. $0.64a^2+0.16ab+0.04b^2$

69. x^2-6x+9

71. $25x^2-60xy+36y^2$

73. $4n^2-\frac{4}{3}n+\frac{1}{9}$

75. $x^8y^4+6x^4y^2+9$

77. $4x^4-12x^2y^3+9y^6$

79. $8a^5+40a^4b+50a^4b^2$

81. $x^4-x^2y^2$

83. x^4-1

85. $a^4-2a^2b^2+b^4$

87. $4x^2+12xy+9y^2-25$

89. $4k^2=12k+4hk-6h+h^2+9$

91. $x^{4a}-y^{4b}$

93. $101\cdot 99=(100+1)(100-1)=10000-1=9999$

95. $505\cdot 495=(500+5)(500-5)=250000-25=249975$

97. x^2-x-12

99. $(fg)(x)=15x^2-28x+12$

101. $(fg)(x)=-3x^4+8x^3+22x^2-45x$

103. $(PR)(x)=x^3-2x^2-4x+8$

105. $(PQ)(a)=2a^3-8a$

107. $(PQ)(3)=30$

109. $(QR)(x)=2x^2-4x$

111. $(QR)(a+1)=2a^2-2$

113. $P(2a+3)=4a^2+12a+5$

115. $4x^3-40x^2+96x$

P3 Exercises

1. False; When dividing powers with the same bases, we subtract exponents. So, the quotient will be a fourth-degree polynomial.

3. $4x^2-3x+1$

5. $2xy-6$

7. $-3a^3+5a^2-4a$

9. $8-\frac{9}{x}+\frac{3}{2x^2}$

11. $\frac{2b}{a}+\frac{5}{3}+\frac{3c}{a}$ **13.** $y+5$ **15.** $t-4 \quad \mathrm{R}\ -21$

17. $2a^2-a+2 \quad \mathrm{R}\ 6$ **19.** $2z^2-4z+1 \quad \mathrm{R}\ -10$ **21.** $3x+1 \quad \mathrm{R}\ -3x-7$

23. $3k^2+4k+1$ **25.** $\frac{5}{4}t+1 \quad \mathrm{R}\ -5$ **27.** p^2+p+1

29. $y^3-2y^2+4y-8 \quad \mathrm{R}\ 32$ **31.** $Q(x)=2x^2-x+6;\ R(x)=4$

33. $\left(\frac{f}{g}\right)(x)=3x-2;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\{0\}$ **35.** $\left(\frac{f}{g}\right)(x)=x-6;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\{-6\}$

37. $\left(\frac{f}{g}\right)(x)=x+1;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\left\{\frac{3}{2}\right\}$ **39.** $\left(\frac{f}{g}\right)(x)=4x^2-10x+25;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\left\{-\frac{5}{2}\right\}$

41. $\left(\frac{R}{Q}\right)(x)=\frac{x-2}{2x}$ **43.** $\left(\frac{R}{P}\right)(x)=\frac{1}{x+2},\ x\neq 2$ **45.** $\left(\frac{R}{Q}\right)(0)=DNE$

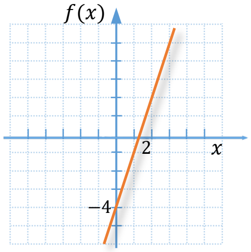
47. $\left(\frac{R}{P}\right)(-2)=DNE$ **49.** $\left(\frac{P}{R}\right)(a)=a+2$ **51.** $\frac{1}{2}\left(\frac{Q}{R}\right)(x)=\frac{x}{x-2}$

53. **a.** $L=3x-2$ **b.** 10 m

P4 Exercises

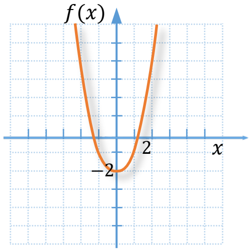
1. False; it's the shape of a basic parabola with a vertex at $(\mathbf{0},\mathbf{3})$.

3.



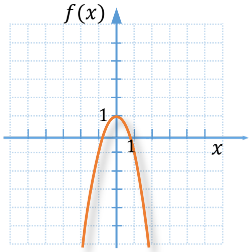
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

5.



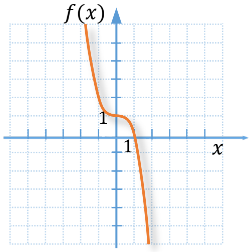
Domain: $\mathbb{R}$
Range: $[-2,\infty)$

7.



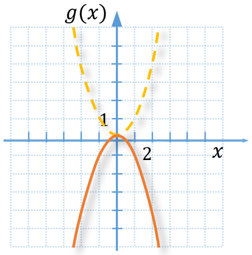
Domain: $\mathbb{R}$
Range: $(-\infty,1]$

9.



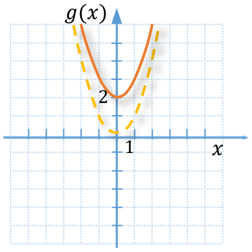
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

11.



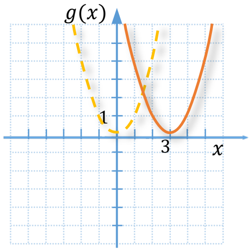
symmetry in x-axis

13.



translation: 2 up

15.



translation: 3 to the right

Factoring - ANSWERS

F1 Exercises

1. false

3. Both are correct but the second one is preferable as the binomial factor has integral coefficients.

5. $7a^3b^5$

7. $x(x-3)$

9. $(x-2y)$

11. $x^{-4}(x+2)^{-2}$

13. $8k(k^2+3)$

15. $-6a^2(6a^2-a-3)$

17. $5x^2y^2(y-2x)$

19. $(a-2)(y^2-3)$

21. $2n(n-2)$

23. $(4x-y)(4x+1)$

25. $-(p-3)(p^2-10p+19)$

27. $k^{-4}(k^2+2)$

29. $-p^{-5}(2p^3-p^2-3)$

31. $-x^{-2}y^{-3}(2xy-5)$

33. $(a^2-7)(2a+1)$

35. $-(xy+3)(x-2)$

37. $(x^2-y)(x-y)$

39. $-(y-3)(x^2+z^2)$

41. $(x-6)(y+3)$

43. $(x^2-a)(y^2-b)$

45. $(x^n+1)(y-3)$

47. $2(s+1)(3r-7)$

49. $x(x-1)(x^3+x^2-1)$

51. no, as $(2xy^2-4)$ can still be factored to $2(xy^2-2)$

53. $p=\frac{2M}{q+r}$

55. $y=\frac{x}{3-w}$

57. $A=(4-\pi)x^2$

59. $A=(\pi-1)r^2$

F2 Exercises

1. no

3. All of them; however, the preferable answer is $-(2x-3)(x+5)$.

5. $x-3$

7. $x-5y$

9. $(x+3)(x+4)$

11. $(y+8)(y-6)$

13. not factorable

15. $(m-7)(m-8)$

17. $-(n+9)(n-2)$

19. $(x-2y)(x-3y)$

21. $-(x+3)(x-7)$

23. $n^2(n+2)(n-15)$

25. $-2(x-10)(x-4)$

27. $y(x^2+12)(x^2-5)$

29. $-5(t^{13}+8)(t^{13}-1)$

31. $-n(n^4+16)(n^4-3)$

33. $\pm 12, \pm 13, \pm 15, \pm 20, \pm 37$

35. $3x-4$

37. $3x-5$

39. $(2y+1)(3y-2)$

41. $(6t-1)(t-6)$

43. $(6n+5)(7n-5)$

45. $-2(2x-3)(3x+5)$

47. $(6x+5y)(3x+2y)$

49. $-(2n+5)(4n-3)$

51. $2x^2(2x-1)(x+3)$

53. $(9xy-4)(xy+1)$

55. $(2p^2-7q)^2$

57. $(2a+9)(a+5)$

59. $\pm 3, \pm 4, \pm 11, \pm 17, \pm 28, \pm 59$

61. $(3x+2)$ feet

F3 Exercises

1. difference of squares
3. neither
5. difference of cubes
7. difference of squares
9. perfect square
11. difference of cubes
13. $25x^2 + 100 = 25(x^2 + 4)$; The sum of squares is factorable in integral coefficients only if the two terms have a common factor.
15. $(x + y)(x - y)$
17. $(x - y)(x^2 + xy + y^2)$
19. $(2z - 1)^2$
21. not factorable
23. $(5 - y)(25 + 5y + y^2)$
25. $(n + 10m)^2$
27. $(3a^2 + 5b^3)(3a^2 - 5b^3)$
29. $(p^2 - 4q)(p^4 + 4p^2q + 16q^2)$
31. $(7p + 2)^2$
33. $r^2(r + 3)(r - 3)$
35. $\frac{1}{8}(1 - 2a)(1 + 2a + 4a^2)$ or $\left(\frac{1}{2} - a\right)\left(\frac{1}{4} + \frac{1}{2}a + a^2\right)$
37. not factorable
39. $x^2(4x^2 + 11y^2)(4x^2 - 11y^2)$
41. $-(ab + 5c^2)(a^2b^2 - 5abc^2 + 25c^4)$
43. $(3a^4 - 8b)^2$
45. $(x + 8)(x - 6)$
47. $2t(t - 4)(t^2 + 4t + 16)$
49. $(x^n + 3)^2$
51. $(4z^2 + 1)(2z + 1)(2z - 1)$
53. $5(3x^2 + 15x + 25)$
55. $0.01(5z - 7)^2$ or $(0.5z - 0.7)^2$
57. $-3y(2x - y)$
59. $4(3x^2 + 4)$
61. $2(x - 5a)^2$
63. $(y + 6 + 3a)(y + 6 - 3a)$
65. $(m + 2)(m^2 - 2m + 4)(m - 1)(m^2 + m + 1)$
67. $(a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$
69. $(x^2 + 1)(x + 3)(x - 3)$
71. $(a + b + 3)(a - b - 3)$
72. $z(3xy + 4z)(xy + 7z)$
75. $(x^2 + 1)(x + 1)(x - 1)^3$
77. $c(c^w + 1)^2$

F4 Exercises

1. true
3. false
5. false
7. $x \in \{-4, 1\}$
9. $x \in \left\{-\frac{4}{5}, -\frac{1}{3}\right\}$
11. $x \in \{-6, -3\}$
13. $x \in \left\{-\frac{7}{2}, 1\right\}$
15. $x \in \{-6, 0\}$
17. $x \in \{4\}$
19. $x \in \left\{\frac{5}{2}\right\}$
21. $x \in \{-8, 4\}$
23. $x \in \left\{\frac{1}{3}, 3\right\}$
25. $x \in \left\{-2, \frac{8}{9}\right\}$
27. $x \in \{0, 6\}$
29. $x \in \{-4, 2\}$
31. $x \in \{1, 5\}$

33. $x \in \left\{-\frac{15}{8}, -1\right\}$

35. $x \in \{-5, 0, 3\}$

37. $x \in \left\{-\frac{8}{5}, 0, \frac{8}{5}\right\}$

39. $x \in \{-5, -1, 1, 5\}$

41. $x \in \{0, 2, 4\}$

43. $x \in \{-3, -1, 3\}$

45. $x \in \left\{-2, -\frac{2}{5}, 2\right\}$

47. 3; $\{-3,0,3\}$; Do not divide by x as x can be equal to zero. Also, $\sqrt{x^2} = |x|$ so in the last step, we should obtain $x = \pm 3$. The safest way to solve polynomial equations is by factoring and using the zero-product property.

49. $x \in \left\{\frac{1}{2}, 7\right\}$

51. $x \in \left\{-3, \frac{7}{3}\right\}$

53. $s = \frac{5-2p}{r+3}$

55. $r = \frac{R}{E-1}$

57. $t = \frac{4}{c+2}$

59. 8 seconds

61. -12 or 13

63. width = 9 cm; length = 16 cm

65. width = 7 m; height = 10 m

67. 7 m by 12 m

69. 2 cm

71. 9 in

Rational Expressions and Functions - ANSWERS

RT1 Exercises

1. true

3. true

5. true

7. false

9. false

11. false

13. $\frac{1}{64}$

15. $\frac{1}{512}$

17. $-\frac{125}{81}$

19. $\frac{3}{8}$

21. $-\frac{14}{x^{11}}$

23. $-\frac{36}{x^{12n}}$

25. $-\frac{4}{x^3}$

27. $3n^4m^2$

29. $\frac{3x^2}{2y^2}$

31. $-\frac{b^{15}}{27a^6}$

33. $\frac{27}{8x^9y^3}$

35. $\frac{x^{10}y^5}{5^{10}}$

37. $\frac{64}{x^{24}y^{12}}$

39. $\frac{4k^5}{m^2}$

41. $-\frac{5^3y^3}{x^{30}}$

43. $-\frac{1}{3^8x^8y^8}$

45. $\frac{1}{5^{a^2}}$

47. $3n^x$

49. x^{b+5}

51. $2.6 \cdot 10^{10}$

53. $1.05 \cdot 10^{-8}$

55. 670,000,000

57. 2,000,000,000,000

59. $1048576 = 1.05 \cdot 10^6$

61. $1.3338 \cdot 10^{-10}$

63. $5 \cdot 10^{-5}$

65. $2.5 \cdot 10^7$

67. $1.25 \cdot 10^3$

69. 18,108.11 \$/person

71. $1.59 \cdot 10^7$ ft³/min; $3.816 \cdot 10^8$ ft³/day

73. 81 times

RT2 Exercises

1. false

3. true

5. $f(-1) = \frac{1}{3},\ f(0) = 0,\ f(2) = undefined$

7. $f(-1) = \frac{1}{2},\ f(0) = \frac{1}{3},\ f(2) = undefined$

9. $6;\ D = \mathbb{R} \setminus \{6\};\ D = (-\infty, 6) \cup (6, \infty)$

11. $\frac{4}{5};\ D = \mathbb{R} \setminus \left\{\frac{4}{5}\right\};\ D = \left(-\infty, \frac{4}{5}\right) \cup \left(\frac{4}{5}, \infty\right)$

13. none; $D = \mathbb{R};\ D = (-\infty, \infty)$

15. $-7, -5;\ D = \mathbb{R} \setminus \{-7, -5\};\ D = (-\infty, -7) \cup (-7, -5) \cup (-5, \infty)$

17. **b.**, **d.**, and **e.** are equivallent to -1

19. $\frac{8a^2}{b^2}$

21. -1

23. 1

25. $\frac{4x-5}{7}$

27. $\frac{y-3}{y+3}$

29. $\frac{6}{7}$

31. $-\frac{m+5}{4}$

33. $\frac{t+5}{t-5}$

35. $\frac{x-8}{x+4}$

37. $\frac{x^2+xy+y^2}{x+y}$

39. $10ab^2$

41. $\frac{3}{2y^4}$

43. $\frac{10}{9a^2}$

45. $-\frac{y+5}{2y}$

47. $(2a-1)(3a-8)$

49. $\frac{x^2-16}{x(x+3)}$

51. $\frac{1}{b(b-1)}$

53. $\frac{x(3x+2)}{(3x+1)(3x-2)}$

55. $\frac{a^2+ab+b^2}{a-b}$

57. $\frac{x^2+4x+16}{(x+4)^2}$

59. $\frac{1}{2x+3y}$

61. $-\frac{7x+3}{7}$

63. $\frac{15}{y^2}$

65. $\frac{2b}{a+2b}$

67. $\frac{x-6}{x+5}$

69. $f(x)\cdot g(x)=\frac{2(x-4)}{(x+1)^2}$; $f(x)\div g(x)=\frac{x-4}{2x^2}$

71. $f(x)\cdot g(x)=-(x-3)^2$; $f(x)\div g(x)=-\frac{(x-4)^2}{(x+3)^2}$

RT3 Exercises

1. **a.** 18; **b.** 18

3. 36; $\frac{41}{36}$

5. 240; $\frac{221}{240}$

7. $72a^5b^4$

9. $x(x+2)(x-2)$

11. $(x-1)^2$

13. $y(x+y)(x-y)$

15. $(x+1)^2(x-5)$

17. $(x-3)^2(2x+1)(x-1)$

19. $6x^3(x+2)^2(x-2)$

21. true; $\frac{1}{3-x}$ is opposite to $\frac{1}{x-3}$

23. false; $\frac{3}{4}+\frac{x}{5}=\frac{3\cdot 5+4x}{20}=\frac{4x+15}{20}$

25. $\frac{8}{a+1}$

27. $\frac{3n-3}{n-2}$

29. $\frac{1}{a+7}$

31. a^2+ab+b^2

33. $\frac{2x^2-x+14}{(x+3)(x-4)}$

35. $\frac{(x+y)^2}{(x+y)(x-y)}$

37. $\frac{y-34}{20(y+2)}$

39. $\frac{4y+17}{y^2-4}$

41. $\frac{x(3x+19)}{(x-4)(x-2)(x+3)}$

43. $\frac{3y^2+7y+14}{(2y-5)(y+2)(y-1)}$

45. $\frac{2x^2-13x+7}{(x+3)(x-3)(x-1)}$

47. $\frac{-y}{(y+3)(y-1)}$

49. $\frac{-(14y^2+3y-3)}{(2y+1)(2y-1)}$

51. $\frac{6+x^2}{3x^3}$

53. $\frac{x-14}{(x+1)(x-4)}$

55. $\frac{-(2x^2+5x-2)}{(x+2)(x+1)}$

57. $(f+g)(x)=\frac{x^2+x+8}{(x+2)(x-3)}$; $(f-g)(x)=\frac{x^2-7x-8}{(x+2)(x-3)}$

59. $(f+g)(x)=\frac{3x^2-2x+3}{(x-1)^2(x+3)}$; $(f-g)(x)=\frac{3x^2-4x-3}{(x-1)^2(x+3)}$

61. every 12th day

63. $\frac{100(P_1-P_0)}{P_0}$

RT4 Exercises

1. $\frac{5}{16}$

3. $-\frac{111}{160}$

5. xy^2

7. $\frac{a-1}{4a+1}$

9. $\frac{-9(x-4)}{2(x+3)}$

11. $\frac{2y-x}{2y+x}$

13. $\frac{a^2(b-3)}{b^2(a-1)}$

15. $\frac{-(2x+y)}{x}$

17. $\frac{n-3}{n}$

19. $\frac{1}{a(a-h)}$

21. $\frac{4}{5}$

23. $\frac{a+b}{ab}$

25. $\frac{(x-3)(x+1)}{x^2+x-1}$

27. $\frac{-ab(a-b)}{a^2-ab+b^2}$

29. The expressions $\frac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$ and $\frac{x+y}{x^2+y^2}$ are **<u>not</u>** equivalent, as if we assume for example that $x = 1$ and $y = 2$, the first expression results in $\frac{5}{6}$ while the second results in $\frac{3}{5}$. Notice that the powers with negative exponents can't be 'shifted to a different level' due to the addition in the numerator and denominator. Only powers that are <u>factors</u> of the numerator or denominator can be 'shifted to a different level' to change the sign of their exponents.

31. $\frac{x+1}{3x}$

33. $\frac{n}{n+1}$

35. $\frac{-2(2a-h)}{a^2(a+h)^2}$

37. $\frac{1}{(a-2)(a+h-2)}$

39. $\frac{-3x-2}{x-2}$

RT5 Exercises

1. $\mathbb{R}$

3. $\mathbb{R}\setminus\{-4,11\}$

5. $\mathbb{R}\setminus\{-5,5,7\}$

7. $x = \frac{17}{2}$

9. $x \in \{-8,-1\}$

11. $r = 2$

13. $r = 30$

15. $y = 3$

17. $x = -5$

19. $x \in \{-3, 1\}$

21. $y = -3$

23. $k = \frac{5}{4}$

25. $y = 4$

27. $x = \frac{1}{5}$

29. $x = \frac{31}{5}$

31. $x = -2$

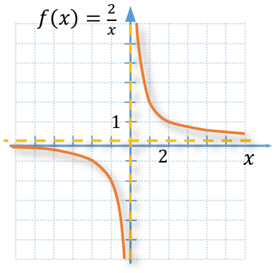
33. $x = 2$

35. $x \in \left\{-\frac{1}{3}, 5\right\}$

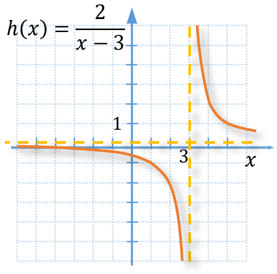
37. $x \in \left\{-\frac{5}{2}, 3\right\}$

39. $x \in \{-2, 6\}$

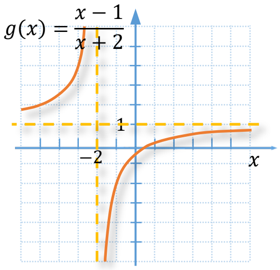
41. $D = \mathbb{R}\setminus\{0\}$; range $= \mathbb{R}\setminus\{0\}$;
VA: $x = 0$; HA: $y = 0$



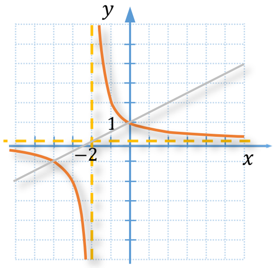
43. $D = \mathbb{R}\setminus\{3\}$; range $= \mathbb{R}\setminus\{0\}$;
VA: $x = 3$; HA: $y = 0$



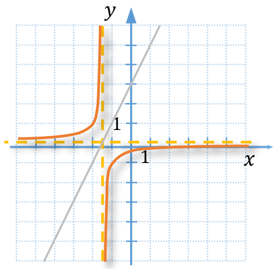
45. $D = \mathbb{R} \setminus \{-2\}$; range $= \mathbb{R} \setminus \{1\}$;
VA: $x = -2$; HA: $y = 1$



47. $g(x) = \frac{2}{x+2}$
VA: $x = -2$; HA: $y = 0$



49. $g(x) = \frac{-1}{2x+3}$
VA: $x = -\frac{3}{2}$; HA: $y = 0$



51. $x \in \left\{-1, \frac{1}{2}\right\}$

53. **a.** $D(10) = 0.9$ If a smoker is 10 times more likely to die of lung cancer than a non-smoker, then 90% of deaths is caused by smoking.

b. $x = 2$

c. The incidence rate is 0 if a smoker is as likely to die of lung cancer as a nonsmoker.

RT6 Exercises

1. $q = 15$

3. factorization of k

5. $a = \frac{F}{m}$

7. $d_1 = \frac{W_1 d_2}{W_2}$

9. $t = \frac{2s}{v_1+v_2}$

11. $R = \frac{r_1 r_2}{r_1+r_2}$

13. $q = \frac{fp}{p-f}$

15. $v = \frac{PVt}{Tp}$

17. $b = \frac{2A}{h} - a$ or $b = \frac{2A-ah}{h}$

19. $s = \frac{Rg}{g-R}$

21. $n = \frac{IE}{E-Ir}$

23. $r = \frac{Re}{E-e}$

25. $R = \frac{V}{\pi h^2} + \frac{h}{3}$ or $R = \frac{3V+\pi h^3}{3\pi h^2}$

27. $h = \frac{2R^2 g}{V^2} - R$ or $h = \frac{2R^2 g - V^2 R}{V^2}$

29. 12.375 kg

31. 77 km

33. ~1142 zebras

35. ~155 white-tailed eagles

37. $PR = 6; PS = 3; SR = 4.2$

39. ~17.8 km/h

41. 4.8 km/h

43. 50 km/h

45. 1275 km

47. 2 km

49. $\frac{4(x+y)}{xy}$

51. 15 hr

53. 24 hr

55. 2450 people

57. 20 km

59. 12 hours

61. 1.4 m

63. 2651 km

65. ~1802 N

Radicals and Radical Functions - ANSWERS

RD1 Exercises

1. 7

3. not a real number

5. 0.04

7. 4

9. 0.2

11. $\frac{1}{0.03}$

13. 0.2

15. not a real number

17. **a.** negative **b.** not a real number **c.** 0

19. 15

21. $|x|$

23. $9|x|$

25. $|x+3|$

27. $|x-2|$

29. -5

31. $-5a$

33. $5|x|$

35. $y-3$

37. $|2a-b|$

39. $|a+1|^3$

41. $-k^5$

43. 18.708

45. 1.710

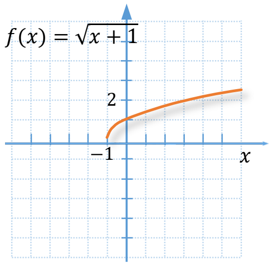
47. 8

49. 11

51. 50

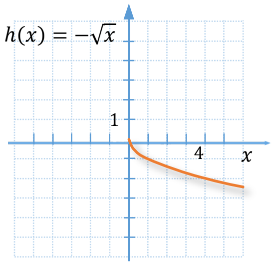
53. 14 m by 7 m; 42 m

55. $D=[-1,\infty)$
range $=[0,\infty)$



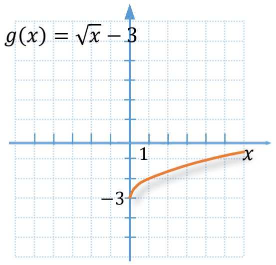
Translation: 1 step to the left

57. $D=[0,\infty)$
range $=(-\infty,0]$



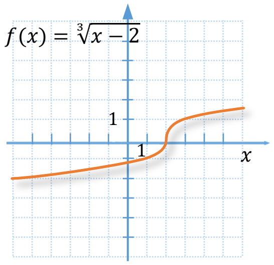
Reflection in x-axis

59. $D=[0,\infty)$
range $=[-3,\infty)$



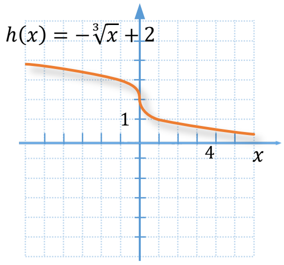
Translation: 3 steps down

61. $D=\mathbb{R}$
range $=\mathbb{R}$



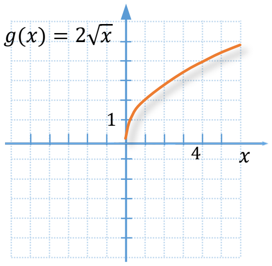
Translation: 2 steps to the right

63. $D=[0,\infty)$
range $=(-\infty,0]$

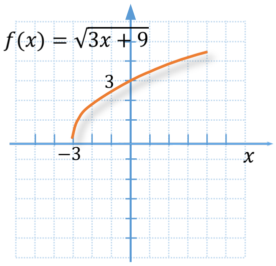


Reflection in x-axis

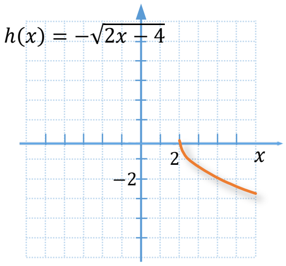
65. $D=[0,\infty)$
range $=[0,\infty)$



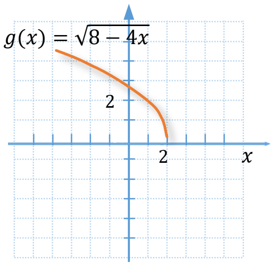
67. $D = [-3, \infty)$
range $= [0, \infty)$



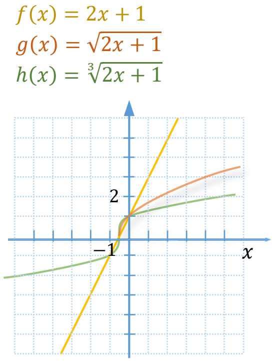
69. $D = [2, \infty)$
range $= (-\infty, 0]$



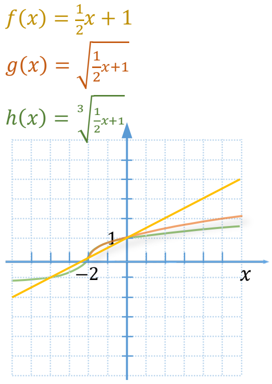
71. $D = (-\infty, 2]$
range $= [0, \infty)$



73.



75.



77. ~186 cm

79. 3.25 m

81. $700\sqrt{15}$ m²

RD2 Exercises

1. **a.-B.; b.-A.; c.-C.; d.-F.; e.-D.; f.-E.**
3. 2
5. -343
7. $-\frac{1}{10}$
9. $\frac{8}{27}$
11. not a real number
13. -2
15. $5^{\frac{1}{2}}$
17. x^3
19. $4x^2$
21. $5x^{-\frac{5}{2}}$
23. 32
25. $\sqrt[5]{x^3}$
27. $\sqrt[3]{9}$
29. $\frac{2}{\sqrt{x}}$
31. $3^{\frac{7}{8}}$
33. $2^{\frac{3}{4}}$
35. $5^{\frac{5}{4}}$
37. $x^{\frac{1}{2}} \cdot y^{\frac{10}{3}}$
39. $\frac{x^{\frac{5}{9}}}{y^{\frac{1}{2}}}$
41. $5x^{\frac{4}{15}}$
43. $\sqrt[3]{x}$
45. y^{-3} or $\frac{1}{y^3}$
47. $\sqrt[3]{9}$
49. $2y^2$
51. $2x^2\sqrt[3]{2y^2}$
53. $2x\sqrt{y}$

55. $\sqrt[6]{5^5}$

57. $\sqrt[6]{9a^5}$

59. $x\sqrt{x}$

61. $\frac{\sqrt{x}}{x^2}$ or $\frac{1}{x\sqrt{x}}$

63. $\frac{2}{\sqrt[12]{x^5}}$

65. $\sqrt[12]{xy}$

67. $\sqrt[24]{x}$

69. $\sqrt[8]{x^3}$

71. To treat an equation as an identity, the equation must be true for <u>all</u> variable values in the domain. The fact that the equation is true for specific values does not guarantee that it is true for all values of x and y. A counterexample: Let $x = y = 2$. Then $\sqrt[n]{2^n + 2^n} = \sqrt[n]{2 \cdot 2^n} = 2\sqrt[n]{2} \neq 2 + 2 = 4$.

73. 30 beats per minute

RD3 Exercises

1. 5

3. $3\sqrt{2}$

5. $30\sqrt{3}$

7. $3x^4\sqrt{2}$

9. $4x^3y\sqrt{6xy}$

11. $2x^2$

13. $3\sqrt{2}$

15. $\sqrt{6}$

17. $2b\sqrt{b}$

19. $4x\sqrt{y}$

21. 2

23. $2a\sqrt[3]{b}$

25. $12x^2y^4\sqrt{y}$

27. $-5a^2b^3c^4$

29. $\frac{m^2n^5}{2}$

31. $a^3b^3\sqrt{7a}$

33. $2x^2y^3\sqrt[5]{2x^2}$

35. $-3a^3b^2\sqrt[4]{2a^3b^2}$

37. $\frac{4}{7}$

39. $\frac{11}{y}$

41. $\frac{3a\sqrt[3]{a^2}}{4}$

43. $\frac{2x^3}{yz^4}$

45. $\sqrt{6}$

47. $-x^2\sqrt{x}$

49. $\frac{-\sqrt{xy}}{x^2y}$

51. $\frac{x^2\sqrt[6]{x}}{yz^2}$

53. This is not correct as the radical of a sum is <u>not</u> the sum of radicals. We can simplify it by factoring the radicand: $\sqrt{x^3 + x^2} = \sqrt{x^2(x+1)} = |x|\sqrt{x+1}$

55. $\sqrt[10]{x^7}$

57. $2\sqrt[15]{2^4}$ or $2\sqrt[15]{16}$

59. $\sqrt[4]{x}$

61. $\frac{\sqrt[15]{2^7a^{11}}}{a}$

63. $\sqrt[6]{2x^5}$

65. $\sqrt[12]{x^{11}}$

67. $\sqrt{6}$

69. $\sqrt{n^2 - 9}$

71. $2\sqrt{31}$

73. $2\sqrt{5}$

75. $\frac{\sqrt{41}}{7}$

77. $2\sqrt{38}$

79. $\sqrt{p^2 + q^2}$

81. ~7.05 meters

83. $(-4, 0)$ and $(4, 0)$

85. 30 m

RD4 Exercises

1. No. The equation must be true for all $x \geq 0$.

3. $7\sqrt{3}$

5. $13y\sqrt{3x}$

7. $14\sqrt{2} + 2\sqrt{3}$

9. $11\sqrt[3]{2}$

11. $(1 + 6a)\sqrt{5a}$

13. $(4x - 6)\sqrt{x}$ or $2(2x - 3)\sqrt{x}$

15. $24\sqrt{2x}$

17. $(x + 1)\sqrt[3]{6x}$

19. $-8n\sqrt{2}$

21. $(6ab^2 - 9ab)\sqrt{ab}$
or $3ab(2b - 3)\sqrt{ab}$

23. $5x\sqrt[4]{xy}$

25. $-x\sqrt[3]{2x} + \sqrt{2}$

27. $\sqrt{x + 3}$

29 $(5 - x)\sqrt{x - 1}$

31. $\frac{3\sqrt{3}}{4}$

33. $\frac{4a\sqrt[4]{a}}{9}$

35. Error: cannot add unlike radicals (see line 3). Correct solution: $2\sqrt{2} + 2\sqrt[3]{2} = 2\left(\sqrt{2} + \sqrt[3]{2}\right)$

37. $3\sqrt{5} - 10$

39. $9 - 2\sqrt{5}$

41. -6

43. 1

45. -13

47. $30 - 10\sqrt{5}$

49. $a - 25b$

51. $9 + 6\sqrt{2}$

53. $38 + 12\sqrt{10}$

55. $22 - 13\sqrt{3}$

57. $\sqrt[3]{4y^2} - 4\sqrt[3]{2y} - 5$

59. 1

61. $(f + g)(x) = 13x\sqrt{5x}$; $(fg)(x) = 150x^3$

63. $\frac{\sqrt{10}}{4}$

65. $2\sqrt{6}$

67. $-\sqrt{5}$

69. $\frac{\sqrt{10y}}{8}$

71. $\frac{y\sqrt[3]{9x^2y}}{3x^2}$

73. $\sqrt[4]{pq^3}$

75. $\frac{6-\sqrt{2}}{2}$

77. $6 + 2\sqrt{6}$

79. $\frac{3\sqrt{5}-2\sqrt{3}}{11}$

81. $\sqrt{m} - 2$

83. $\frac{3+4\sqrt{3x}+4x}{3-4x}$

85. $\frac{2a+2\sqrt{ab}}{a-b}$

87. $1 - 2\sqrt{5}$

89 $\frac{2-9\sqrt{2}}{3}$

91. $\frac{6-2\sqrt{6p}}{3}$

93. Yes. $\frac{\sqrt{3}-1}{1+\sqrt{3}}$ after rationalization of the denominator becomes $2 - \sqrt{3}$.

95. $2\sqrt{3} \approx 3.5$ cm

RD5 Exercises

1. False, as the radicals do not contain a variable.

3. True, as the radical cannot be negative.

5. $x = \frac{39}{7}$

7. $x = \frac{2}{3}$

9. no solution

11. $x = -27$

13. $y = 19$

15. $a = \frac{1}{25}$

17. $r = 5$

19. $y = 18$

21. $x = 9$

23. $x \in \{-1, 3\}$

25. $y = 4$

27. $x = 5$

29. not correct, as $(8 - x)^2 = 64 - 16x + x^2$

31. $x = 2$

33. $p = 9$

35. No solution

37. $t = -1$

39. No solution

41. $n = 3$

43. $n = -2$

45. $a \in \{2, 6\}$

47. No solution

49. $m = 2$

51. $x \in \left\{-1, \frac{1}{3}\right\}$

53. $x \in \{1,9\}$

55. $x = \frac{4}{9}$

57. $k \in \{-2, -1\}$

59. $x \in \{-5, 5\}$

61. $a \in \left\{0, \frac{125}{4}\right\}$

63. $L = CZ^2$

65. $m = \frac{2K}{V^2}$

67. $F = \frac{Mm}{r^2}$

69. $C = \frac{1}{4\pi^2 F^2 L}$

71. $r = \frac{a}{4\pi^2 N^2}$

73. 189 cm

75. 22 m

RD6 Exercises

1. The mistake is in the first step – the product rule for radicals cannot be used, so we need to convert into i notation before multiplying: $\sqrt{-3} \cdot \sqrt{-15} = \sqrt{3}\, i \cdot \sqrt{15}\, i = \sqrt{45}\, i^2 = 3\sqrt{5}\,(-1) = \mathbf{-3\sqrt{5}}$

3. Both are correct because $(8i)^2 = 8^2 \cdot -1 = -64$ and $(-8i)^2 = (-8)^2 \cdot -1 = -64$.

5. $10i$

7. $7\sqrt{2}\, i$

9. -7

11. $-7\sqrt{3}$

13. $-4\sqrt{2}\, i$

15. $24\sqrt{10}\, i$

17. $-73 + 31i$

19. $-90 - 46\sqrt{3}\, i$

21. 112

23. $18 + 2i$

25. $6 - 82i$

27. $-13 + 84i$

29. 181

31. 1

33. i

35. $-i$

37. $2 - 2\sqrt{14}\, i$

39. $\frac{1}{5} + \frac{\sqrt{39}}{10} i$

41. $\frac{6}{5} i$

43. $\frac{15}{74} - \frac{21}{74} i$

45. $\frac{7}{25} + \frac{24}{25} i$

47. $\frac{97}{137} + \frac{27}{137} i$

49. Yes, because $(-2i)^2 = (-2)^2 i^2 = -4$.

51. Yes, because substituting $x = 3 - 2i$ into the equation gives $(3 - 2i)^2 - 6(3 - 2i) + 13 = (9 - 12i + 4i^2) - 18 + 12i + 13 = 4 + 4i^2 = 4 - 4 = 0$.

53. No, because substituting $x = 5 + i$ into the equation gives $(5 + i)^2 + 5(5 + i) + 60 = (25 + 10i + i^2) + 25 + 5i + 60 = 110 + 15i + i^2 = 110 + 15i - 1 = 109 + 15i \neq 0$.

Quadratic Equations and Functions - ANSWERS

Q1 Exercises

1. False

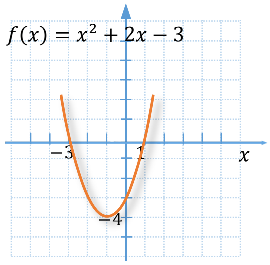
3. True

5. False

7. False

9. **a)**

x	$f(x)$
1	0
0	-3
-1	-4
-2	-3
-3	0



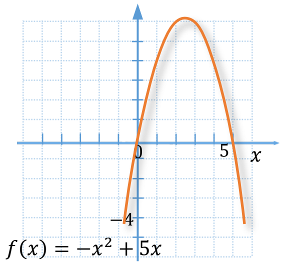
b) $(-3,0)$, $(1,0)$

c) $x \in \{-3,1\}$

The solutions are the first coordinates of the x-intercepts.

11. **a)**

x	$f(x)$
0	0
1	4
2	6
$\frac{5}{2}$	6.25
3	6
4	4
5	0



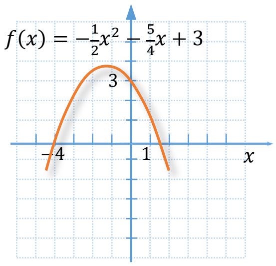
b) $(0,0)$, $(5,0)$

c) $x \in \{0,5\}$

The solutions are the first coordinates of the x-intercepts.

13. **a)**

x	$f(x)$
-4	0
-2	3.5
-1	3.75
0	3
1	1.25
2	-1.5



b) $(-4,0)$, $\left(\frac{3}{2},0\right)$

c) $x \in \left\{-4,\frac{3}{2}\right\}$

The solutions are the first coordinates of the x-intercepts.

15. $x \in \{-4\sqrt{2}, 4\sqrt{2}\}$

17. $n \in \{-2\sqrt{6}, 2\sqrt{6}\}$

19. $y \in \{-2\sqrt{10}, 2\sqrt{10}\}$

21. $x \in \{-7,1\}$

23. $t \in \left\{\frac{-2-2\sqrt{3}}{5}, \frac{-2+2\sqrt{3}}{5}\right\}$

25. $y \in \{-4 - 2\sqrt{11}, -4 + 2\sqrt{11}\}$

27. $y \in \left\{\frac{44}{5}, \frac{56}{5}\right\}$

29. $x \in \left\{\frac{-3-5i}{4}, \frac{-3+5i}{4}\right\}$

31. $x \in \left\{\frac{1-3\sqrt{2}}{2}, \frac{1+3\sqrt{2}}{2}\right\}$

33. $y \in \{0,3\}$

35. $n = -2$

37. $y \in \left\{\frac{-7-\sqrt{53}}{2}, \frac{-7+\sqrt{53}}{2}\right\}$

39. $a \in \{-1 - \sqrt{2}\,i, -1 + \sqrt{2}\,i\}$

41. $x \in \{6 - 2\sqrt{5}, 6 + 2\sqrt{5}\}$

43. $x \in \left\{\frac{-1-\sqrt{7}}{3}, \frac{-1+\sqrt{7}}{3}\right\}$

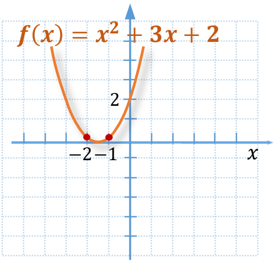
45. $x \in \left\{\frac{4-\sqrt{3}}{3}, \frac{4+\sqrt{3}}{3}\right\}$

47. $x \in \left\{\frac{2-\sqrt{3}}{3}, \frac{2+\sqrt{3}}{3}\right\}$

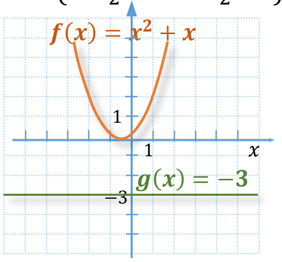
49. $x \in \left\{\frac{1-2\sqrt{19}}{5}, \frac{1+2\sqrt{19}}{5}\right\}$

51. $x \in \{1 - 2\sqrt{2}, 1 + 2\sqrt{2}\}$

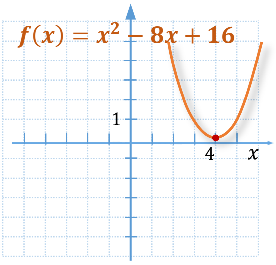
53. $x \in \{-2, -1\}$



55. $x \in \left\{\frac{-1-\sqrt{11}\,i}{2}, \frac{-1+\sqrt{11}\,i}{2}\right\}$



57. $x = 4$



59. $a = 1 - \sqrt{5} \approx -1.24$, or $a = 1 + \sqrt{5} \approx 3.24$

61. $x = \frac{-5-\sqrt{11}}{2} \approx -4.16$, or $x = \frac{-5+\sqrt{11}}{2} \approx -0.84$

63. $y = \frac{-1-\sqrt{7}}{6} \approx -0.27$, or $y = \frac{-1+\sqrt{7}}{6} \approx 0.61$

65. $x = \frac{17-\sqrt{249}}{10} \approx 0.12$, or $x = \frac{17-\sqrt{249}}{10} \approx 3.28$

67. $x = \frac{5-\sqrt{7}}{6} \approx 0.39$, or $x = \frac{5+\sqrt{7}}{6} \approx 1.27$

69. 2 rational solutions; factoring possible

71. 2 real solutions; use quadratic formula

73. 1 double rational solution; factoring possible

75. $k = 25$

77. No, as the product of a rational and irrational number is irrational. This would contradict the fact that the quadratic equation has integral coefficients.

79. $x = 1 \pm \sqrt{10}$

81. $x = \frac{5 \pm 2\sqrt{6}}{2}$

83. $x \in \{-3, 2\}$

85. $x = -1 \pm 2i$

87. $x \in \left\{-\frac{3}{2}, 1\right\}$

89. $x = 5 \pm \sqrt{53}$

Q2 Exercises

1. The solution is incorrect as the question calls for the values of x not a.

3. $x \in \{-\sqrt{5}, -\sqrt{2}, \sqrt{2}, \sqrt{5}\}$

5. $x \in \left\{\frac{1}{4}, 16\right\}$

7. $a \in \{-1, 2\}$

9. $x = 9$

11. $x \in \{-1, 3, 1 - \sqrt{2}, 1 + \sqrt{2}\}$

13. $x = 8$

15. $u \in \left\{-\frac{8}{3}, -1\right\}$

17. $x \in \{-1 \pm \sqrt{2}, 3 \pm \sqrt{10}\}$

19. $r = \pm\sqrt{\frac{V}{\pi h}}$

21. $s = \pm\sqrt{\frac{3}{Vh}}$

23. $s = \pm\sqrt{\frac{kq_1q_2}{N}}$

25. $H = \pm\sqrt{\frac{703W}{I}}$

27. $r = \frac{-\pi h \pm \sqrt{\pi^2 h^2 + 2\pi A}}{2\pi}$

29. $a = \pm\frac{bt}{\sqrt{1-t^2}}$ **31.** $I = \frac{E\pm\sqrt{E^2-4PR}}{2R}$ **33.** $v = \frac{c\sqrt{m^2-m_0^2}}{m}$ **35.** $\frac{(r+R)\sqrt{pR}}{R}$

37. **a.** $r - c$ **b.** $r + c$ **39.** $7 + 2\sqrt{35},\ 10 + 2\sqrt{35},$ and $17 + 2\sqrt{35}$

41. 5 ft by 12 ft **43.** 10 ft 3 in **45.** 9 in by 13 in **47.** $10\sqrt{2}$ m by $5\sqrt{2}$ m

49. 1.5 ft **51.** 12 cm **53.** 7 cm by 13 cm **55.** 60 km/h

57. Skyhawk at 250 km/h; Mooney Bravo at 350 km/h **59.** ~10.8 km/h

61. 800 km/h and 840 km/h **63.** 7 hr 32 min

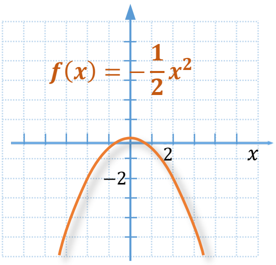
65. Helen: ~16 hr 31 min; Monica: ~15 hr 31 min **67.** smaller-diameter pipe: 2 hr;
larger-diameter pipe: 3 hr

69. ~3.8 sec **71.** 4.2%

Q3 Exercises

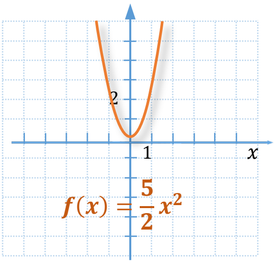
1. **a.-III; b.-I; c.-IV; d.-II** **3.** **a.-II; b.-III; c.-I; d.-IV**

5. wider; opens down



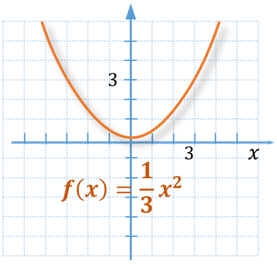
Range = $(-\infty, 0]$

7. narrower; opens up



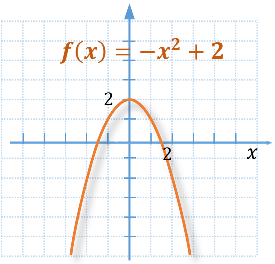
Range = $[0, \infty)$

9. narrower; opens up



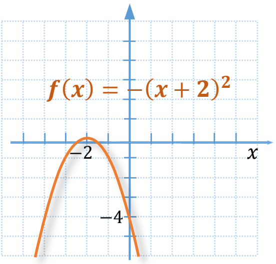
Range = $[0, \infty)$

11. S_x; shift 2 units up



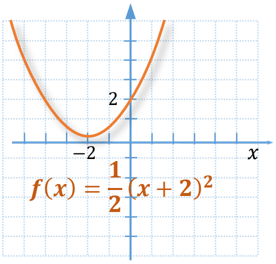
Domain = $\mathbb{R}$
Range = $(-\infty, 2]$
Axis of symmetry: $x = 0$

13. S_x; shift 2 units to the left



Domain = $\mathbb{R}$
Range = $(-\infty, 0]$
Axis of symmetry: $x = -2$

15. vertical dilation by $\frac{1}{2}$;
shift 2 units to the left



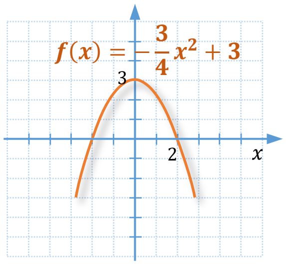
Domain = $\mathbb{R}$
Range = $[0, \infty)$
Axis of symmetry: $x = -2$

17. vertex = $(0,3)$

shape of $\frac{3}{4}x^2$; opens down

x-int.: $(-2,0)$, $(2,0)$

y-int.: $(0,3)$



S_x; vertical dilation by $\frac{3}{4}$;

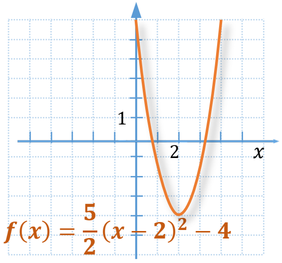
shift 3 units up

19. vertex = $(2,-4)$

shape of $\frac{5}{2}x^2$; opens up

x-int.: $\left(\frac{10-2\sqrt{10}}{5}, 0\right)$, $\left(\frac{10+2\sqrt{10}}{5}, 0\right)$

y-int.: $(0,6)$



vertical dilation by $\frac{5}{2}$;

shift 2 units to the right;

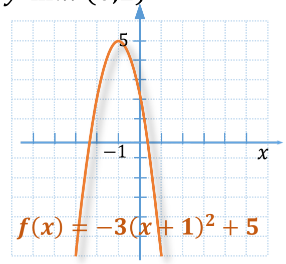
shift 4 units down

21. vertex = $(-1,5)$

shape of $3x^2$; opens down

x-int.: $\left(\frac{-3-\sqrt{15}}{3}, 0\right)$, $\left(\frac{-3+\sqrt{15}}{3}, 0\right)$

y-int.: $(0,2)$



S_x; vertical dilation by 3;

shift 1 unit to the left;

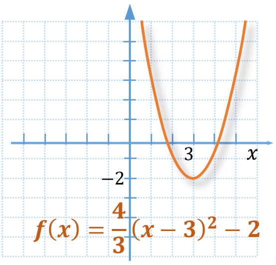
shift 5 units up

23. vertex = $(3,-2)$

shape of $\frac{4}{3}x^2$; opens up

x-int.: $(-2,0)$, $(2,0)$

y-int.: $(0,3)$

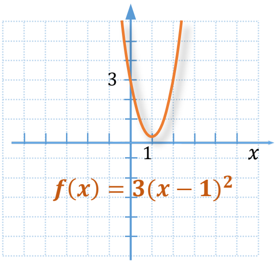


vertical dilation by $\frac{4}{3}$;

shift 3 units to the right;

shift 2 units down

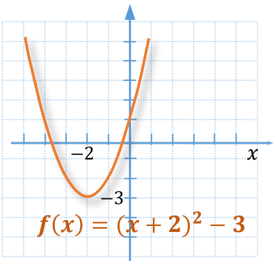
25. vertex = $(1,0)$



Minimum value = 0;

Range = $[0,\infty)$

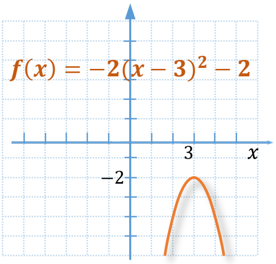
27. vertex = $(-2,-3)$



Minimum value = -3;

Range = $[-3,\infty)$

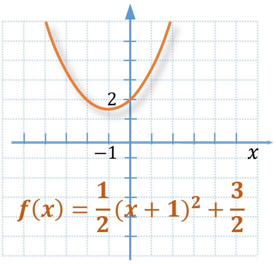
29. vertex = $(3,-2)$



Maximum value = -2;

Range = $(-\infty,-2]$

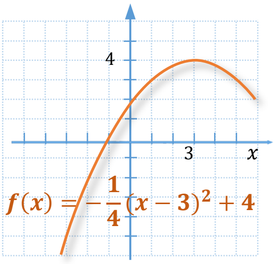
31. vertex = $\left(-1,\frac{3}{2}\right)$



Minimum value = $\frac{3}{2}$;

Range = $\left[\frac{3}{2},\infty\right)$

33. vertex = $(3,4)$



Maximum value = 4;

Range = $(-\infty,4]$

35. $f(x) = (x+3)^2 - 4$

37. $f(x) = 2(x-1)^2 - 5$

39. $f(x) = -3(x+2)^2 + 6$

Q4 Exercises

1. $f(x) = (x+3)^2 + 1;\ V(-3,1)$

3. $f(x) = \left(x + \frac{1}{2}\right)^2 - \frac{13}{4};\ V(-\frac{1}{2}, -\frac{13}{4})$

5. $f(x) = -\left(x - \frac{7}{2}\right)^2 + \frac{61}{4};\ V\left(\frac{7}{2}, \frac{61}{4}\right)$

7. $f(x) = -3(x-1)^2 + 15;\ V(1,15)$

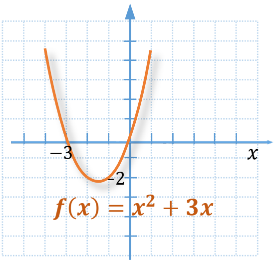
9. $f(x) = \frac{1}{2}(x+3)^2 - \frac{11}{2};\ V\left(-3, -\frac{11}{2}\right)$

11. $V\left(\frac{3}{2}, -\frac{11}{4}\right)$

13. $V(1,8)$

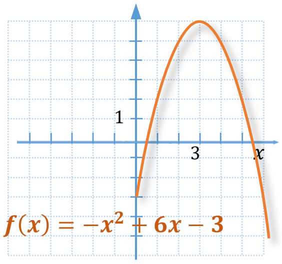
15. $V(-1,-23)$

17. $V\left(-\frac{3}{2}, -\frac{9}{4}\right)$; opens up
shape of x^2



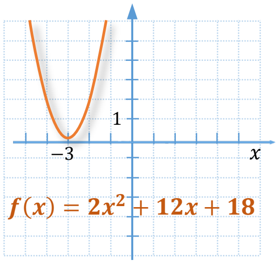
$D = \mathbb{R}$; Range $= \left[-\frac{9}{4}, \infty\right)$

19. $V(3,6)$; opens down
shape of x^2



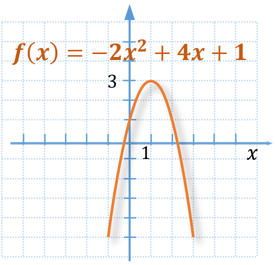
$D = \mathbb{R}$; Range $= (-\infty, 6]$

21. $V(-3,0)$; opens up
shape of $2x^2$



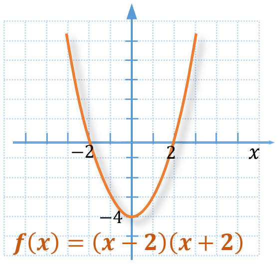
$D = \mathbb{R}$; Range $= [0, \infty)$

23. $V(1,3)$; opens down
shape of $2x^2$



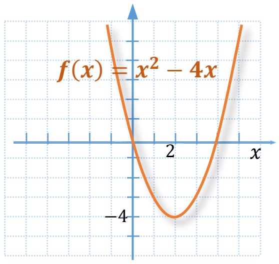
$D = \mathbb{R}$; Range $= (-\infty, 3]$

25. zeros: -2, 2; $V(0,-4)$;
opens up; shape of x^2



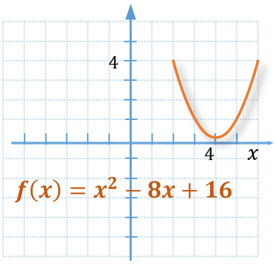
Minimum value $= -4$
Minimum occurs at $x = 0$

27. zeros: 0, 4; $V(2,-4)$;
opens up; shape of x^2



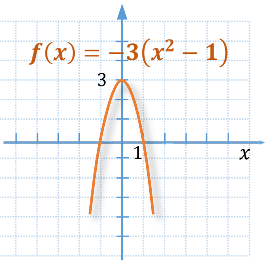
Minimum value $= -4$
Minimum occurs at $x = 2$

29. zero: 4; $V(4,0)$;
opens up; shape of x^2



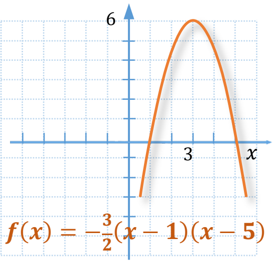
Minimum value $= 0$
Minimum occurs at $x = 4$

31. zero: $-1, 1$; $V(0,3)$;
opens down; shape of $3x^2$



Maximum value $= 3$
Maximum occurs at $x = 0$

33. zeros: $1, 5$; $V(3,6)$;
opens down; shape of $\frac{3}{2}x^2$



Maximum value $= 6$
Maximum occurs at $x = 3$

35. $f(x) = x(5x - 2)$

37. $f(x) = \frac{3}{4}(x - 1)(x - 4)$

39. $x(3x - 1) = 0$

41. $(x - 2)^2 = 0$

43. By observing the second coordinate of the vertex in combination with the opening. For example, the second coordinate "+'ve" with openning up means no x-intercepts while the second coordinate "+'ve" with openning down indicates 2 x-intercepts.

45. true

47. true

49. true

51. 30.625 m; 5 sec

53. 20; \$150

55. 16, 16

57. 4 m by 8 m

59. **a.** $P(n) = 60 - 2n$ **b.** $R(n) = (60 - 2n)n$ **c.** 15 **d.** 450\$

Trigonometry - ANSWERS

T1 Exercises

1. 20.075°

3. 274.304°

5. 15.168°

7. $18^\circ 0' 45''$

9. $65^\circ 0' 5''$

11. $175^\circ 23' 58''$

13. $83^\circ 59'$

15. $33^\circ 50'$

17. $28^\circ 03' 03''$

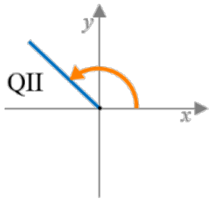
19. $60^\circ, 150^\circ$

21. $45^\circ, 135^\circ$

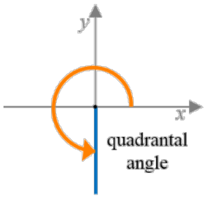
23. $74^\circ 30', 164^\circ 30'$

25. $180 - \theta^\circ$

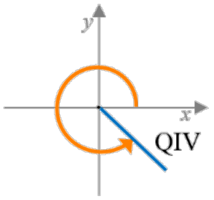
27.



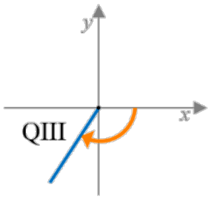
29.



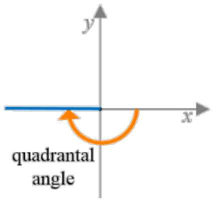
31.



33.



35.



37. 15°

39. 135°

41. $30^\circ + k \cdot 360^\circ$

43. $k \cdot 360^\circ$

45. $\alpha^\circ + k \cdot 360^\circ$

47. 7.5°

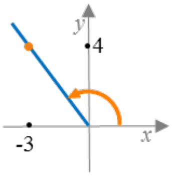
T2 Exercises

1. $\sin\theta = \frac{3}{5}, \quad \cos\theta = \frac{4}{5}, \quad \tan\theta = \frac{3}{4}, \quad \csc\theta = \frac{5}{3}, \quad \sec\theta = \frac{5}{4}, \quad \cot\theta = \frac{4}{3}$

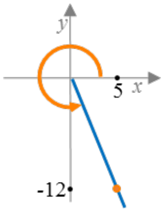
3. $\sin\theta = \frac{\sqrt{3}}{2}, \quad \cos\theta = \frac{1}{2}, \quad \tan\theta = \sqrt{3}, \quad \csc\theta = \frac{2\sqrt{3}}{3}, \quad \sec\theta = 2, \quad \cot\theta = \frac{\sqrt{3}}{3}$

5. $\sin\theta = \frac{n}{\sqrt{n^2+4}} = \frac{n\sqrt{n^2+4}}{n^2+4}, \quad \cos\theta = \frac{2}{\sqrt{n^2+4}} = \frac{2\sqrt{n^2+4}}{n^2+4}, \quad \tan\theta = \frac{n}{2}, \quad \csc\theta = \frac{\sqrt{n^2+4}}{n}, \quad \sec\theta = \frac{\sqrt{n^2+4}}{2}, \quad \cot\theta = \frac{2}{n}$

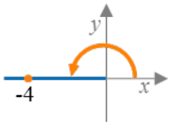
7. $\sin\theta = \frac{4}{5}, \quad \cos\theta = -\frac{3}{5}, \quad \tan\theta = -\frac{4}{3}$



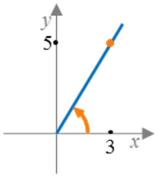
9. $\sin\theta = -\frac{12}{13}, \quad \cos\theta = \frac{5}{13}, \quad \tan\theta = -\frac{12}{5}$



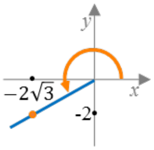
11. $\sin\theta = 0, \quad \cos\theta = -1, \quad \tan\theta = 0$



13. $\sin\theta = \frac{5\sqrt{34}}{34}, \quad \cos\theta = \frac{3\sqrt{34}}{34}, \quad \tan\theta = \frac{5}{3}$



15. $\sin\theta = -\frac{1}{2}, \quad \cos\theta = -\frac{\sqrt{3}}{2}, \quad \tan\theta = \frac{\sqrt{3}}{3}$



17. sine, cosine, cosecant and secant are negative, tangent and cotangent are positive

19. negative

21. negative

23. positive

25. positive

27. negative

29. 1

31. -1

33. 0

35. $undefined$

37. 0

39. $\cos\beta = -\frac{\sqrt{5}}{3}$

$\tan\beta = \frac{2\sqrt{5}}{5}$

T3 Exercises

1. 0.6000

3. -0.9106

5. $\frac{\sqrt{2}}{2}$

7. $\frac{\sqrt{3}}{2}$

9. $\frac{1}{2}$

11. 1

13. $\cos 67.5°$

15. 82°

17. 13°

19. 6°

21. QIII and QIV

23. QII

25. QIV

27. negative

29. negative

31. positive

33. positive

35. $\frac{\sqrt{3}}{2}$

37. $\frac{1}{2}$

39. $-\frac{\sqrt{3}}{2}$

41. 1

43. $60°, 300°$

45. $60°, 120°$

47. $135°, 225°$

49. $150°, 330°$

51. $\sin\alpha = -\frac{4}{5}$
$\tan\alpha = -\frac{4}{3}$

T4 Exercises

1. 52.2°

3. 68.4°

5. 60°

7. $\angle B = 54°,\ b \simeq 16.5,\ c \simeq 20.4$

9. $\angle A \simeq 31.0°,\ \angle B \simeq 59.0°,\ c \simeq 17.5$

11. $\angle A \simeq 74.4°,\ \angle B \simeq 15.6°,\ b \simeq 2.6$

13. $a = 2\sqrt{3},\ b = 6\sqrt{3},\ d = 4\sqrt{3},\ h = 6$

15. $a = 5,\ b = \frac{5}{2},\ h = \frac{5\sqrt{3}}{2},\ s = 5$

17. $32\sqrt{3}$ cm

19. 23°

21. 700 m

23. 317 m

25. 1101 km; direction of 107° (or **S73°E**)

27. 552 m; 447 m

29. 29.6 m

31. 237 m

T5 Exercises

1.	$\angle P = 39°,\ p \simeq 15.3\text{ cm},\ s \simeq 22.8\text{ cm}$	**3.**	$\angle A \simeq 25.9°,\ \angle C \simeq 18.1°,\ c \simeq 19.3\text{ ft}$
5.	$\angle I \simeq 19.8°,\ i \simeq 8.8\text{ cm},\ \angle J \simeq 122.2$	**7.**	$b = 10,\ \angle C = 120°,\ c \simeq 17.3$
9.	$\angle A \simeq 25.6°,\ a \simeq 10.5,\ \angle B \simeq 9.4°$	**11.**	$\angle X \simeq 40.6°,\ y \simeq 18.4\text{ m},\ \angle Z \simeq 54.4°$
13.	$p \simeq 19.8\text{ m},\ \angle R \simeq 33.1°,\ \angle S \simeq 129.9°$	**15.**	$\angle I \simeq 48.5°,\ \angle J \simeq 86.3°,\ \angle K \simeq 45.2°$
17.	$\angle A \simeq 17°,\ \angle B \simeq 103°,\ c \simeq 8.9$	**19.**	$\angle A \simeq 34.7°,\ \angle B \simeq 48.1°,\ \angle C \simeq 97.2°$

21. No, because the ratio of sines of angles is not the same as the ratios of those angles.
For instance, $\frac{\sin 90°}{\sin 45°} = \sqrt{2} \neq \frac{90°}{45°} = 2.$

23.	127 m	**25.**	8.1 km; 11.0 km
27.	~ 6.4 m	**29.**	~ 351 m from A; ~ 295 from B
31.	~ 777 km; direction: ~ 279°2′	**33.**	~26°
35.	~76 m	**37.**	~ 1199 m^2
39.	~ 69°	**41.**	~247.3 m^2